where

$$d_j(\omega) = d_{jo}e^{j\alpha_j}, \quad y_i(\omega) = y_{io}e^{j\beta_i} \tag{3.17}$$

Here the use of ω (and not $j\omega$) as the argument of $d_j(\omega)$ and $y_i(\omega)$ implies that these are complex numbers, representing at each frequency ω the magnitude and phase of the sinusoidal signals in (3.13) and (3.14).

The overall response to simultaneous input signals of the same frequency in several input channels is, by the superposition principle for linear systems, equal to the sum of the individual responses, and we have from (3.16)

$$y_i(\omega) = g_{i1}(j\omega)d_1(\omega) + g_{i2}(j\omega)d_2(\omega) + \cdots = \sum_j g_{ij}(j\omega)d_j(\omega) \tag{3.18}$$

or in matrix form

$$\boxed{y(\omega) = G(j\omega)d(\omega)} \tag{3.19}$$

where

$$d(\omega) = \begin{bmatrix} d_1(\omega) \\ d_2(\omega) \\ \vdots \\ d_m(\omega) \end{bmatrix} \quad \text{and} \quad y(\omega) = \begin{bmatrix} y_1(\omega) \\ y_2(\omega) \\ \vdots \\ y_l(\omega) \end{bmatrix} \tag{3.20}$$

represent the vectors of sinusoidal input and output signals.

Example 3.2 *Consider a 2×2 multivariable system where we simultaneously apply sinusoidal signals of the same frequency ω to the two input channels:*

$$d(t) = \begin{bmatrix} d_1(t) \\ d_2(t) \end{bmatrix} = \begin{bmatrix} d_{10}\sin(\omega t + \alpha_1) \\ d_{20}\sin(\omega t + \alpha_2) \end{bmatrix} \tag{3.21}$$

The corresponding output signal is

$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} y_{10}\sin(\omega t + \beta_1) \\ y_{20}\sin(\omega t + \beta_2) \end{bmatrix} \tag{3.22}$$

which can be computed by multiplying the complex matrix $G(j\omega)$ by the complex vector $d(\omega)$:

$$y(\omega) = G(j\omega)d(\omega); \quad y(\omega) = \begin{bmatrix} y_{10}e^{j\beta_1} \\ y_{20}e^{j\beta_2} \end{bmatrix}, \; d(\omega) = \begin{bmatrix} d_{10}e^{j\alpha_1} \\ d_{20}e^{j\alpha_2} \end{bmatrix} \tag{3.23}$$

3.3.2 Directions in multivariable systems

For a SISO system, $y = Gd$, the gain at a given frequency is simply

$$\frac{|y(\omega)|}{|d(\omega)|} = \frac{|G(j\omega)d(\omega)|}{|d(\omega)|} = |G(j\omega)|$$

The gain depends on the frequency ω, but since the system is linear it is independent of the input magnitude $|d(\omega)|$.

Things are not quite as simple for MIMO systems where the input and output signals are both vectors, and we need to "sum up" the magnitudes of the elements in each vector by use of some norm, as discussed in Appendix A.5.1. If we select the vector 2-norm, the usual measure of length, then at a given frequency ω the magnitude of the vector input signal is

$$\|d(\omega)\|_2 = \sqrt{\sum_j |d_j(\omega)|^2} = \sqrt{d_{10}^2 + d_{20}^2 + \cdots} \tag{3.24}$$

and the magnitude of the vector output signal is

$$\|y(\omega)\|_2 = \sqrt{\sum_i |y_i(\omega)|^2} = \sqrt{y_{10}^2 + y_{20}^2 + \cdots} \tag{3.25}$$

The *gain* of the system $G(s)$ for a particular input signal $d(\omega)$ is then given by the ratio

$$\frac{\|y(\omega)\|_2}{\|d(\omega)\|_2} = \frac{\|G(j\omega)d(\omega)\|_2}{\|d(\omega)\|_2} = \frac{\sqrt{y_{10}^2 + y_{20}^2 + \cdots}}{\sqrt{d_{10}^2 + d_{20}^2 + \cdots}} \tag{3.26}$$

