Self-optimizing control Theory

«How to put optimization into the control layer by selecting the right controlled variable c»

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Outline

Skogestad procedure for control structure design:

I. Top Down
- Step S1: Define operational objective (cost) and constraints
- Step S2: Identify degrees of freedom and optimize operation for disturbances
- Step S3: Implementation of optimal operation
  - Control active constraints
  - Control self-optimizing variables for unconstrained, c=Hy
- Step S4: Where set the production rate? (Inventory control)

II. Bottom Up
- Step S5: Regulatory control: What more to control (secondary CV's)?
- Step S6: Supervisory control
- Step S7: Real-time optimization

Step S3: Implementation of optimal operation

- Optimal operation for given d*:

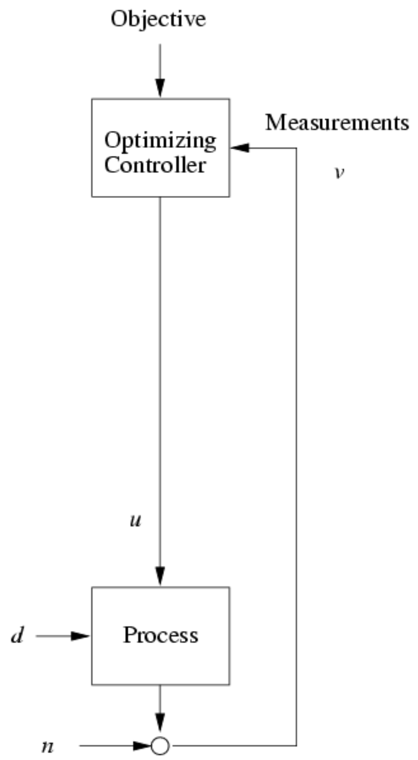
$$\left.\begin{array}{ll} & \min_u J(u,x,d) \\ \text{subject to:} & \\ \text{Model equations:} & f(u,x,d)=0 \\ \text{Operational constraints:} & g(u,x,d)<0 \end{array}\right\} \rightarrow u_{opt}(d)$$

Problem: Usally cannot keep u_{opt} constant because disturbances d change

How should we adjust the degrees of freedom (u)?

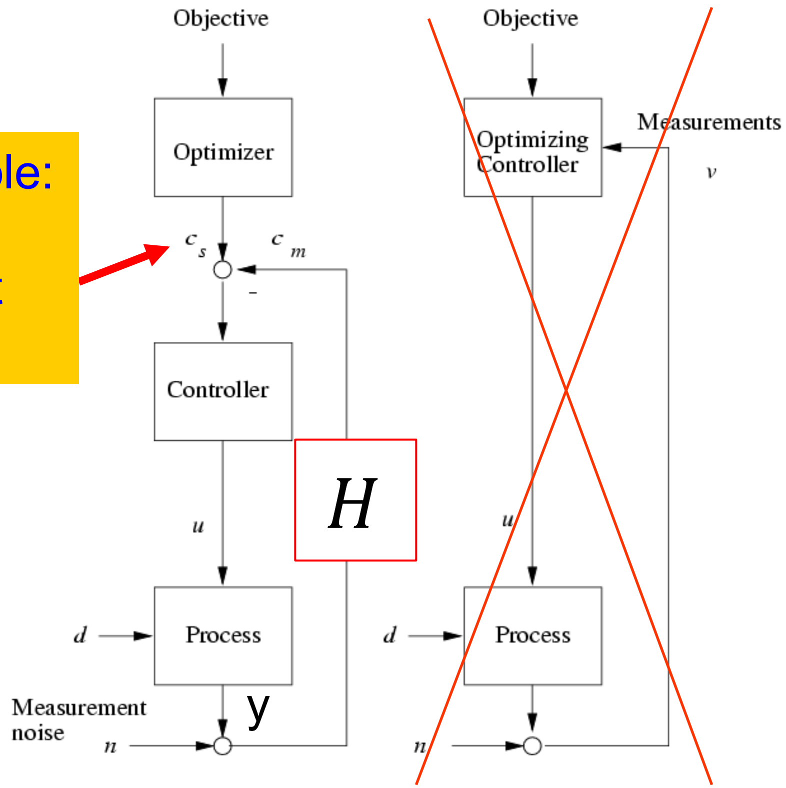
What should we control?

