Input-output Controllability Analysis

Idea: Find out how well the process can be controlled - without having to design a specific controller

Note: Some processes are impossible to control

Reference: S. Skogestad, ``A procedure for SISO controllability analysis - with application to design of pH neutralization processes'', *Comp.Chem.Engng.*, **20**, 373-386, 1996.

Example: First-order with delay process

$$g(s) = k\frac{e^{-\theta s}}{1+\tau s}; \quad G_d(s) = k_d\frac{e^{-\theta_d s}}{1+\tau_d s}$$

$+$ Measurement delays: $\theta_m, \quad \theta_{md}$.

Problem: What values are desired for good controllability?

Qualitative results:

	Feedback control	Feedforward control
k	[illegible]	[illegible]
τ	[illegible]	[illegible]
θ	[illegible]	[illegible]
k_d	[illegible]	[illegible]
τ_d	[illegible]	[illegible]
θ_d	[illegible]	[illegible]
θ_m	[illegible]	[illegible]
θ_{md}	[illegible]	[illegible]

WANT TO QUANTIFY!

Example: First-order with delay process

$$g(s) = k\frac{e^{-\theta s}}{1+\tau s}; \quad G_d(s) = k_d\frac{e^{-\theta_d s}}{1+\tau_d s}$$

$$+ \text{Measurement delays: } \theta_m, \quad \theta_{md}.$$

Problem: What values are desired for good controllability?

Qualitative results:

	Feedback control	Feedforward control
k	Large	Large
τ	Small	Small
θ	Small	Small
k_d	Small	Small
τ_d	Large	Large
θ_d	No effect	Large
θ_m	Small	No effect
θ_{md}	No effect	Small

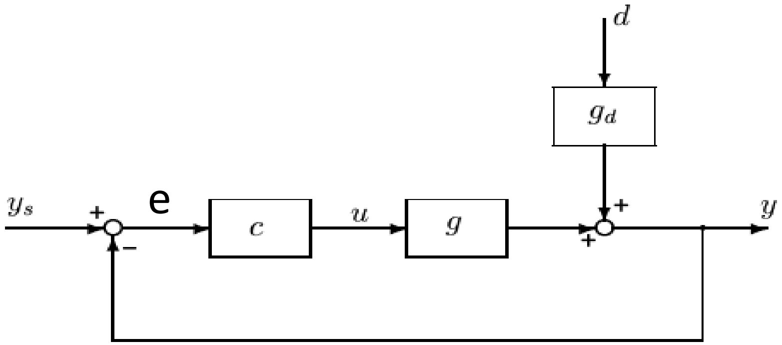
Want fast and large response from input (MV) to output (CV) (rows k, τ, θ)

Want slow and small response from disturbance (DV) to output (CV) (rows k_d, τ_d, θ_d)

Opposite for feedback and feedforward (rows θ_m, θ_{md})

WANT TO QUANTIFY!

Recall: Closed-loop frequency response



No control ($c = 0$): $e_{OL} = y_s - y = y_s - g_d d$

With control: $e = y_s - y = Sy_s - Sg_d d = Se_{OL}$

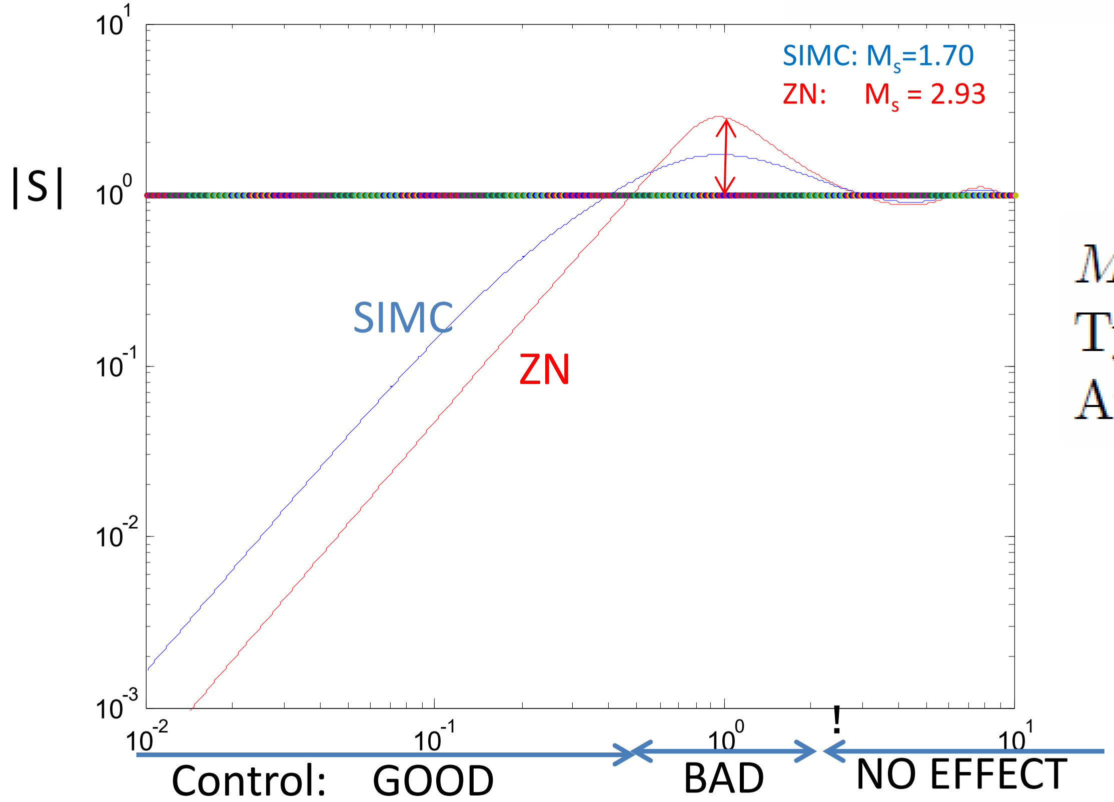
$S = \frac{1}{1+L}$ - sensitivity function = effect of control

$L = gc$ - loop transfer function

Low ω where $|S| < 1$: Control reduces error

Intermediate ω where $|S| > 1$: Control increases error

High ω where $S = 1$ ($L \to 0$): Control has no effect



M_s = peak of $|S|$

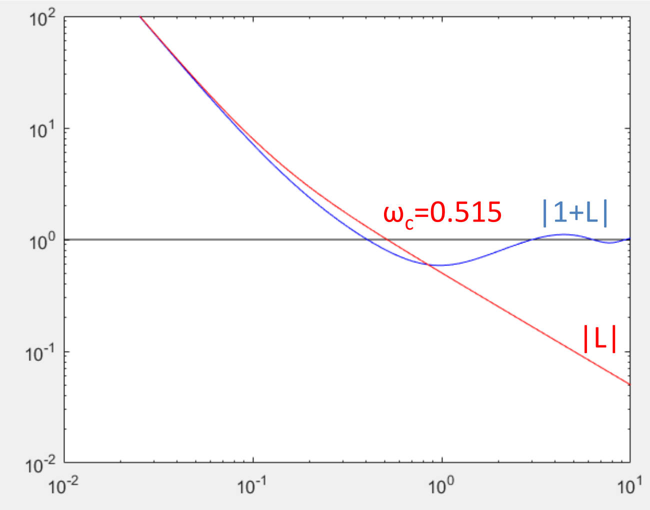
Typical requirement: $M_s < 2$

At stability limit: $M_s \to \infty$

```
s=tf('s')
g = exp(-s)/s;
Kc=0.707;  taui=3.33; c = Kc*(1+1/(taui*s)); L1 = g*c;  % ZN
Kc=0.5; taui=8; c = Kc*(1+1/(taui*s)); L2 = g*c;   % SIMC
w = logspace(-2,1,1000);
[mag1,phase]=bode(1/(1+L1),w);
[mag2,phase]=bode(1/(1+L2),w);
L0=1*exp(-s); [mag0,phase]=bode(L0,w); % Trick to make a line for 1
figure(1), loglog(w,mag1(:),'red',w,mag2(:),'blue',w,mag0(:),'black')
axis([0.01,10,0.001,10])
```