Again the gain depends on the frequency ω, and again it is independent of the input magnitude $\|d(\omega)\|_2$. However, for a MIMO system there are additional degrees of freedom and the gain depends also on the *direction* of the input d.[1] The maximum gain as the direction of the input is varied is the maximum singular value of G,

$$\max_{d\neq 0} \frac{\|Gd\|_2}{\|d\|_2} = \max_{\|d\|_2=1} \|Gd\|_2 = \bar{\sigma}(G) \tag{3.27}$$

whereas the minimum gain is the minimum singular value of G,

$$\min_{d\neq 0} \frac{\|Gd\|_2}{\|d\|_2} = \min_{\|d\|_2=1} \|Gd\|_2 = \underline{\sigma}(G) \tag{3.28}$$

The first identities in (3.27) and (3.28) follow because the gain is independent of the input magnitude for a linear system.

Example 3.3 *For a system with two inputs, $d = \begin{bmatrix} d_{10} \\ d_{20} \end{bmatrix}$, the gain is in general different for the following five inputs:*

$$d_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ d_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ d_3 = \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix},\ d_4 = \begin{bmatrix} 0.707 \\ -0.707 \end{bmatrix},\ d_5 = \begin{bmatrix} 0.6 \\ -0.8 \end{bmatrix}$$

(which all have the same magnitude $\|d\|_2 = 1$ but are in different directions). For example, for the 2×2 system

$$G_1 = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} \tag{3.29}$$

[1] The term *direction* refers to a normalized vector of unit length.

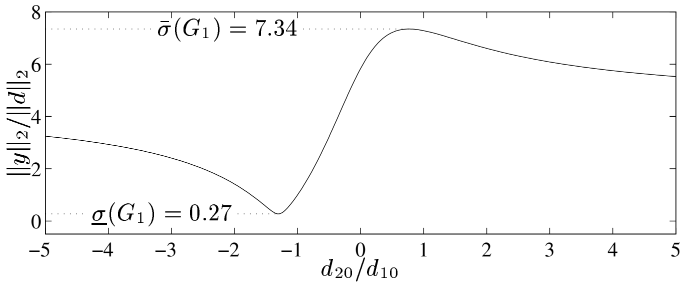


Figure 3.5: Gain $\|G_1 d\|_2/\|d\|_2$ as a function of d_{20}/d_{10} for G_1 in (3.29)

(a constant matrix) we compute for the five inputs d_j the following output vectors

$$y_1 = \begin{bmatrix} 5 \\ 3 \end{bmatrix},\ y_2 = \begin{bmatrix} 4 \\ 2 \end{bmatrix},\ y_3 = \begin{bmatrix} 6.36 \\ 3.54 \end{bmatrix},\ y_4 = \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix},\ y_5 = \begin{bmatrix} -0.2 \\ 0.2 \end{bmatrix}$$

and the 2-norms of these five outputs (i.e. the gains for the five inputs) are

$$\|y_1\|_2 = 5.83,\ \|y_2\|_2 = 4.47,\ \|y_3\|_2 = 7.30,\ \|y_4\|_2 = 1.00,\ \|y_5\|_2 = 0.28$$

This dependency of the gain on the input direction is illustrated graphically in Figure 3.5 where we have used the ratio d_{20}/d_{10} as an independent variable to represent the input direction. We see that, depending on the ratio d_{20}/d_{10}, the gain varies between 0.27 and 7.34. These are the minimum and maximum singular values of G_1, respectively.

3.3.3 Eigenvalues are a poor measure of gain

Before discussing in more detail the singular value decomposition we want to demonstrate that the magnitudes of the eigenvalues of a transfer function matrix, e.g. $|\lambda_i(G(j\omega)|$, do *not* provide a useful means of generalizing the SISO gain, $|G(j\omega)|$. First of all, eigenvalues can only be computed for square systems, and even then they can be very misleading. To see this, consider the system $y = Gd$ with

$$G = \begin{bmatrix} 0 & 100 \\ 0 & 0 \end{bmatrix} \tag{3.30}$$

which has both eigenvalues λ_i equal to zero. However, to conclude from the eigenvalues that the system gain is zero is clearly misleading. For example, with an input vector $d = [0\ \ 1]^T$ we get an output vector $y = [100\ \ 0]^T$.