“Optimizing Control” (EMPC)



“Self-Optimizing Control”

c=self-optimizing variable: Constant setpoints give acceptable loss (at least on short time scale)



What should we control?
(What is c? What is H?)

$c = Hy$

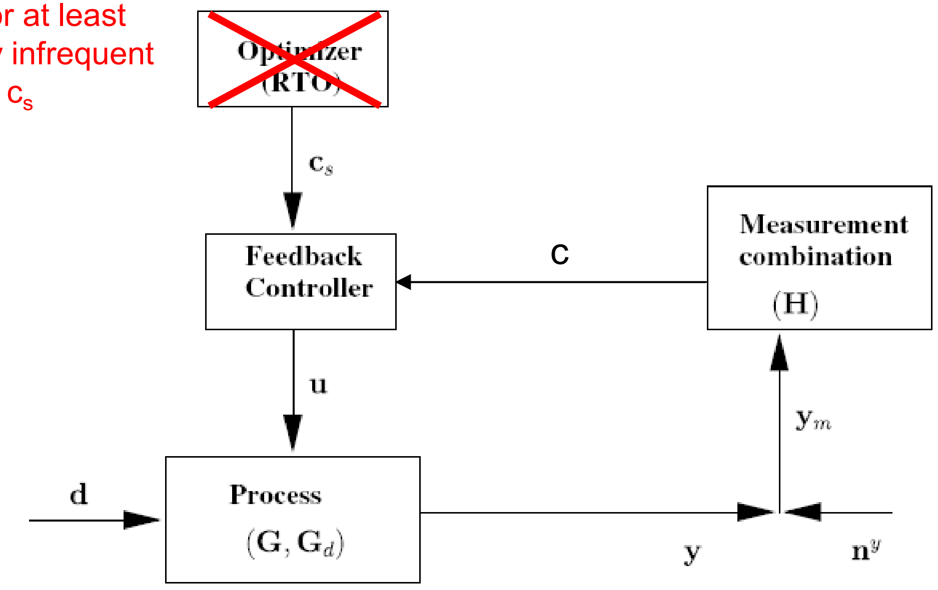
H: Nonsquare matrix

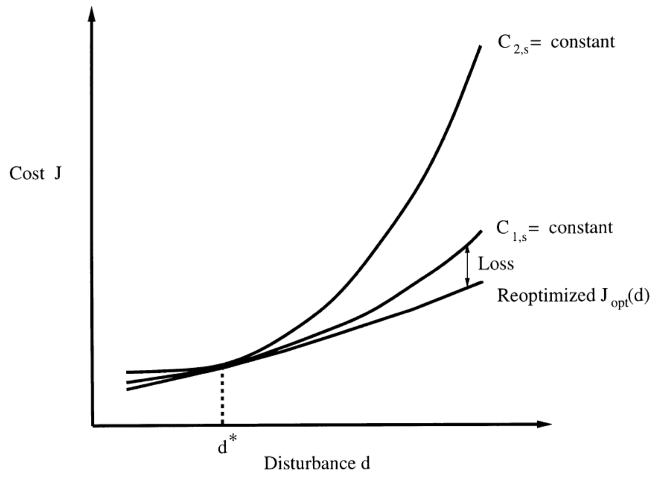
- Usually prefer single measurements as c’s (simple)– H is selection matrix of 0’s and 1’s
- H can also be full matrix (measurement combinations)