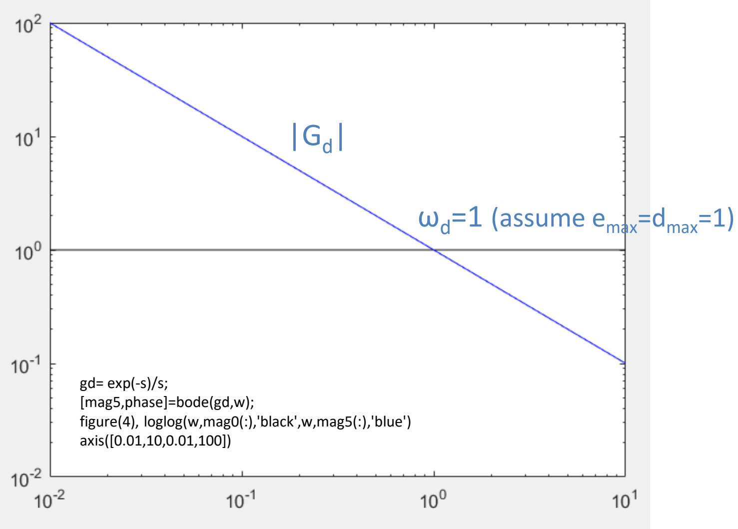
Performance requirement for disturbances with feedback control

- $e = y - y_s$ (opposite sign compared to previous slide, doesn't matter here because we look at magnitude)
- Recall: $e = S\, G_d\, d$. $S = 1/(1+L)$; $L = GC$
- Performance requirement: Want $|e(t)| < e_{max}$ for worst-case disturbance: $d(t) = d_{max} \sin(\omega t)$ (at any frequency)
- So want $|S\, G_d\, d_{max}| < e_{max}$ (at all frequencies)
- Or $|1+L|\, e_{max} > |G_d|\, d_{max}$ (at all frequencies)
- At low frequency where $|L|$ is large:
  - set $|1+L| \approx |L|$
    - A bit «optimistic» around $|L|=1$, but OK (see figure)
  - Performance requirement becomes: $|L| > |G_d|\, d_{max}/e_{max}$
  - so at least we need $\omega_c > \omega_d$
  - where ω_c and ω_d are defined as: $|L(j\omega_c)| = 1$, $|G_d(j\omega_d)| = e_{max}/d_{max}$
  - This can also be used to tune the controller: $\tau_c < 1/\omega_d$ (approx)
- Normally I assume the variables (and model (G, G_d)) have been scaled such that $e_{max}=1$, $d_{max}=1$.




```
[mag3,phase]=bode((1+L2),w);
[mag4,phase]=bode(L2,w);
Plot |L| and |1+L1
figure(3), loglog(w,mag0(:),'black',w,mag3(:),'blue',w,mag4(:),'red')
axis([0.01,10,0.01,100])
```

- ω_d is the frequency up to which control is needed to get $|e| < e_{max}$ for disturbances (faster disturbances are averaged out by the process)
- ω_c is the frequency up to which feedback is effective ($|L|>1$)
- This means we need $\omega_c > \omega_d$ (Rule 1)



QESTION: What about example on right?

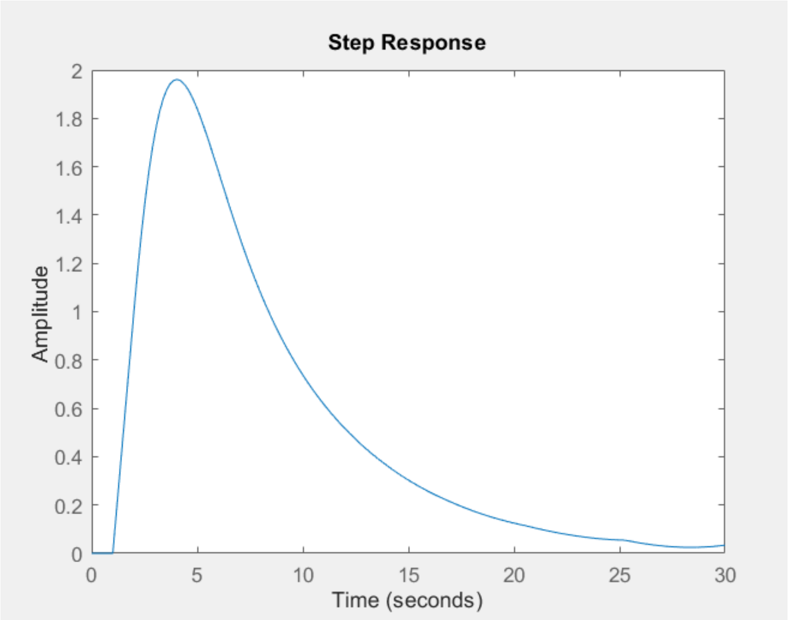
Was SIMC-tuned with $\tau_c = 1$ which happens to be $= 1/\omega_d$

– but resulting ω_c=0.515 is NOT larger than ω_d=1

- So does not look promising for sinusoidal disturbance
- Let's check step response (which is not sinusoid)

Check: Step disturbance

- $G = G_d = e^{-s}/s$
- SIMC PI-controller with $\tau_c=1$
  - $K_c=0.5$, $\tau_I = 8$




```
s=tf('s');
g = exp(-s)/s; gd=g; % Input disturbance
Kc=0.5; taui=8; c = Kc*(1+1/(taui*s));
L2 = g*c; S2=1/(1+L2);
step(S2*gd,30)
```

As expected from frequency analysis we get peak $e(t) \approx 2 > 1$ ($so\ not\ OK$)

- Would be OK with Gd = 0.5 e^{-s}/s which would give $\omega_d=0.5 < \omega_c=0.515$

Input usage

- $y = Gu + G_d d$
- To reject a disturbance d (and achieve y=0) we need for both feedforward and feedback

  $u = - G^{-1} G_d d$
- Assume $|d|=d_{max}$ and we must have $|u|<u_{max}$
- This means that we must require to avoid input saturation

  $|G^{-1}|\cdot|G_d|\ d_{max} < u_{max}$

  or: $|G|\ u_{max} > |G_d|\ d_{max}$ (at all frequencies where we need control)

- Normally I assume the variables)and model (G, Gd)) have been scaled such that $e_{max}=1$, $d_{max}=1$, $u_{max}=1$.
  - The requirement to avoid input saturation then becomes:

    $|G| > |G_d|$ (at all frequencies where we need control)