The "problem" is that the eigenvalues measure the gain for the special case when the inputs and the outputs are in the same direction, namely in the direction of the eigenvectors. To see this let t_i be an eigenvector of G and consider an input $d = t_i$. Then the output is $y = Gt_i = \lambda_i t_i$ where λ_i is the corresponding eigenvalue. We get

$$\|y\|/\|d\| = \|\lambda_i t_i\|/\|t_i\| = |\lambda_i|$$

so $|\lambda_i|$ measures the gain in the direction t_i. This may be useful for stability analysis, but not for performance.

To find useful generalizations of $|G|$ for the case when G is a matrix, we need the concept of a *matrix norm*, denoted $\|G\|$. Two important properties which must be satisfied for a matrix norm are the *triangle inequality*

$$\|G_1 + G_2\| \leq \|G_1\| + \|G_2\| \tag{3.31}$$

and the multiplicative property

$$\|G_1 G_2\| \leq \|G_1\| \cdot \|G_2\| \tag{3.32}$$

(see Appendix A.5 for more details). As we may expect, the magnitude of the largest eigenvalue, $\rho(G) \triangleq |\lambda_{\max}(G)|$ (the spectral radius), does *not* satisfy the properties of a matrix norm; also see (A.116).

In Appendix A.5.2 we introduce several matrix norms, such as the Frobenius norm $\|G\|_F$, the sum norm $\|G\|_{\text{sum}}$, the maximum column sum $\|G\|_{i1}$, the maximum row sum $\|G\|_{i\infty}$, and the maximum singular value $\|G\|_{i2} = \bar{\sigma}(G)$ (the latter three norms are induced by a vector norm, e.g. see (3.27); this is the reason for the subscript i). Actually, the choice of matrix norm is not critical as they for a $l \times m$ matrix at most differ by a factor $\sqrt{ml}$, see (A.119)-(A.124. We will in this book use all of the above norms, each depending on the situation. However, in this chapter we will mainly use the induced 2-norm, $\bar{\sigma}(G)$. Notice that $\bar{\sigma}(G) = 100$ for the matrix in (3.30).

Exercise 3.5 *Compute the spectral radius and the five matrix norms mentioned above for the matrices in (3.29) and (3.30).*

3.3.4 Singular value decomposition

The singular value decomposition (SVD) is defined in Appendix A.3. Here we are interested in its physical interpretation when applied to the frequency response of a MIMO system $G(s)$ with m inputs and l outputs.

Consider a fixed frequency ω where $G(j\omega)$ is a constant $l \times m$ complex matrix, and denote $G(j\omega)$ by G for simplicity. Any matrix G may be decomposed into its singular value decomposition, and we write

$$G = U \Sigma V^H \tag{3.33}$$

where

Σ is an $l \times m$ matrix with $k = \min\{l, m\}$ non-negative singular values, σ_i, arranged in descending order along its *main diagonal*; the other entries are zero. The singular values are the positive square roots of the eigenvalues of $G^H G$, where G^H is the complex conjugate transpose of G.

$$\sigma_i(G) = \sqrt{\lambda_i(G^H G)} \tag{3.34}$$

U is an $l \times l$ unitary matrix of output singular vectors, u_i,

V is an $m \times m$ unitary matrix of input singular vectors, v_i,

In short, any matrix may be decomposed into an input rotation V, a diagonal scaling matrix Σ, and an output rotation U. This is illustrated by the SVD of a real 2×2 matrix which can always be written in the form

$$G = \underbrace{\begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} \cos\theta_2 & \pm\sin\theta_2 \\ -\sin\theta_2 & \pm\cos\theta_2 \end{bmatrix}^T}_{V^T} \tag{3.35}$$

where the angles θ_1 and θ_2 depend on the given matrix. From (3.35) we see that the matrices U and V involve rotations and that their columns are orthonormal.