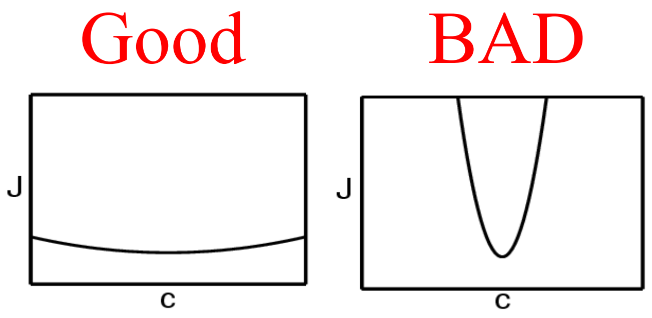
Self-optimizing control

Self-optimizing control is when we can achieve an acceptable loss with constant setpoint values for the controlled variables

Delete - or at least need only infrequent update of c_s







(b) Flat optimum: Implementation easy

(c) Sharp optimum: Sensitive to implementation erros

Optimal operation of runner

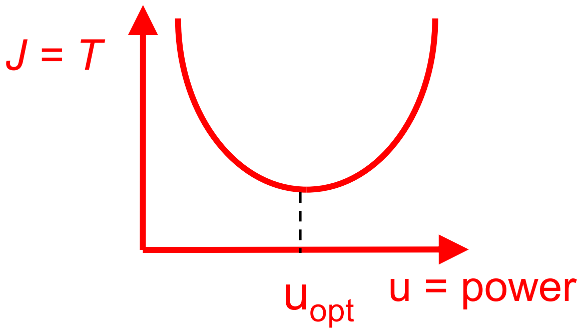
- Cost to be minimized: $J = T$ (total time)
- One degree of freedom: $u =$ power
- What should we control?

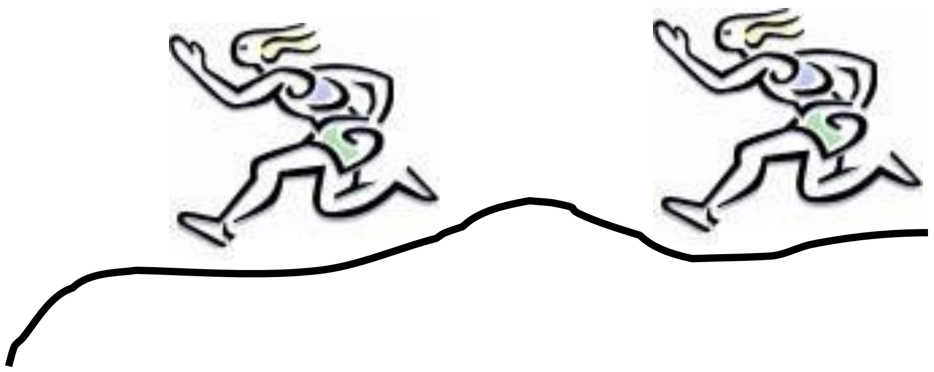
1. Sprinter case

- 100 meters run. $J = T$
- Active constraint control:
  - Maximum speed ("no thinking required")
  - CV = power (at max)

2. Marathon runner case

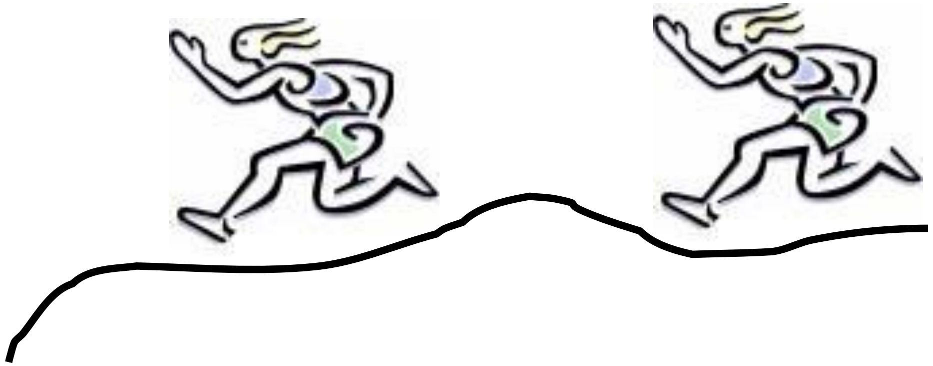
- 40 km run. $J = T$ (total time)
- What should we control? CV = ?
- Unconstrained optimum:



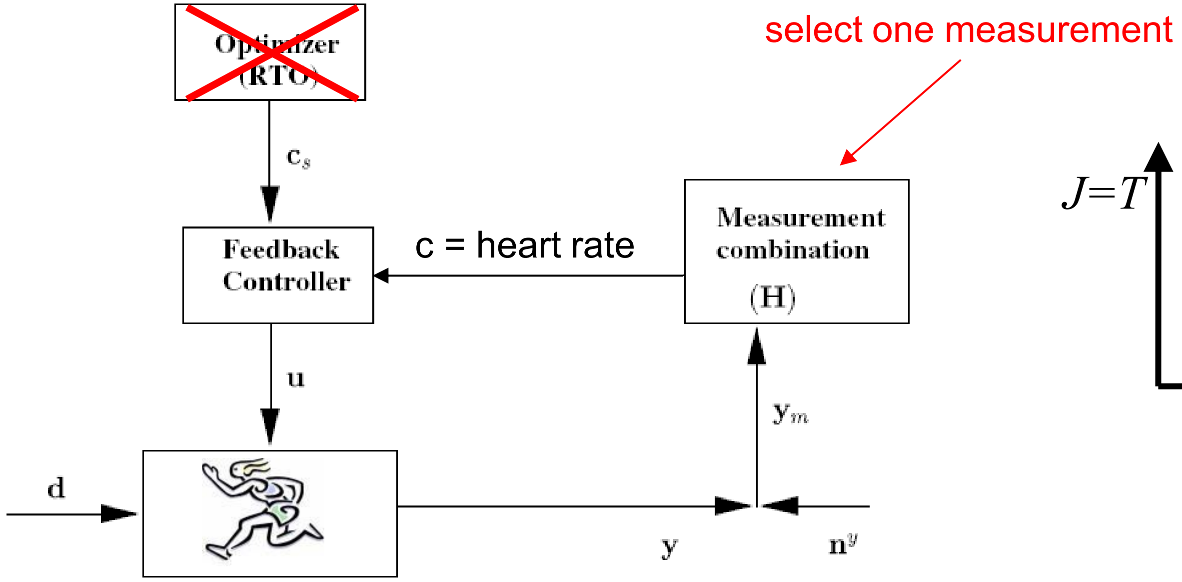


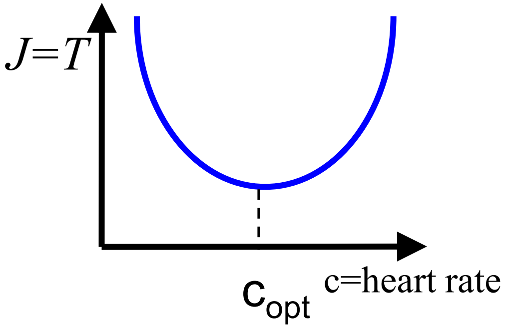
Self-optimizing control: Marathon

- Any self-optimizing variable (to control at constant setpoint)?
  - c_1 = distance to leader of race (not optimal and not always feasible)
  - c_2 = speed (not always feasible, similar to controlling cost J=T, speed = 42 km/T)
  - c_3 = heart rate
  - c_4 = «pain» = level of lactate in muscles