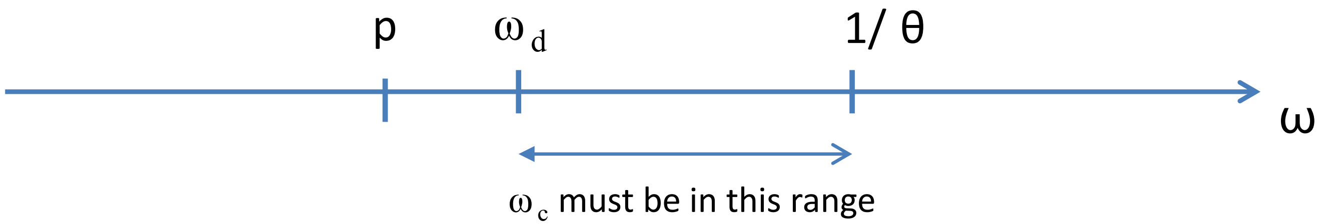
Controllability rules (approximate)

Rule 1. Need $\omega_c > \omega_d$ for disturbance rejection
Rule 2. Need $\omega_c < 1/\theta$ for robustness
Rule 3. Need $\omega_c > p$ for stabilization (g(s)=1/(s-p))
Rule 4. Need $|g|\cdot|\Delta u_{max}| > |g_d|\cdot|\Delta d|$ to avoid constraints

- ω_d = frequency where $|g_d(j\omega_d)|\,\Delta d/\Delta e_{max} = 1$ *($|g_d(j\omega_d)|=1$ in scaled units)*
- $\omega_c = 1/\tau_c$ (bandwidth frequency; frequency where |L| crosses 1 from above)
  - Note: This τ_c is close to but not idential to the τ_c used in SIMC
- Θ = effective delay
- p = unstable pole
- Δe_{max} = max allowed output error
- Δu_{max} = max input change (because of input constraints)
- Δd = max expected disturbance

- Combined Rules 1 & 2: Must require $\omega_d\,\theta < 1$
- Rule 1 is for typical case where $|g_d|$ is highest at low frequencies
  - The more exact rule 1 is: Need $|Sg_d|\Delta d < \Delta e_{max}$ at all frequencies. Note that $|S| \approx 1/|L|$ at low frequencies
- Rule 4: Applies at frequencies where control is needed (up to ω_d). *In scaled units the rule simplifies to $|g| > |g_d|$.*
- *Scaled units:*
  - *Maximum allowed control error $\Delta e_{max} = 1$.*
  - *Maximum input change, $\Delta u_{max} = 1$.*
  - *Maximum expected disturbamce $\Delta d = 1$*

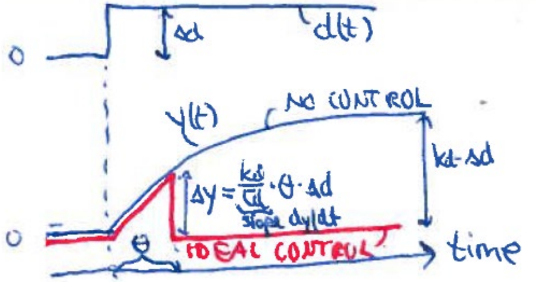
This situation is OK according to rules 1-3:



Example: Controllability requirements for first-order process

- Assume process (g) has effective delay θ
- Assume maximum allowed output change (error) is Δy_{max}
- Consider response to disturbance, $g_d = k_d/(\tau_d s+1)$

- Time domain analysis
  - For step Δd : Output reaches $\Delta y = (k_d \theta / \tau_d) \Delta d$ at time θ (approximately; see figure)
  - If this is larger than acceptable (Δy_{max}) then we are in trouble
  - To be controllable, we must require
    $(k_d \theta / \tau_d) < \Delta e_{max} / \Delta d$



- Check with more general Rules in frequency domain:
  - The controllability requirement is (Rule 1&2): $\omega_d \theta < 1$
  - where $|g_d(j\omega_d)| = \Delta e_{max}/\Delta d$
  - Asymptote for g_d at $\omega > 1/\tau_d$: $|g_d(j\omega)| = k_d/\tau_d\omega$
  - So $\omega_d = k_d/(\tau_d \Delta e_{max} / \Delta d)$
  - And $\omega_d \theta < 1$ gives the expected controllability requirement: $(k_d \theta / \tau_d) < \Delta e_{max} / \Delta d$

- In addition we must avoid input saturation. We have: $\Delta y = g_d \Delta d + g \Delta u$
- So to get $\Delta y = 0$ without exceeding constraint Δu_{max}, we must require (Rule 4)

At all frequencies $\omega < \omega_d$ (where we need control) :	$\lvert g(jw) \Delta u_{max} \rvert > \lvert g_d(jw) \Delta d \rvert$
At steady state:	$\lvert k \Delta u_{max} \rvert > \lvert k_d \Delta d \rvert$
Initial response (approximately):	$\lvert k/\tau \Delta u_{max} \rvert > \lvert k_d/\tau_d \Delta d \rvert$

Controllability analysis

- Use of controllability analysis
  - To avoid spending time on impossible control problem
  - To help design the process (e.g., size buffer tanks)
- Also useful for tuning.
  - τ_c = SIMC tuning parameter
  - Must for acceptable controllability have:

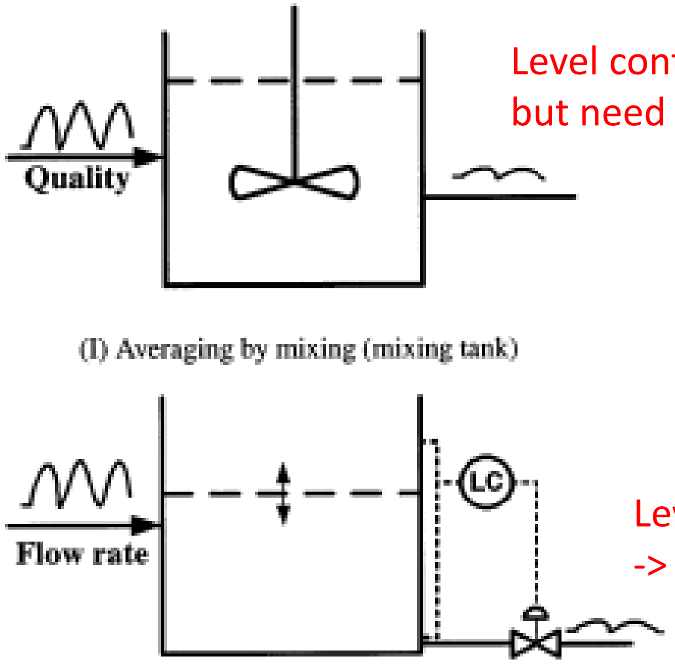
$$\theta \leq \tau_c \leq \frac{1}{\omega_d}$$

- Note
  - Tight control: $\tau_{c,min} = \theta$
  - "Smooth" control: $\tau_{c,max} = 1/\omega_d$

ω_d is defined as frequency where $|g_d(j\omega_d)| = \Delta e_{max}/\Delta d$

If process is not controllable: Need to change the design

- For example, dampen disturbance by adding buffer tank:



Level control unimportant, but need good mixing

Level control is NOT tight -> level varies

Integral action is not recommended for averaging level control

(II) Averaging level control (surge tank)

Figure 1. Two types of buffer tanks.