The singular values are sometimes called the principal values or principal gains, and the associated directions are called principal directions. In general, the singular values must be computed numerically. For 2×2 matrices however, analytic expressions for the singular values are given in (A.36).

Caution. It is standard notation to use the symbol U to denote the matrix of *output* singular vectors. This is unfortunate as it is also standard notation to use u (lower case) to represent the *input* signal. The reader should be careful not to confuse these two.

Input and output directions. The column vectors of U, denoted u_i, represent the *output directions* of the plant. They are orthogonal and of unit length (orthonormal), that is

$$\|u_i\|_2 = \sqrt{|u_{i1}|^2 + |u_{i2}|^2 + \ldots + |u_{il}|^2} = 1 \tag{3.36}$$

$$u_i^H u_i = 1, \quad u_i^H u_j = 0, \quad i \neq j \tag{3.37}$$

Likewise, the column vectors of V, denoted v_i, are orthogonal and of unit length, and represent the *input directions*. These input and output directions are related through the singular values. To see this, note that since V is unitary we have $V^H V = I$, so (3.33) may be written as $GV = U\Sigma$, which for column i becomes

$$Gv_i = \sigma_i u_i \tag{3.38}$$

where v_i and u_i are vectors, whereas σ_i is a scalar. That is, if we consider an *input* in the direction v_i, then the *output* is in the direction u_i. Furthermore, since $\|v_i\|_2 = 1$ and $\|u_i\|_2 = 1$ we see that the i'th singular value σ_i gives directly the gain of the matrix G in this direction. In other words

$$\sigma_i(G) = \|Gv_i\|_2 = \frac{\|Gv_i\|_2}{\|v_i\|_2} \tag{3.39}$$

Some advantages of the SVD over the eigenvalue decomposition for analyzing gains and directionality of multivariable plants are:

1. The singular values give better information about the gains of the plant.
2. The plant directions obtained from the SVD are orthogonal.
3. The SVD also applies directly to non-square plants.

Maximum and minimum singular values. As already stated, it can be shown that the largest gain for *any* input direction is equal to the maximum singular value

$$\bar{\sigma}(G) \equiv \sigma_1(G) = \max_{d \neq 0} \frac{\|Gd\|_2}{\|d\|_2} = \frac{\|Gv_1\|_2}{\|v_1\|_2} \tag{3.40}$$

and that the smallest gain for any input direction (excluding the "wasted" inputs in the nullspace of G for cases with more inputs than outputs[2]) is equal to the minimum singular value

$$\underline{\sigma}(G) \equiv \sigma_k(G) = \min_{d \neq 0} \frac{\|Gd\|_2}{\|d\|_2} = \frac{\|Gv_k\|_2}{\|v_k\|_2} \tag{3.41}$$

where $k = \min\{l, m\}$. Thus, for any vector d, not in the nullspace of G, we have that

$$\underline{\sigma}(G) \leq \frac{\|Gd\|_2}{\|d\|_2} \leq \bar{\sigma}(G) \tag{3.42}$$

Define $u_1 = \bar{u}, v_1 = \bar{v}, u_k = \underline{u}$ and $v_k = \underline{v}$. Then it follows that

$$G\bar{v} = \bar{\sigma}\bar{u}, \qquad G\underline{v} = \underline{\sigma}\,\underline{u} \tag{3.43}$$

The vector $\bar{v}$ corresponds to the input direction with largest amplification, and $\bar{u}$ is the corresponding output direction in which the inputs are most effective. The directions involving $\bar{v}$ and $\bar{u}$ are sometimes referred to as the "strongest", "high-gain" or "most important" directions. The next most important directions are associated with v_2 and u_2, and so on (see Appendix A.3.5) until the "least important", "weak" or "low-gain" directions which are associated with $\underline{v}$ and $\underline{u}$.