Conclusion Marathon runner



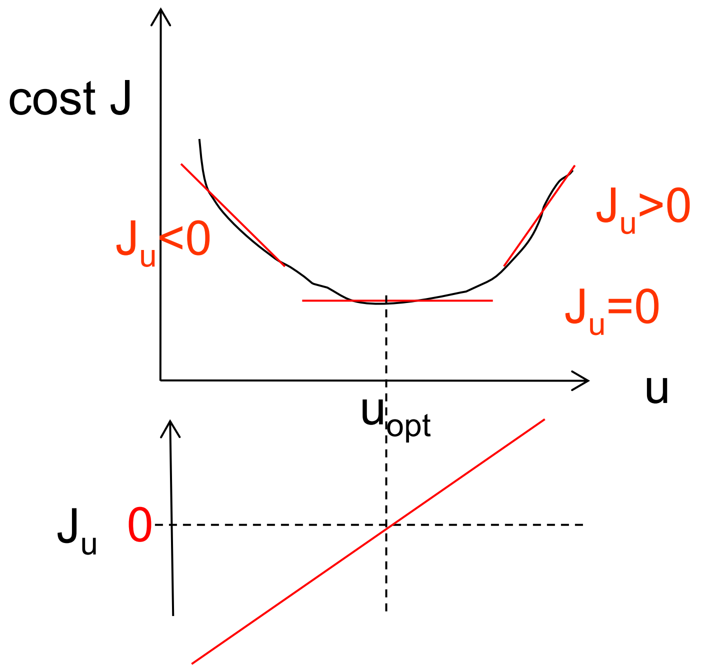


- CV = heart rate is good “self-optimizing” variable
- Simple and robust implementation
- Disturbances are indirectly handled by keeping a constant heart rate
- <u>May</u> have infrequent adjustment of setpoint (c_s)

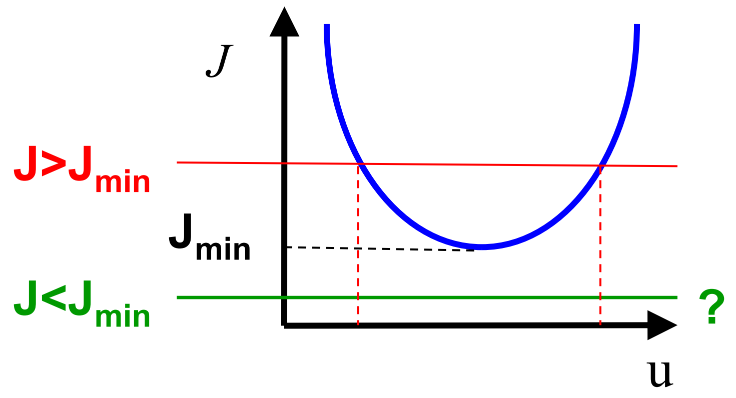
The ideal "self-optimizing" variable is the gradient, J_u

$$c = \Delta J/\Delta u = Ju$$

- Keep gradient at zero for all disturbances ($c = J_u=0$)
- Problem: Usually no measurement of gradient



*I.J. Halvorsen, S. Skogestad, Indirect on-line optimization through setpoint control, in: AIChE 1997 Annual Meeting, Los Angeles; paper 194h.
*I.J. Halvorsen, S. Skogestad, J.C. Morud, V. Alstad, Optimal selection of controlled variables, Industrial & Engineering Chemistry Research 42 (14) (2003) 3273–3284



