the weight w_{P2} so that more emphasis is placed on output 2. We do this by increasing the bandwidth requirement in output channel 2 by a factor of 100:

$$\text{Design 2}: \quad M_1 = M_2 = 1.5; \quad \omega^*_{B1} = 0.25, \; \omega^*_{B2} = 25$$

This yields an $\mathcal{H}_\infty$ norm for N of 2.92. In this case we see from the dashed line in Figure 3.10(b) that the response for output 2 (y_2) is excellent with no inverse response. However, this comes at the expense of output 1 (y_1) where the response is somewhat poorer than for Design 1.

Design 3. *We can also interchange the weights w_{P1} and w_{P2} to stress output 1 rather than output 2. In this case (not shown) we get an excellent response in output 1 with no inverse response, but output 2 responds very poorly (much poorer than output 1 for Design 2). Furthermore, the $\mathcal{H}_\infty$ norm for N is 6.73, whereas it was only 2.92 for Design 2.*

Thus, we see that it is easier, for this example, to get tight control of output 2 than of output 1. This may be expected from the output direction of the RHP-zero, $\underline{u} = \begin{bmatrix} 0.89 \\ -0.45 \end{bmatrix}$, which is mostly in the direction of output 1. We will discuss this in more detail in Section 6.5.1.

Remark 1 We find from this example that we can direct the effect of the RHP-zero to either of the two outputs. This is typical of multivariable RHP-zeros, but there are cases where the RHP-zero is associated with a particular output channel and it is *not* possible to move its effect to another channel. The zero is then called a "pinned zero" (see Section 4.6).

Remark 2 It is observed from the plot of the singular values in Figure 3.10(a), that we were able to obtain by Design 2 a very large improvement in the "good" direction (corresponding to $\underline{\sigma}(S)$) at the expense of only a minor deterioration in the "bad" direction (corresponding to $\bar{\sigma}(S)$). Thus Design 1 demonstrates a shortcoming of the $\mathcal{H}_\infty$ norm: only the worst direction (maximum singular value) contributes to the $\mathcal{H}_\infty$ norm and it may not always be easy to get a good trade-off between the various directions.

3.6 Condition number and RGA

Two measures which are used to quantify the degree of directionality and the level of (two-way) interactions in MIMO systems, are the condition number and the relative gain array (RGA), respectively. We here define the two measures and present an overview of their *practical use*. Some algebraic properties of the condition number and the RGA are given in Appendix A.4.1.

3.6.1 Condition number

We define the *condition number* of a matrix as the ratio between the maximum and minimum singular values,

$$\gamma(G) \stackrel{\triangle}{=} \bar{\sigma}(G)/\underline{\sigma}(G) \tag{3.67}$$

A matrix with a large condition number is said to be *ill-conditioned*. For a non-singular (square) matrix $\underline{\sigma}(G) = 1/\bar{\sigma}(G^{-1})$, so $\gamma(G) = \bar{\sigma}(G)\bar{\sigma}(G^{-1})$. It then follows from (A.120) that the condition number is large if both G and G^{-1} have large elements.

The condition number depends strongly on the scaling of the inputs and outputs. To be more specific, if D_1 and D_2 are diagonal scaling matrices, then the condition numbers of the matrices G and D_1GD_2 may be arbitrarily far apart. In general, the matrix G should be scaled on physical grounds, for example, by dividing each input and output by its largest expected or desired value as discussed in Section 1.4.

One might also consider minimizing the condition number over all possible scalings. This results in the *minimized or optimal condition number* which is defined by

$$\gamma^*(G) = \min_{D_1, D_2} \gamma(D_1 G D_2) \tag{3.68}$$

and can be computed using (A.74).

The condition number has been used as an input-output controllability measure, and in particular it has been postulated that a large condition number indicates sensitivity to uncertainty. This is not true in general, but the reverse holds; if the condition number is small, then the multivariable effects of uncertainty are not likely to be serious (see (6.74)).

If the condition number is large (say, larger than 10), then this may *indicate* control problems:

1. A large condition number $\gamma(G) = \bar{\sigma}(G)/\underline{\sigma}(G)$ may be caused by a small value of $\underline{\sigma}(G)$, which is generally undesirable (on the other hand, a large value of $\bar{\sigma}(G)$ need not necessarily be a problem).
2. A large condition number may mean that the plant has a large minimized condition number, or equivalently, it has large RGA-elements which indicate fundamental control problems; see below.
3. A large condition number *does* imply that the system is sensitive to “unstructured” (full-block) input uncertainty (e.g. with an inverse-based controller, see (8.135)), but this kind of uncertainty often does not occur in practice. We therefore *cannot* generally conclude that a plant with a large condition number is sensitive to uncertainty, e.g. see the diagonal plant in Example 3.12.