Example 3.3 continued. *Consider again the system (3.29) with*

$$G_1 = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} \tag{3.44}$$

The singular value decomposition of G_1 is

$$G_1 = \underbrace{\begin{bmatrix} 0.872 & 0.490 \\ 0.490 & -0.872 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 7.343 & 0 \\ 0 & 0.272 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 0.794 & -0.608 \\ 0.608 & 0.794 \end{bmatrix}^H}_{V^H}$$

The largest gain of 7.343 is for an input in the direction $\bar{v} = \begin{bmatrix} 0.794 \\ 0.608 \end{bmatrix}$. The smallest gain of 0.272 is for an input in the direction $\underline{v} = \begin{bmatrix} -0.608 \\ 0.794 \end{bmatrix}$. This confirms the findings on page 82.

[2] For a "fat" matrix G with more inputs than outputs ($m > l$), we can always choose a nonzero input d in the nullspace of G such that $Gd = 0$.

Note that the directions in terms of by the singular vectors are not unique, in the sense that the elements in each pair of vectors (u_i, v_i) may be multiplied by a complex scalar c of magnitude 1 ($|c| = 1$). This is easily seen from (3.38). For example, we may change the sign of the vector $\bar{v}$ (multiply by $c = -1$) provided we also change the sign of the vector $\bar{u}$. Also, if you use Matlab to compute the SVD of the matrix in (3.44) (`g1=[5 4; 3 2 ]; [u,s,v]=svd(g1)`), then you will probably find that the signs of the elements in U and V are different from those given above.

Since in (3.44) both inputs affect both outputs, we say that the system is *interactive*. This follows from the relatively large off-diagonal elements in G_1. Furthermore, the system is *ill-conditioned*, that is, some combinations of the inputs have a strong effect on the outputs, whereas other combinations have a weak effect on the outputs. This may be quantified by the *condition number*; the ratio between the gains in the strong and weak directions; which for the system in (3.44) is $\gamma = \bar{\sigma}/\underline{\sigma} = 7.343/0.272 = 27.0$.

Example 3.4 **Shopping cart**. *Consider a shopping cart (supermarket trolley) with fixed wheels which we may want to move in three directions; forwards, sideways and upwards. This is a simple illustrative example where we can easily figure out the principal directions from experience. The strongest direction, corresponding to the largest singular value, will clearly be in the forwards direction. The next direction, corresponding to the second singular value, will be sideways. Finally, the most "difficult" or "weak" direction, corresponding to the smallest singular value, will be upwards (lifting up the cart).*

For the shopping cart the gain depends strongly on the input direction, i.e. the plant is ill-conditioned. Control of ill-conditioned plants is sometimes difficult, and the control problem associated with the shopping cart can be described as follows: Assume we want to push the shopping cart sideways (maybe we are blocking someone). This is rather difficult (the plant has low gain in this direction) so a strong force is needed. However, if there is any uncertainty in our knowledge about the direction the cart is pointing, then some of our applied force will be directed forwards (where the plant gain is large) and the cart will suddenly move forward with an undesired large speed. We thus see that the control of an ill-conditioned plant may be especially difficult if there is input uncertainty which can cause the input signal to "spread" from one input direction to another. We will discuss this in more detail later.

Example 3.5 **Distillation process.** *Consider the following steady-state model of a distillation column*

$$G = \begin{bmatrix} 87.8 & -86.4 \\ 108.2 & -109.6 \end{bmatrix} \tag{3.45}$$

The variables have been scaled as discussed in Section 1.4. Thus, since the elements are much larger than 1 *in magnitude this suggests that there will be no problems with input constraints. However, this is somewhat misleading as the gain in the low-gain direction (corresponding to the smallest singular value) is actually only just above* 1. *To see this consider the SVD of G:*

$$G = \underbrace{\begin{bmatrix} 0.625 & -0.781 \\ 0.781 & 0.625 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 197.2 & 0 \\ 0 & 1.39 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 0.707 & -0.708 \\ -0.708 & -0.707 \end{bmatrix}^H}_{V^H} \tag{3.46}$$