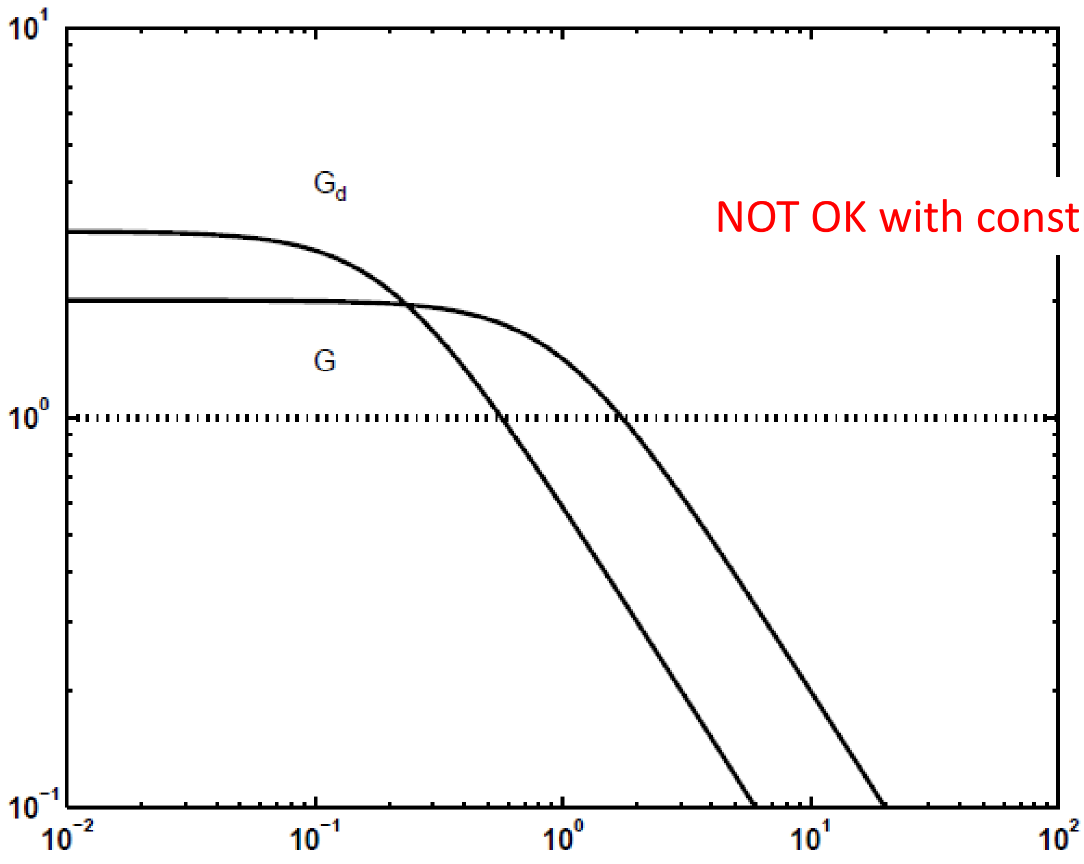
Scaled model

- In all problems below, we assume that models have beed scaled such that
  - Δe_{max}=1
  - Δu_{max}=1
  - Δd =1
  - Define ω_d as frequency where $|G_d(j\omega_d)|=1$.
    – For first-order disturbance model (scaled units): $\omega_d = k_d/\tau_d$

SCALED MODEL

Problem 1

$$G(s) = \frac{2}{s+1} \quad G_d(s) = \frac{3}{5s+1}$$



NOT OK with constraints. (Rule 4 at steady state)

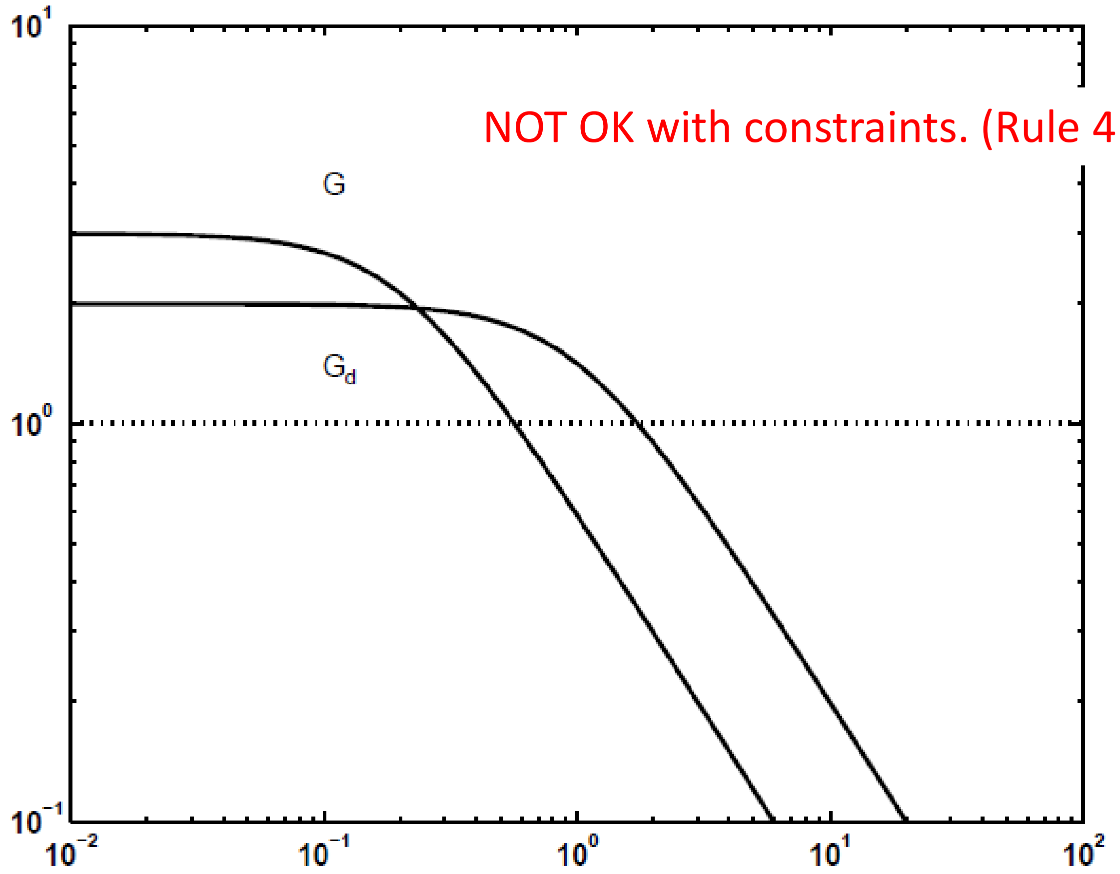
Figure 3: Magnitude of G and G_d.

SCALED MODEL

Problem 2

$$G(s) = \frac{3}{5s+1} \quad G_d(s) = \frac{2}{s+1}$$

NOT OK with constraints. (Rule 4 for ω from about 0.25 to 2)