3.6.2 Relative Gain Array (RGA)

The relative gain array (RGA) of a non-singular square complex matrix G is a square complex matrix defined as

$$\text{RGA}(G) = \Lambda(G) \triangleq G \times (G^{-1})^T \tag{3.69}$$

where $\times$ denotes element-by-element multiplication (the Hadamard or Schur product). For a 2×2 matrix with elements g_{ij} the RGA is

$$\Lambda(G) = \begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix} = \begin{bmatrix} \lambda_{11} & 1-\lambda_{11} \\ 1-\lambda_{11} & \lambda_{11} \end{bmatrix}; \quad \lambda_{11} = \frac{1}{1 - \frac{g_{12}g_{21}}{g_{11}g_{22}}} \tag{3.70}$$

Original interpretation: RGA as an interaction measure

We here follow Bristol (1966), and show that the RGA provides a measure of interactions. Let u_j and y_i denote a particular input-output pair for the multivariable plant $G(s)$, and assume that our task is to use u_j to control y_i. Bristol argued that there will be two extreme cases:

- All other loops open: $u_k = 0, \forall k \neq j$.
- All other loops closed with perfect control: $y_k = 0, \forall k \neq i$.

Perfect control is only possible at steady-state, but it is a good approximation at frequencies within the bandwidth of each loop. We now evaluate for "our" input u_j on "our" output y_i, the gain $\partial y_i / \partial u_j$ for the two extreme cases. We get

$$\text{Other loops open:} \quad \left(\frac{\partial y_i}{\partial u_j}\right)_{u_k=0, k\neq j} = g_{ij} \tag{3.71}$$

$$\text{Other loops closed:} \quad \left(\frac{\partial y_i}{\partial u_j}\right)_{y_k=0, k\neq i} \triangleq \widehat{g}_{ij} \tag{3.72}$$

Here $g_{ij} = [G]_{ij}$ is the ij'th element of G, whereas $\widehat{g}_{ij}$ is the inverse of the ji'th element of G^{-1}

$$\widehat{g}_{ij} = 1/[G^{-1}]_{ji} \tag{3.73}$$

To derive (3.73) note that

$$y = Gu \quad \Rightarrow \quad \left(\frac{\partial y_i}{\partial u_j}\right)_{u_k=0, k\neq j} = [G]_{ij} \tag{3.74}$$

and interchange the roles of G and G^{-1}, of u and y, and of i and j to get

$$u = G^{-1}y \quad \Rightarrow \quad \left(\frac{\partial u_j}{\partial y_i}\right)_{y_k=0, k\neq i} = [G^{-1}]_{ji} \tag{3.75}$$

and (3.73) follows. Bristol argued that the ratio between the gains in (3.71) and (3.72) is a useful measure of interactions, and defined the the ij'th "relative gain" as

$$\lambda_{ij} \triangleq \frac{g_{ij}}{\widehat{g}_{ij}} = [G]_{ij}[G^{-1}]_{ji} \tag{3.76}$$

The Relative Gain Array (RGA) is the corresponding matrix of relative gains. From (3.76) we see that $\Lambda(G) = G \times (G^{-1})^T$ where $\times$ denotes element-by-element multiplication (the Schur product). This is identical to our definition of the RGA-matrix in (3.69).

Intuitively, for decentralized control, we *prefer* to pair variables u_j and y_i so that λ_{ij} is close to 1 at all frequencies, because this means that the gain from u_j to y_i is unaffected by closing the other loops. More precisely, we would like to pair such that the rearranged system, with the pairings along the diagonal, has a RGA matrix close to identity at frequencies near the closed-loop bandwidth (see Pairing Rule 1, page 463). Furthermore, it seems clear thet we should *avoid* pairing on negative steady-state RGA-elements, because otherwise the sign of the steady-state gain may change, and this will yield instability if we have integral action in the loop (see Pairing Rule 2, page 468).