From the first input singular vector, $\bar{v} = [\,0.707 \quad -0.708\,]^T$*, we see that the gain is* 197.2 *when we increase one input and decrease the other input by a similar amount. On the other hand, from the second input singular vector,* $\underline{v} = [\,-0.708 \quad -0.707\,]^T$*, we see that if we change both inputs by the same amount then the gain is only* 1.39*. The reason for this is that the plant is such that the two inputs counteract each other. Thus, the distillation process is ill-conditioned, at least at steady-state, and the condition number is* $197.2/1.39 = 141.7$*. The physics of this example is discussed in more detail below, and later in this chapter we will consider a simple controller design (see Motivating robustness example No. 2 in Section 3.7.2).*

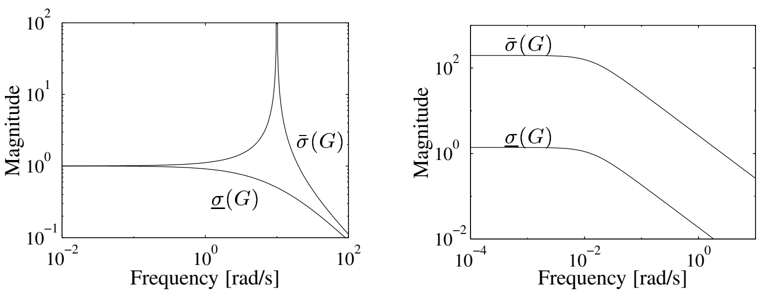
Example 3.6 **Physics of the distillation process.** *The model in (3.45) represents two-point (dual) composition control of a distillation column, where the top composition is to be controlled at* $y_D = 0.99$ *(output* y_1*) and the bottom composition at* $x_B = 0.01$ *(output* y_2*), using reflux L (input* u_1*) and boilup V (input* u_2*) as manipulated inputs (see Figure 10.9 on page 452). Note that we have here returned to the convention of using* u_1 *and* u_2 *to denote the manipulated inputs; the output singular vectors will be denoted by* $\bar{u}$ *and* $\underline{u}$*.*

The $1,1$*-element of the gain matrix G is* 87.8*. Thus an increase in* u_1 *by* 1 *(with* u_2 *constant) yields a large steady-state change in* y_1 *of* 87.8*, that is, the outputs are very sensitive to changes in* u_1*. Similarly, an increase in* u_2 *by* 1 *(with* u_1 *constant) yields* $y_1 = -86.4$*. Again, this is a very large change, but in the opposite direction of that for the increase in* u_1*. We therefore see that changes in* u_1 *and* u_2 *counteract each other, and if we increase* u_1 *and* u_2 *simultaneously by* 1*, then the overall steady-state change in* y_1 *is only* $87.8 - 86.4 = 1.4$*.*

Physically, the reason for this small change is that the compositions in the distillation column are only weakly dependent on changes in the internal flows *(i.e. simultaneous changes in the internal flows* L *and* V*). This can also be seen from the smallest singular value,* $\underline{\sigma}(G) = 1.39$*, which is obtained for inputs in the direction* $\underline{v} = \begin{bmatrix} -0.708 \\ -0.707 \end{bmatrix}$*. From the output singular vector* $\underline{u} = \begin{bmatrix} -0.781 \\ 0.625 \end{bmatrix}$ *we see that the effect is to move the outputs in different directions, that is, to change* $y_1 - y_2$*. Therefore, it takes a large control action to move the compositions in different directions, that is, to make both products purer simultaneously. This makes sense from a physical point of view.*

On the other hand, the distillation column is very sensitive to changes in external flows *(i.e. increase* $u_1 - u_2 = L - V$*). This can be seen from the input singular vector* $\bar{v} = \begin{bmatrix} 0.707 \\ -0.708 \end{bmatrix}$ *associated with the largest singular value, and is a general property of distillation columns where both products are of high purity. The reason for this is that the external distillate flow (which varies as* $V - L$*) has to be about equal to the amount of light component in the feed, and even a small imbalance leads to large changes in the product compositions.*

For dynamic systems the singular values and their associated directions vary with frequency, and for control purposes it is usually the frequency range corresponding to the closed-loop bandwidth which is of main interest. The singular values are usually plotted as a function of frequency in a Bode magnitude plot with a log-scale for frequency and magnitude. Typical plots are shown in Figure 3.6.