Problem 3

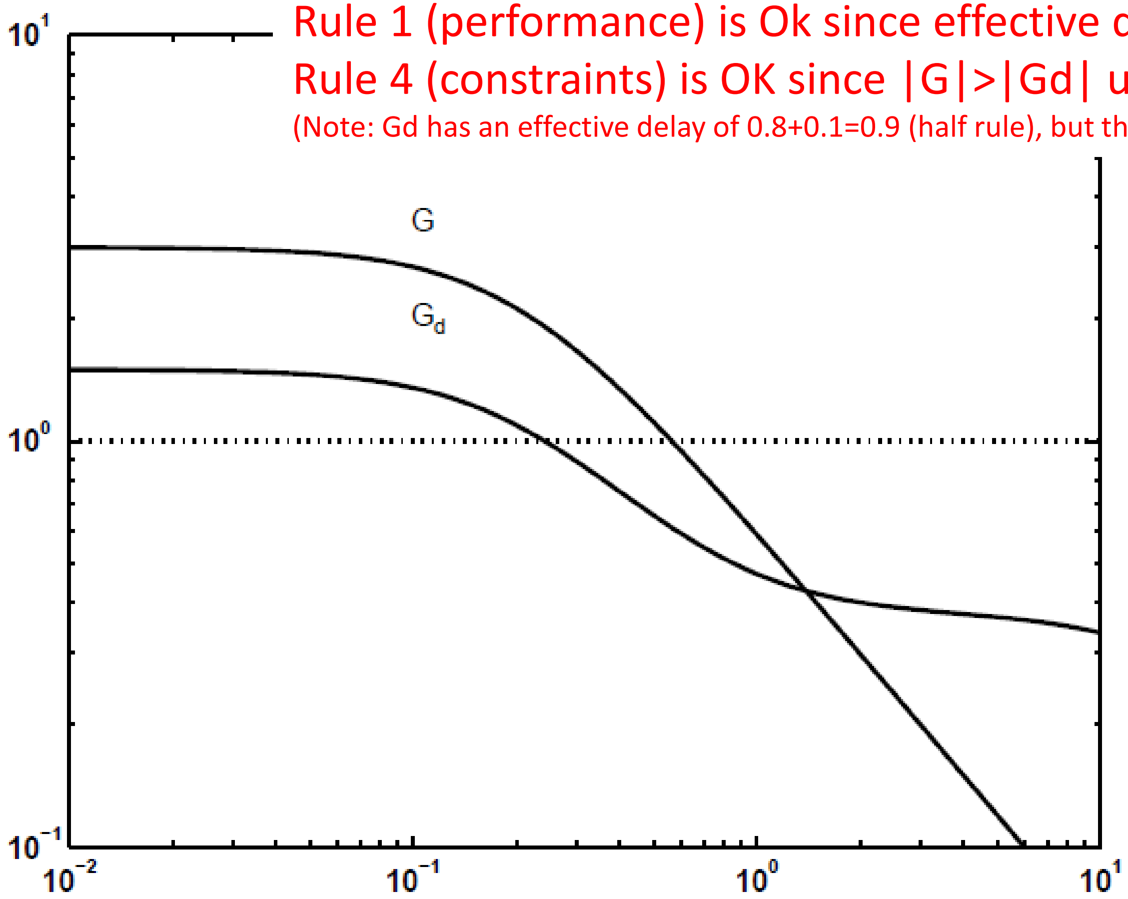
$$G(s) = \frac{3}{5s+1} \qquad G_d(s) = 7.5\frac{s - 0.8}{(s+0.2)(s+20)}$$

OK.

Rule 1 (performance) is Ok since effective delay in G is zero,

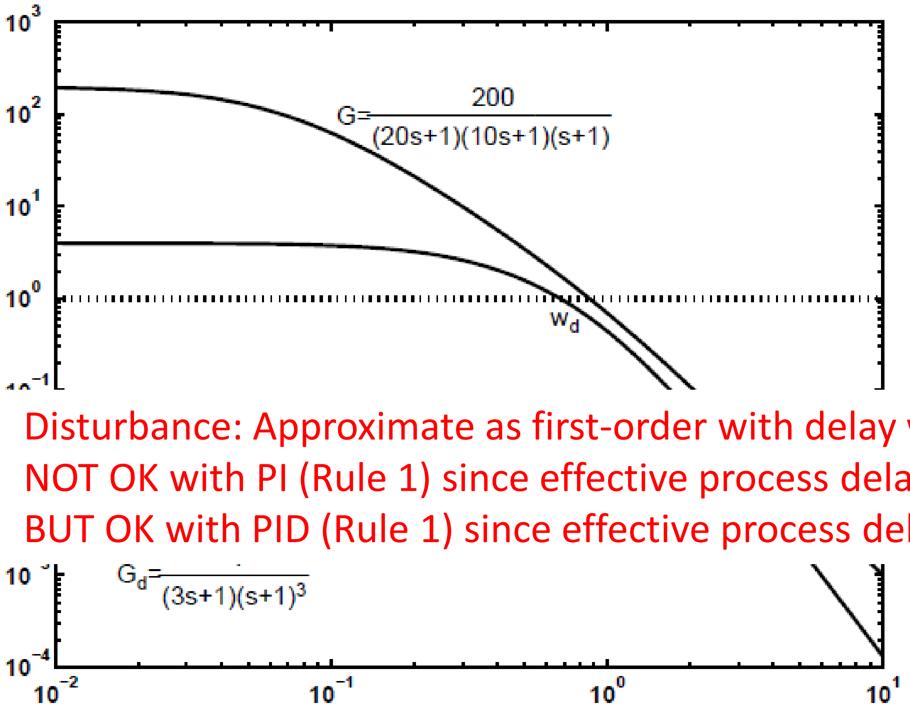
Rule 4 (constraints) is OK since |G|>|Gd| up to frequency ωd where |Gd|=1

(Note: Gd has an effective delay of 0.8+0.1=0.9 (half rule), but the delay in Gd does not matter)



Problem 4

$$G(s) = \frac{200}{(20s+1)(10s+1)(s+1)} \qquad G_d(s) = \frac{4}{(3s+1)((s+1)^3}$$



(a) Magnitude of G and Gd

No problem with constraints, |G|>|Gd|

Disturbances. When does y reach 1 (ω_d)?

What is effective delay?

Disturbance: Approximate as first-order with delay with kd=4, taud=3.5 ⇒ $\omega_d \approx 4/3.5 = 1.14$

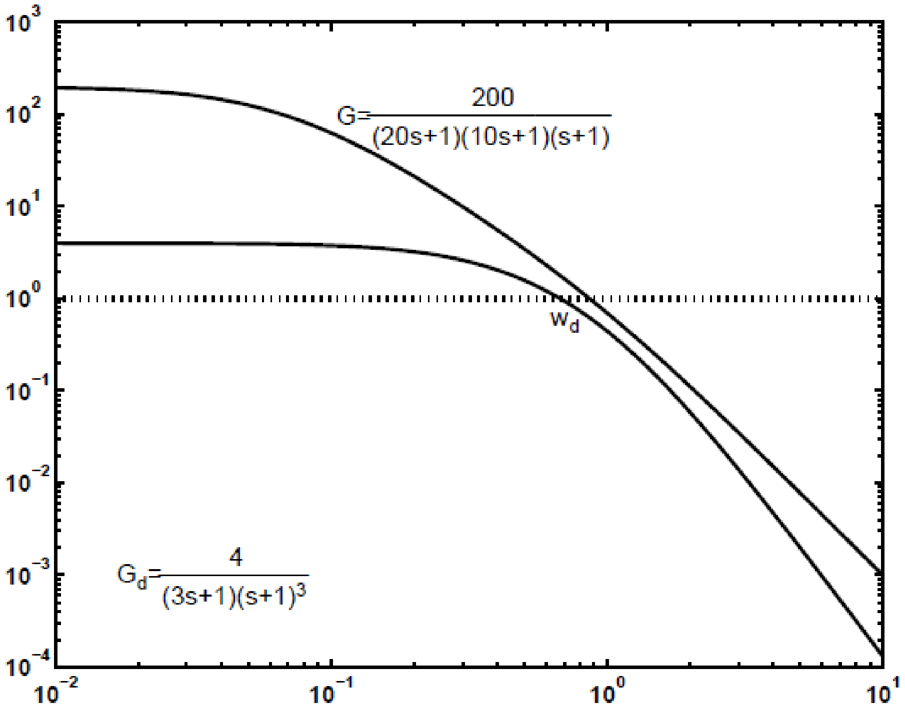
NOT OK with PI (Rule 1) since effective process delay is θ=10/2+1=6 so $\omega_d\,\theta = 6.9 > 1$

BUT OK with PID (Rule 1) since effective process delay is θ=0.5 so $\omega_d\,\theta = 0.6 < 1$