Example 3.9 *Consider a blending process where we mix sugar (u_1) and water (u_2) to make a given amount ($y_1 = F$) of a softdrink with a given sugar fraction ($y_2 = x$). The balances "mass in = mass out" for total mass and sugar mass are*

$$F_1 + F_2 = F$$

$$F_1 = xF$$

Linearlization yields

$$dF_1 + dF_2 = dF$$
$$dF_1 = x^* dF + F^* dx$$

With $u_1 = dF_1, u_2 = dF_2, y_1 = dF$ and $y_2 = dx$ we then get

$$y_1 = u_1 + u_2$$

$$y_2 = \frac{1 - x^*}{F^*} u_1 - \frac{x^*}{F^*} u_2$$

where $x^* = 0.2$ is the nominal sugar fraction and $F^* = 2$ kg/s is the nominal amount. The transfer matrix then becomes

$$G = \begin{bmatrix} 1 & 1 \\ \frac{1-x^*}{F^*} & -\frac{x^*}{F^*} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0.9 & -0.1 \end{bmatrix}$$

and the corresponding RGA-matrix is

$$\Lambda = \begin{bmatrix} x^* & 1 - x^* \\ 1 - x^* & x^* \end{bmatrix} = \begin{bmatrix} 0.2 & 0.8 \\ 0.8 & 0.2 \end{bmatrix}$$

For decetralized control, it then follows from pairing rule 1 ("prefer pairing on RGA-elements close to 1") that we should pair on the off-diagonal elements, that is ise u_1 to control y_2 and use u_2 to control y_1. This corresponds to using the largest stream (water, u_2) to control the amount ($y_1 = F$), which is reasonable from a physical point of view.

Example 3.10 *Consider a 3×3 plant for which we have at steady-state*

$$G = \begin{bmatrix} 16.8 & 30.5 & 4.30 \\ -16.7 & 31.0 & -1.41 \\ 1.27 & 54.1 & 5.40 \end{bmatrix}, \ \Lambda(G) = \begin{bmatrix} 1.50 & 0.99 & -1.48 \\ -0.41 & 0.97 & 0.45 \\ -0.08 & -0.95 & 2.03 \end{bmatrix} \tag{3.77}$$

ssfcc0 For decentralized control, we need to pair on one element in each column or row. It is then clear that the only choice that satisfies pairing rule 2 ("avoid pairing on negative RGA-elements") is to pair on the diagonal elements, that is, use u_1 to control y_1, u_2 to control y_2, and u_3 to control y_3.

Example 3.11 *The following model describes the effect of liquid (u_1) and vapour (u_2) outflow on liquid volume (y_1) and pressure (y_2) in a large pressurized vessel:*

$$G(s) = \frac{0.01}{(s+1.72e-4)(4.32s+1)} \begin{bmatrix} -35.54(s+0.0572) & 1.913 \\ -30.22e+5s & -9.188e+5(s+6.95e-4) \end{bmatrix} \tag{3.78}$$

The RGA-matrix $\Lambda(s)$ depends on frequency. At steady-state ($s = 0$) the 1,2-element in zero, so $\Lambda(0) = I$. Similarly, at high frequencies the two diagonal elements dominate, so $\Lambda(j\infty) = I$. This seems to suggest that the diagonal pairing should be used. However, at intermediate frequencies, the off-diagonal RGA-elements are largest. For example, at frequency $\omega = 0.01$ rad/s the RGA-matrix becomes

```
>> G  = [-35.54*(0.01*j+0.0572) 1.913; -30.22e5*0.01*j -9.188e5*(0.01*j + 6.95e-4)]
>> RGA=G.*inv(G).'
RGA =
   0.2469 + 0.0193i   0.7531 - 0.0193i
   0.7531 - 0.0193i   0.2469 + 0.0193i
```

Thus, the reverse pairing (use u_1 to control y_2, and u_2 to control y_1) is probably best if we use decentralized control and the closed-loop bandwidth is about 0.01 rad/s.

Remark. *The assumption of $y_k = 0$ ("perfect control of y_k") in (3.72) is satisfied at steady-state ($\omega = 0$) provided we have integral action in the loop, but it will generally not hold exactly at other frequencies. Unfortunately, this has led many authors to dismiss the RGA as being "only useful at steady-state" or "only useful if we use integral action". On the contrary, in most cases it is the value of the RGA at frequencies close to crossover which is most important, and both the gain and the phase of the RGA-elements are important. The derivation in (3.71) to (**??**) was included to illustrate one useful interpretation of the RGA, but note that our definition of the RGA in (3.69) is purely algebraic and makes no assumption about "perfect control". The general usefulness of the RGA is further demonstrated by the additional general algebraic and control properties of the RGA listed below.*

Algebraic properties of the RGA

The (complex) RGA-matrix has a number of interesting algebraic properties*, of which the most important are (see Appendix A.4 for more details):*

1. *It is independent of input and output scaling.*
2. *Its rows and columns sum to one.*
3. *The sum-norm of the RGA, $\|\Lambda\|_{\text{sum}}$, is very close to the minimized condition number γ^*; see (A.79). This means that plants with large RGA-elements are always ill-conditioned (with a large value of $\gamma(G)$), but the reverse may not hold (i.e. a plant with a large $\gamma(G)$ may have small RGA-elements).*
4. *A relative change in an element of G equal to the negative inverse of its corresponding RGA-element yields singularity.*
5. *The RGA is the identity matrix if G is upper or lower triangular.*

¿From the last property it follows that the RGA (or more precisely $\Lambda - I$) provides a measure of two-way interaction.