(a) Spinning satellite in (3.85)

(b) Distillation process in (3.90)

Figure 3.6: Typical plots of singular values

Non-Square plant

The singular value decomposition is also useful for non-square plants. For example, consider a plant with 2 inputs and 3 outputs. In this case the third output singular vector, u_3, tells us in which output direction the plant cannot be controlled. Similarly, for a plant with more inputs than outputs, the additional input singular vectors tell us in which directions the input will have no effect.

Example 3.7 *Consider a non-square system with 3 inputs and 2 outputs,*

$$G_2 = \begin{bmatrix} 5 & 4 & 1 \\ 3 & 2 & -1 \end{bmatrix}$$

with singular value decomposition

$$G_2 = \underbrace{\begin{bmatrix} 0.877 & 0.481 \\ 0.481 & -0.877 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 7.354 & 0 & 0 \\ 0 & 1.387 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 0.792 & -0.161 & 0.588 \\ 0.608 & 0.124 & -0.785 \\ 0.054 & 0.979 & 0.196 \end{bmatrix}^H}_{V^H}$$

From our definition, the minimum singular value is $\underline{\sigma}(G_2) = 1.387$, but note that an input d in the direction $v_3 = \begin{bmatrix} 0.588 \\ -0.785 \\ 0.196 \end{bmatrix}$ is in the nullspace of G and yields a zero output, $y = Gd = 0$.

Exercise 3.6 *For a system with m inputs and 1 output, what is the interpretation of the singular values and the associated input directions (V)? What is U in this case?*

3.3.5 Singular values for performance

So far we have used the SVD primarily to gain insight into the directionality of MIMO systems. But the maximum singular value is also very useful in terms of frequency-domain performance and robustness. We here consider performance.

For SISO systems we earlier found that $|S(j\omega)|$ evaluated as a function of frequency gives useful information about the effectiveness of feedback control. For example, it is the gain from a sinusoidal reference input (or output disturbance) to the control error, $|e(\omega)|/|r(\omega)| = |S(j\omega)|$.

For MIMO systems a useful generalization results if we consider the ratio $\|e(\omega)\|_2/\|r(\omega)\|_2$, where r is the vector of reference inputs, e is the vector of control errors, and $\|\cdot\|_2$ is the vector 2-norm. As explained above, this gain depends on the *direction* of $r(\omega)$ and we have from (3.42) that it is bounded by the maximum and minimum singular value of S,

$$\underline{\sigma}(S(j\omega)) \leq \frac{\|e(\omega)\|_2}{\|r(\omega)\|_2} \leq \bar{\sigma}(S(j\omega)) \tag{3.47}$$

In terms of *performance*, it is reasonable to require that the gain $\|e(\omega)\|_2/\|r(\omega)\|_2$ remains small for any direction of $r(\omega)$, including the "worst-case" direction which gives a gain of $\bar{\sigma}(S(j\omega))$. Let $1/|w_P(j\omega)|$ (the inverse of the performance weight) represent the maximum allowed magnitude of $\|e\|_2/\|r\|_2$ at each frequency. This results in the following performance requirement:

$$\begin{aligned} \bar{\sigma}(S(j\omega)) < 1/|w_P(j\omega)|, \ \forall\omega \quad &\Leftrightarrow \quad \bar{\sigma}(w_P S) < 1, \ \forall\omega \\ &\Leftrightarrow \quad \|w_P S\|_\infty < 1 \end{aligned} \tag{3.48}$$

where the $\mathcal{H}_\infty$ norm (see also page 66) is defined as the peak of the maximum singular value of the frequency response

$$\|M(s)\|_\infty \triangleq \max_\omega \bar{\sigma}(M(j\omega)) \tag{3.49}$$

Typical performance weights $w_P(s)$ are given in Section 2.7.2, which should be studied carefully.