SCALED MODEL

Problem 4

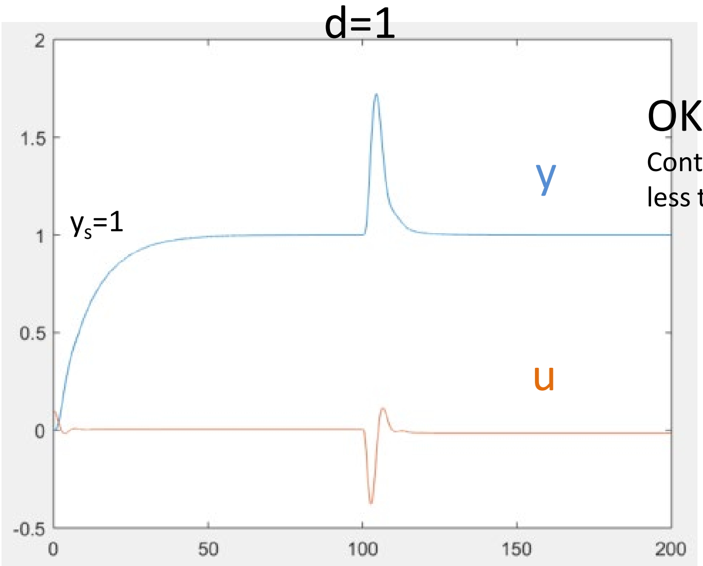
$$G(s) = \frac{200}{(20s+1)(10s+1)(s+1)} \qquad G_d(s) = \frac{4}{(3s+1)((s+1)^3}$$



(a) Magnitude of G and Gd

Check simulation PID:

```
%PID (w/o D-action on setpoint)
g = 200/((20*s+1)*(10*s+1)*(s+1))
gd = 4/((3*s+1)*(s+1)^3)
Kc=(1/200)*20/1,taui=20,taud=10.5
```



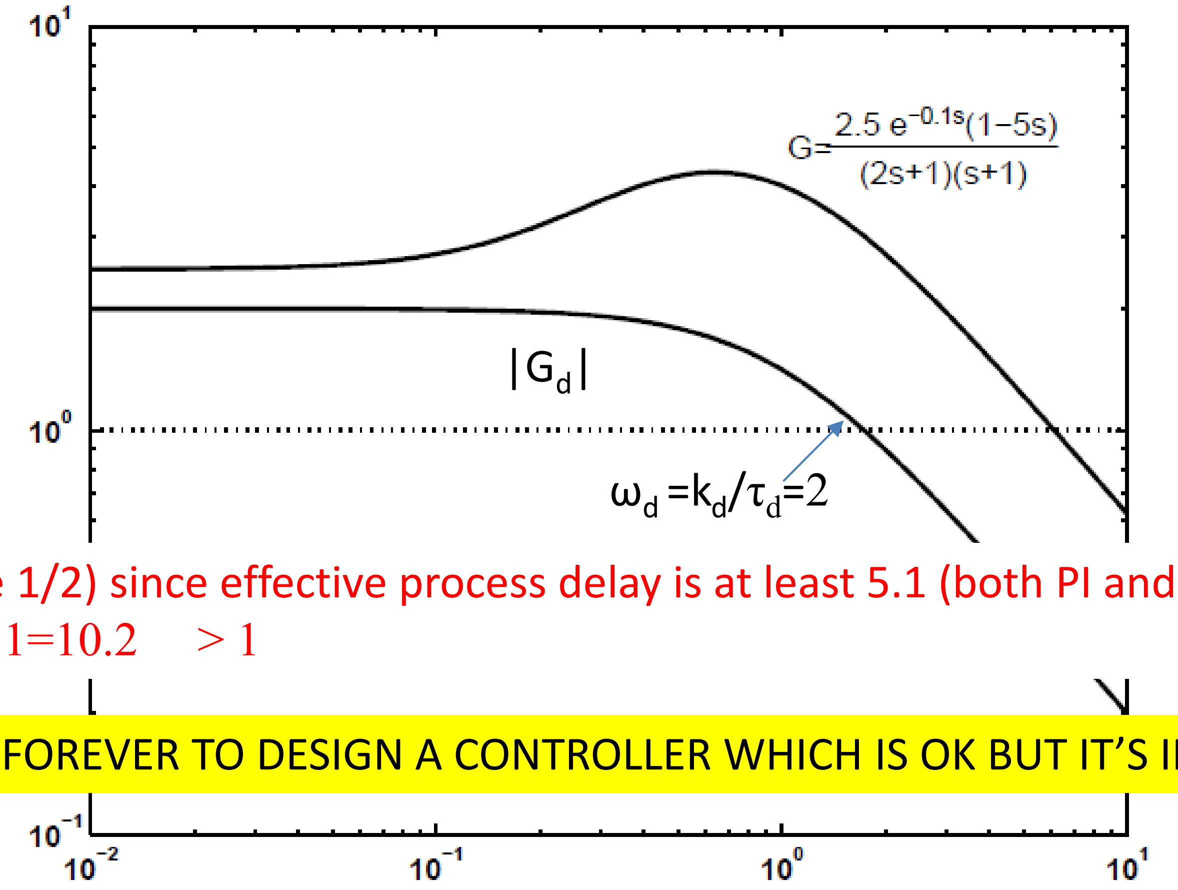
OK!

Control error $e=y-y_s$ alwys less that $e_{max}=1$

SCALED MODEL

Problem 5

$$G(s) = \frac{2.5e^{-0.1s}(1 - 5s)}{(3s+1)((s+1)^3} \quad G_d(s) = \frac{2}{s+1}$$



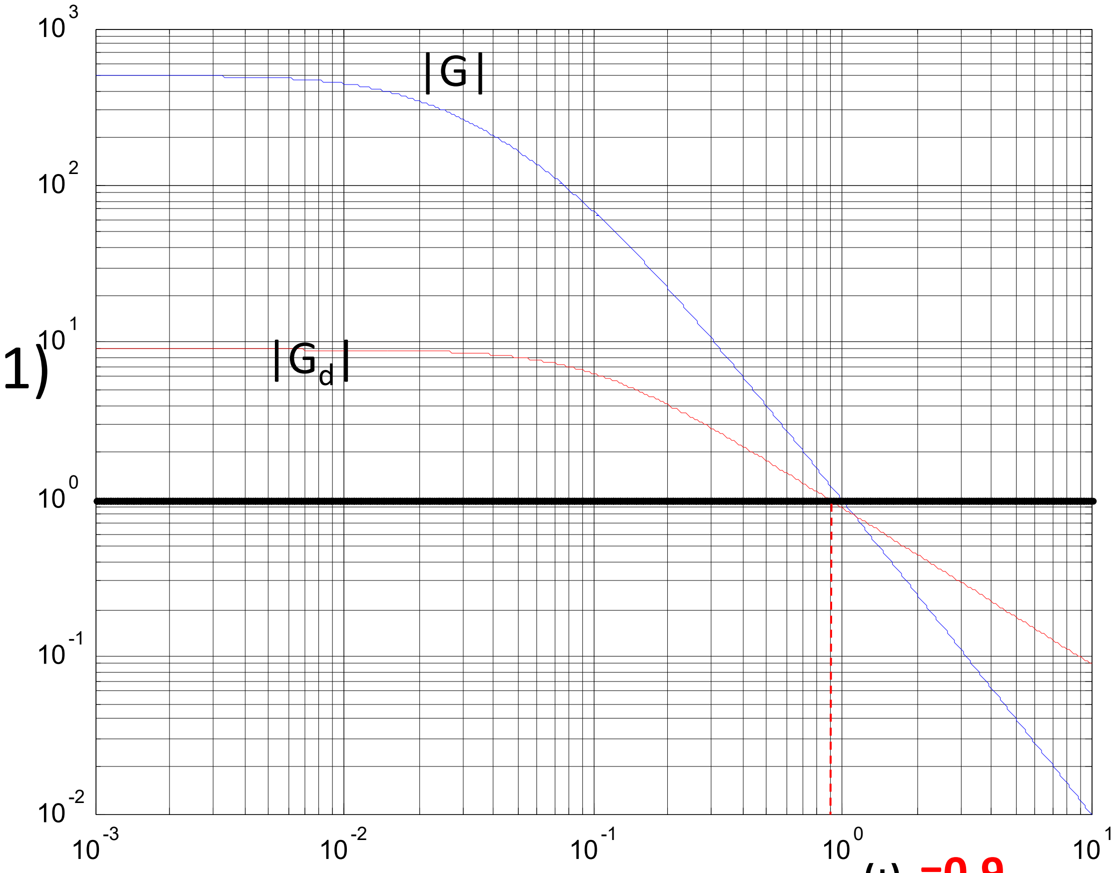
NOT OK (Rule 1/2) since effective process delay is at least 5.1 (both PI and PID),
so $\omega_d\, \theta = 2*5.1 = 10.2 \quad > 1$