Example 3.12 *Consider a diagonal plant for which we have*

$$G = \begin{bmatrix} 100 & 0 \\ 0 & 1 \end{bmatrix}, \ \Lambda(G) = I, \ \gamma(G) = \frac{\bar{\sigma}(G)}{\underline{\sigma}(G)} = \frac{100}{1} = 100, \ \gamma^*(G) = 1 \tag{3.79}$$

Here the condition number is 100 which means that the plant gain depends strongly on the input direction. However, since the plant is diagonal there are no interactions so $\Lambda(G) = I$ and the minimized condition number $\gamma^*(G) = 1$.

Example 3.13 *Consider a triangular plant G for which we get*

$$G = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \ G^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}, \ \Lambda(G) = I, \ \gamma(G) = \frac{2.41}{0.41} = 5.83, \ \gamma^*(G) = 1 \tag{3.80}$$

Note that for a triangular matrix, the RGA is always the identity matrix and $\gamma^*(G)$ is always 1.

Control properties of the RGA

In addition to the algebraic properties listed above, the RGA has a surprising number of useful control properties*:*

1. *The RGA is a good indicator of sensitivity to uncertainty:*
   (a) Uncertainty in the input channels (diagonal input uncertainty). *Plants with large RGA-elements around the crossover frequency are fundamentally difficult to control because of sensitivity to input uncertainty (e.g. caused by uncertain or neglected actuator dynamics). In particular, decouplers or other inverse-based controllers should not be used for plants with large RGA-elements (see page 267).*
   (b) Element uncertainty. *As implied by algebraic property no. 4 above, large RGA-elements imply sensitivity to element-by-element uncertainty. However, this kind of uncertainty may not occur in practice due to physical couplings between the transfer function elements. Therefore, diagonal input uncertainty (which is always present) is usually of more concern for plants with large RGA-elements.*
2. RGA and RHP-zeros. *If the sign of an RGA-element changes from $s = 0$ to $s = \infty$, then there is a RHP-zero in G or in some subsystem of G (see Theorem 10.5, page 467).*
3. Non-square plants. *The definition of the RGA may be generalized to non-square matrices by using the pseudo inverse; see Appendix A.4.2. Extra inputs: If the sum of the elements in a column of RGA is small ($\ll 1$), then one may consider deleting the corresponding input. Extra outputs: If all elements in a row of RGA are small ($\ll 1$), then the corresponding output cannot be controlled (see Section 10.4).*
4. Pairing and diagonal dominance. *The RGA can be used as a measure of diagonal dominance (or more precisely, a ameasure of how easily the inputs or outputs can be scaled to obtain diagonal dominance), by the simple quantity*

$$\textit{RGA-number} = \|\Lambda(G) - I\|_{\text{sum}} \tag{3.81}$$

   For decentralized control we prefer pairings for which the RGA-number at crossover frequencies is close to 0 (see pairing rule 1 on page 463). Similarly, for certain *multivariable design methods, it is simpler to choose the weights and shape the plant if we first rearrange the inputs and outputs to make the plant diagonally dominant with a small RGA-number.*

5. RGA and decentralized control.
   (a) Integrity: *For stable plants avoid input-output pairing on negative steady-state RGA-elements. Otherwise, if the sub-controllers are designed independently each with integral action, then the interactions will cause instability either when all of the loops are closed, or when the loop corresponding to the negative relative gain becomes inactive (e.g. because of saturation) (see Theorem 10.4, page 464).*
   (b) Stability: *Prefer pairings corresponding to an RGA-number close to 0 at crossover frequencies (see page 463).*

Example 3.14 *Consider again the distillation process in (3.45) for which we have at steady-state*

$$G = \begin{bmatrix} 87.8 & -86.4 \\ 108.2 & -109.6 \end{bmatrix},\ G^{-1} = \begin{bmatrix} 0.399 & -0.315 \\ 0.394 & -0.320 \end{bmatrix},\ \Lambda(G) = \begin{bmatrix} 35.1 & -34.1 \\ -34.1 & 35.1 \end{bmatrix} \tag{3.82}$$