The singular values of $S(j\omega)$ may be plotted as functions of frequency, as illustrated later in Figure 3.10(a). Typically, they are small at low frequencies where feedback is effective, and they approach 1 at high frequencies because any real system is strictly proper:

$$\omega \to \infty : \quad L(j\omega) \to 0 \quad \Rightarrow \quad S(j\omega) \to I \tag{3.50}$$

The maximum singular value, $\bar{\sigma}(S(j\omega))$, usually has a peak larger than 1 around the crossover frequencies. This peak is undesirable, but it is unavoidable for real systems.

As for SISO systems we define the bandwidth as the frequency up to which feedback is effective. For MIMO systems the bandwidth will depend on directions, and we have a *bandwidth region* between a lower frequency where the maximum singular value, $\bar{\sigma}(S)$, reaches 0.7 (the low-gain or worst-case direction), and a higher frequency where the minimum singular value, $\underline{\sigma}(S)$, reaches 0.7 (the high-gain or best direction). If we want to associate a single bandwidth frequency for a multivariable system, then we consider the worst-case (low-gain) direction, and define

- *Bandwidth*, ω_B: Frequency where $\bar{\sigma}(S)$ crosses $\frac{1}{\sqrt{2}} = 0.7$ from below.

It is then understood that the bandwidth is at least ω_B for any direction of the input (reference or disturbance) signal. Since $S = (I + L)^{-1}$, (A.53) yields

$$\underline{\sigma}(L) - 1 \leq \frac{1}{\bar{\sigma}(S)} \leq \underline{\sigma}(L) + 1 \tag{3.51}$$

Thus at frequencies where feedback is effective (namely where $\underline{\sigma}(L) \gg 1$) we have $\bar{\sigma}(S) \approx 1/\underline{\sigma}(L)$, and at the bandwidth frequency (where $1/\bar{\sigma}(S(j\omega_B)) = \sqrt{2} = 1.41$) we have that $\underline{\sigma}(L(j\omega_B))$ is between 0.41 and 2.41. Thus, the bandwidth is approximately where $\underline{\sigma}(L)$ crosses 1. Finally, at higher frequencies where for any real system $\underline{\sigma}(L)$ (and $\bar{\sigma}(L)$) is small we have that $\bar{\sigma}(S) \approx 1$.

3.4 Control of multivariable plants

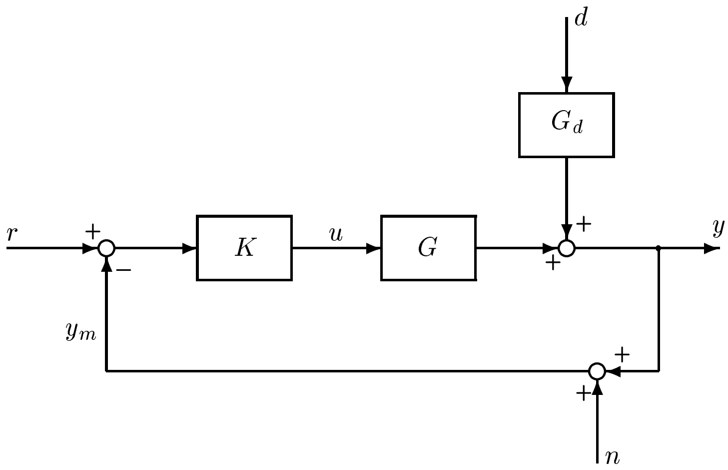


Figure 3.7: One degree-of-freedom feedback control configuration

Consider the simple feedback system in Figure 3.7. A conceptually simple approach to multivariable control is given by a two-step procedure in which we first design a "compensator" to deal with the interactions in G, and then design a *diagonal* controller using methods similar to those for SISO systems. This approach is discussed below.