YOU CAN TRY FOREVER TO DESIGN A CONTROLLER WHICH IS OK BUT IT'S IMPOSSIBLE

SCALED MODEL

PROBLEM 7, g = 500/((50*s+1)*(10*s+1))

gd = 9/(10*s+1)



ω_d=0.9

```
s=tf('s')
g = 500/((50*s+1)*(10*s+1))
gd = 9/(10*s+1)
w = logspace(-3,1,1000);
[mag,phase]=bode(g,w);
[magd,phased]=bode(gd,w);
loglog(w,mag(:),'blue',w,magd(:),'red',w,1,'black'), grid on
```

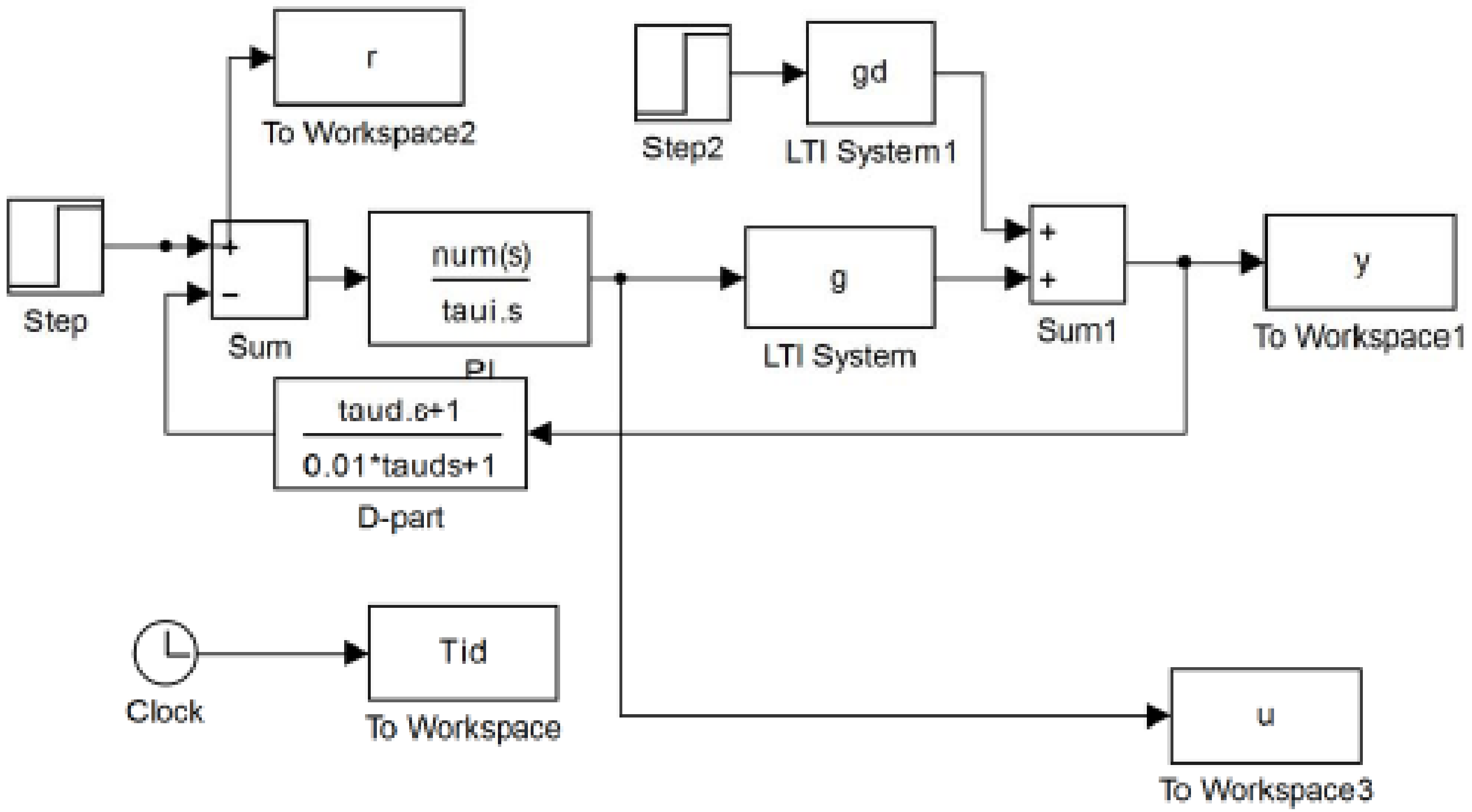
PI- control: θ_{eff} = 5 (from half rule):

$\omega_d \theta = k_d \theta / \tau_d = 9*5/10 = 4.5 \quad > 1$

NOT CONTROLLABLE WITH PI!