In this case $\gamma(G) = 197.2/1.391 = 141.7$ is only slightly larger than $\gamma^*(G) = 138.268$. The magnitude sum of the elements in the RGA-matrix is $\|\Lambda\|_{\rm sum} = 138.275$. This confirms (A.80) which states that, for 2×2 systems, $\|\Lambda(G)\|_{\rm sum} \approx \gamma^*(G)$ when $\gamma^*(G)$ is large. The condition number is large, but since the minimum singular value $\underline{\sigma}(G) = 1.391$ is larger than 1 this does not by itself imply a control problem. However, the large RGA-elements indicate control problems, and fundamental control problems are expected if analysis shows that $G(j\omega)$ has large RGA-elements also in the crossover frequency range. (Indeed, the idealized dynamic model (3.90) used below has large RGA-elements at all frequencies, and we will confirm in simulations that there is a strong sensitivity to input channel uncertainty with an inverse-based controller).

Example 3.15 *Consider again the plant in (**??**) with*

$$G = \begin{bmatrix} 16.8 & 30.5 & 4.30 \\ -16.7 & 31.0 & -1.41 \\ 1.27 & 54.1 & 5.40 \end{bmatrix},\ \Lambda(G) = \begin{bmatrix} 1.50 & 0.99 & -1.48 \\ -0.41 & 0.97 & 0.45 \\ -0.08 & -0.95 & 2.03 \end{bmatrix} \tag{3.83}$$

ssfcc0 and $\gamma = 69.6/1.63 = 42.6$ and $\gamma^* = 7.80$. The magnitude sum of the elements in the RGA is $\|\Lambda\|_{\rm sum} = 8.86$ which is close to γ^* as expected from (A.79). Note that the rows and the columns of Λ sum to 1. Since $\underline{\sigma}(G)$ is larger than 1 and the RGA-elements are relatively small, this steady-state analysis does not indicate any particular control problems for the plant.

Remark. *The plant in (**??**) represents the steady-state model of a fluid catalytic cracking (FCC) process. A dynamic model of the FCC process in (**??**) is given in Exercise 6.16.*

Example 3.16 *Consider the plant*

$$G(s) = \frac{1}{5s+1}\begin{pmatrix} s+1 & s+4 \\ 1 & 2 \end{pmatrix} \tag{3.84}$$

We find that $\lambda_{11}(\infty) = 2$ *and* $\lambda_{11}(0) = -1$ *have different signs. Since none of the diagonal elements have RHP-zeros we conclude from Theorem 10.5 that* $G(s)$ *must have a RHP-zero. This is indeed true and* $G(s)$ *has a zero at* $s = 2$.

Assume we use decentralized control with integral action in each loop, and want to pair on one or more negative steady-state RGA-elements. This may happen because this pairing is preferred for dynamic reasons. What will happen? Will the system be unstable? No, not necessarily. We may, for example, tune one loop at a time in a sequential manner (usually starting with the fastest loops), and we will end up with a stable overall system. However, due to the negative RGA-element there will be some hidden problem, because the system is not decentralized integral controllable (DIC). This is discussed in more detail on page 464 in Chapter10.1. The stability of the overall system then depends on one or more of the individual loops being in service. This means that detuning one or more of the individual loops may result in instability for the overall system.

For a detailed analysis of achievable performance of the plant (input-output controllability analysis), one must also consider the singular values, as well as the RGA and condition number as functions of frequency. In particular, the crossover frequency range is important. In addition, disturbances and the presence of unstable (RHP) plant poles and zeros must be considered. All these issues are discussed in much more detail in Chapters 5 and 6 where we address achievable performance and input-output controllability analysis for SISO and MIMO plants, respectively.

3.7 Introduction to MIMO robustness

To motivate the need for a deeper understanding of robustness, we present two examples which illustrate that MIMO systems can display a sensitivity to uncertainty not found in SISO systems. We focus our attention on diagonal input uncertainty, which is present in any real system and often limits achievable performance because it enters between the controller and the plant.

3.7.1 Motivating robustness example no. 1: Spinning Satellite

Consider the following plant (Doyle, 1986; Packard et al., 1993) which can itself be motivated by considering the angular velocity control of a satellite spinning about one of its principal axes:

$$G(s) = \frac{1}{s^2 + a^2} \begin{bmatrix} s - a^2 & a(s+1) \\ -a(s+1) & s - a^2 \end{bmatrix}; \quad a = 10 \tag{3.85}$$

A minimal, state-space realization, $G = C(sI - A)^{-1}B + D$, *is*

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|cc} 0 & a & 1 & 0 \\ -a & 0 & 0 & 1 \\ \hline 1 & a & 0 & 0 \\ -a & 1 & 0 & 0 \end{array}\right] \tag{3.86}$$