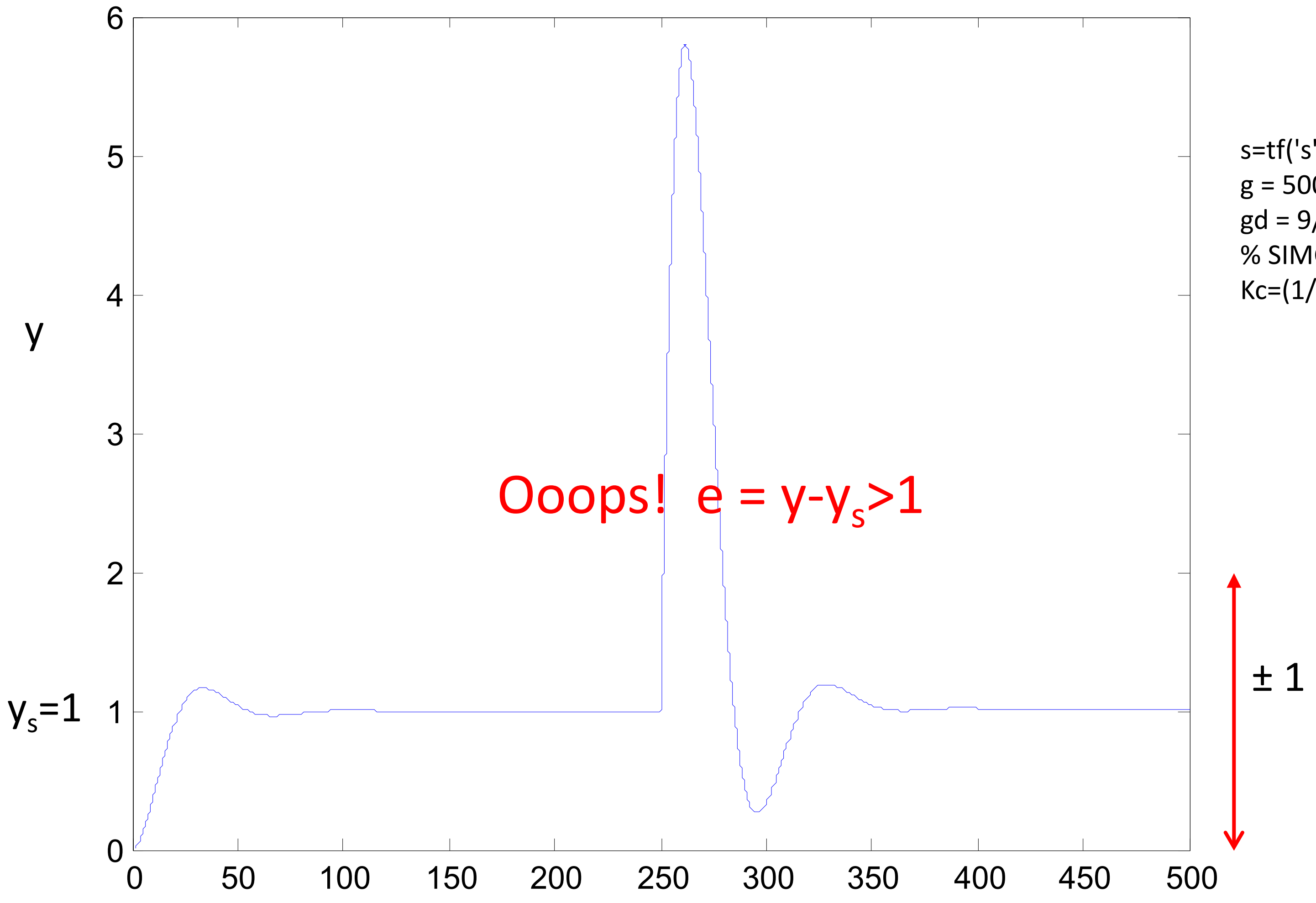
PID-control : θ_{eff} = 0. Controllable!

CHECK CONTROLLABILITY ANALYSIS WITH SIMULATIONS



SCALED MODEL

Problem 7: PI control not acceptable*

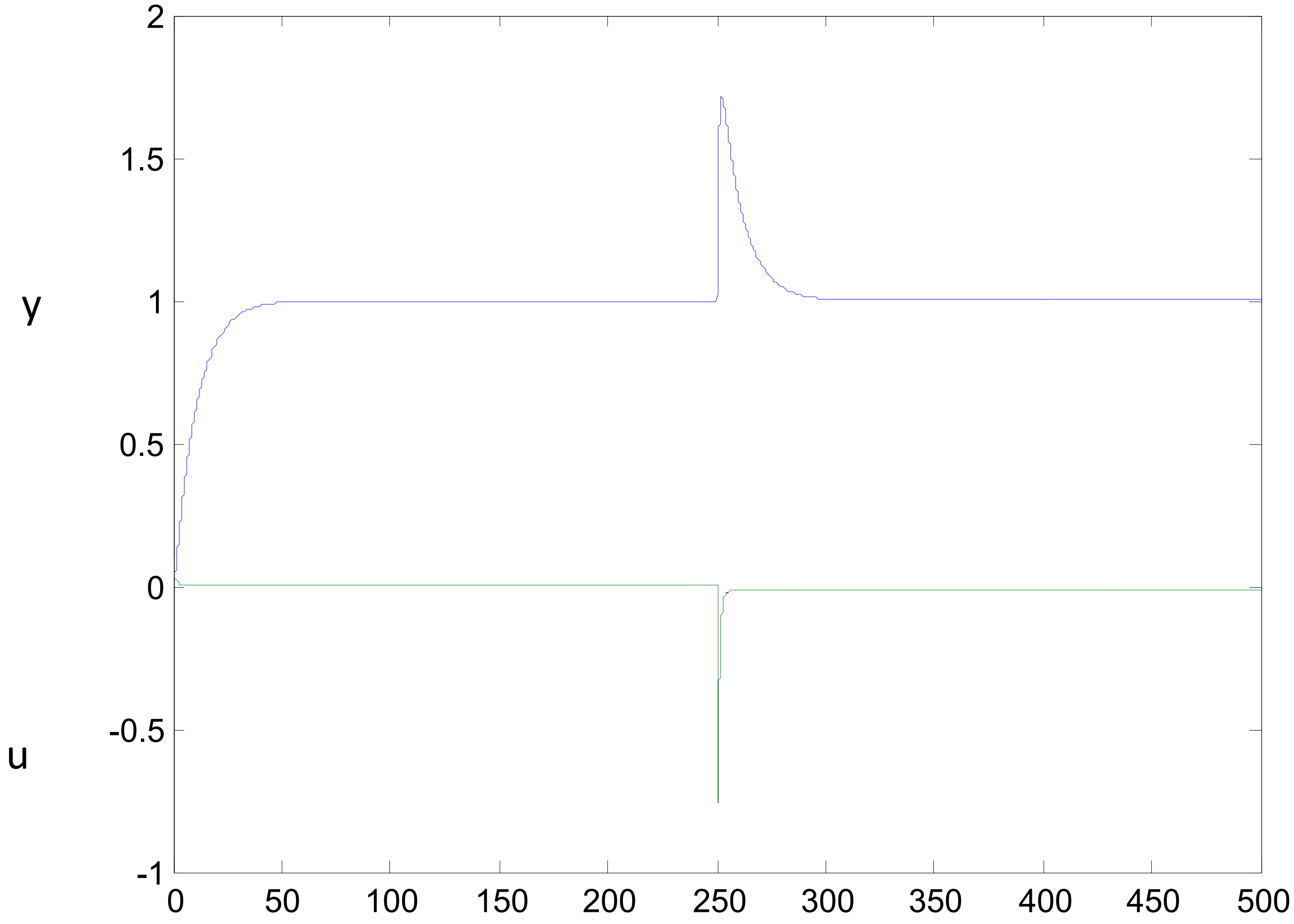



```
s=tf('s')
g = 500/((50*s+1)*(10*s+1))
gd = 9/(10*s+1)
% SIMC-PI with tauc=theta=5
Kc=(1/500)*(55/(5+5)); taui=55; taud=0;
```

*As expected since need $\omega_c > \omega_d = 0.9$, but can only achieve $\omega_c < 1/\theta = 1/5 = 0.2$

SCALED MODEL

Problem 7: PID control acceptable: e and u are within ±1



g = 500/((50*s+1)*(10*s+1))
gd = 9/(10*s+1)
%SIMC-PID (cascade form) with tauc=1/wd=1:
Kc=(1/500)*(50/(1+0)); taui=50; taud=10;

Exam.

- Tuesday 10 Dec. 2024. 9-13 (Physical)
- One sheet with own notes (both sides OK; printed OK)
- Simple calculator
- Note: Remember to state clearly all assumptions you make.
- General: Look through the whole exam before you start, read the questions carefully!

Q&A session: Thursday 05 Dec. 14-16, (H1)

(please send questions before by email: sigurd.skogestad@ntnu.no)