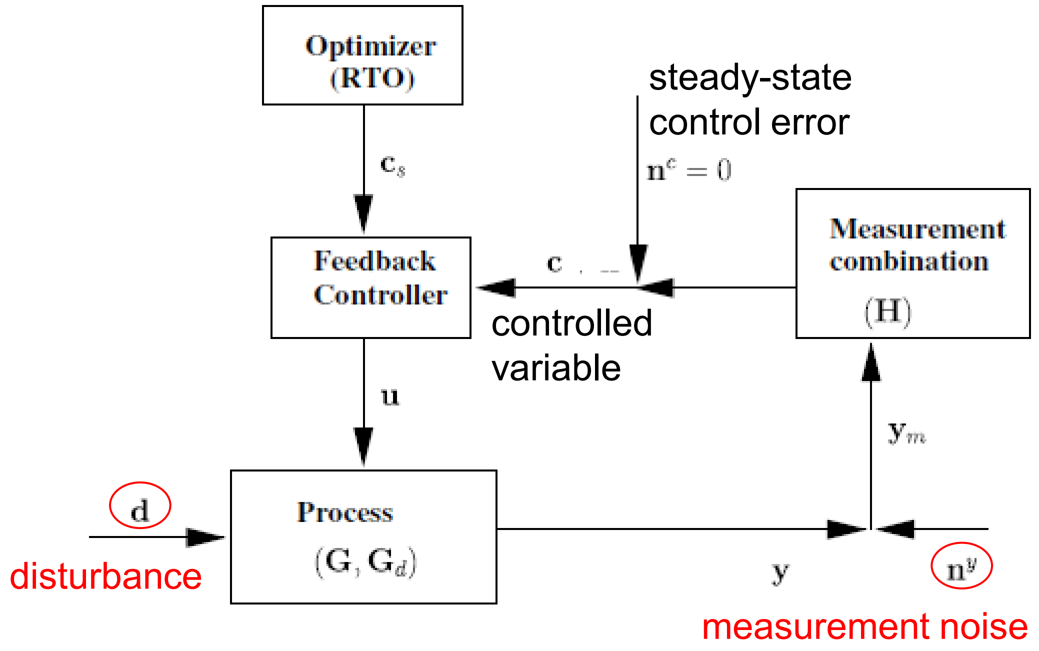
Unconstrained optimum: NEVER try to control a variable that reaches max or min at the optimum

- In particular, never try to control directly the cost J
- Assume we want to minimize J (e.g., J = V = energy) - and we make the stupid choice os selecting CV = V = J
  - **Then setting $J < J_{min}$: Gives infeasible operation (cannot meet constraints)**
  - **and setting $J > J_{min}$: Forces us to be nonoptimal (two steady states: may require strange operation)**

Measurements or mesurement combinations

Ideally: $c = J_u$

In practice: $c = Hy$



- Single measurements:

$$\mathbf{c} = \mathbf{H}\mathbf{y} \qquad \mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

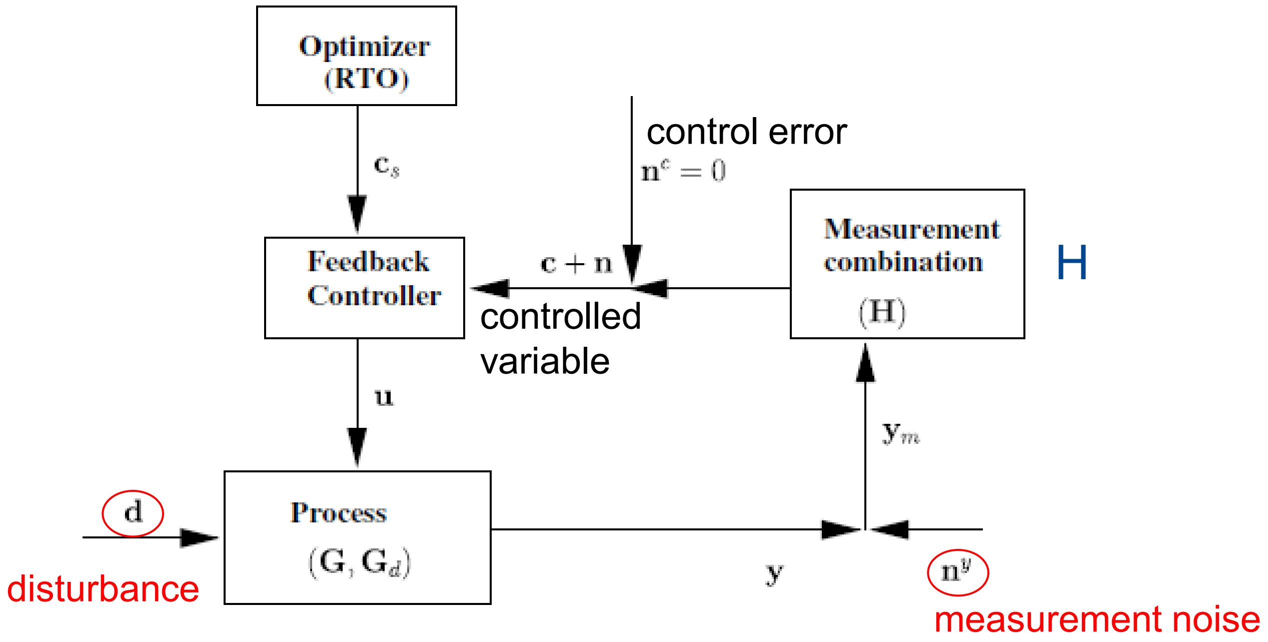
- Combinations of measurements:

$$\mathbf{c} = \mathbf{H}\mathbf{y} \qquad \mathbf{H} = \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \end{bmatrix}$$

Optimal measurement combination

$$\Delta c = h_1 \Delta y_1 + h_2 \Delta y_2 + \cdots = H \Delta y$$

- Candidate measurements (y): Include also inputs u



No measurement noise (n^y=0)

Nullspace method

Theorem

Given a sufficient number of measurements ($n_y \geq n_u + n_d$) and no measurement noise, select **H** such that

$$\mathbf{HF} = 0$$

where

$$\mathbf{F} = \frac{\partial \mathbf{y}^{opt}}{\partial \mathbf{d}}$$

Controlling $\mathbf{c} = \mathbf{Hy}$ to zero yields locally zero loss from optimal operation.

Proof: Given $\partial y^{opt} = F\, \partial d$, and $c = Hy$:

$\partial c^{opt} = H\, \partial y^{opt} = HF\, \partial d$

To make $\partial c^{opt} = 0$ for any ∂d, we must have $HF = 0$.

V. Alstad, S. Skogestad, Null space method for selecting optimal measurement combinations as controlled variables, Industrial & Engineering Chemistry Research 46 (2007) 846–853.

Jäschke, J., Cao, Y., & Kariwala, V. (2017). Self-optimizing control–A survey. Annual Reviews in Control, 43, 199–223.

Nullspace method (HF=0): Analytic expression for H and proof that it gives J_u=0

$$J_u = J_{uu}\Delta u + J_{ud}\Delta d = [J_{uu}\, J_{ud}]\begin{bmatrix}\Delta u\\ \Delta d\end{bmatrix}$$

$$\Delta y = [G^y\, G^y_d]\begin{bmatrix}\Delta u\\ \Delta d\end{bmatrix} = \tilde{G}_y\begin{bmatrix}\Delta u\\ \Delta d\end{bmatrix} \rightarrow \begin{bmatrix}\Delta u\\ \Delta d\end{bmatrix} = \tilde{G}_y^{+}\Delta y$$

Formula for F:

$$J_u^{opt} = J_{uu}\Delta u^{opt} + J_{ud}\Delta d = 0 \rightarrow \Delta u^{opt} = -J_{uu}^{-1}J_{ud}\Delta d$$

$$\Delta y^{opt} = \tilde{G}_y\begin{bmatrix}\Delta u^{opt}\\ \Delta d\end{bmatrix} = \tilde{G}_y\begin{bmatrix}-J_{uu}^{-1}J_{ud}\\ I\end{bmatrix}\Delta d$$

$$\rightarrow F = \tilde{G}_y\begin{bmatrix}-J_{uu}^{-1}J_{ud}\\ I\end{bmatrix}$$

Let $H = [J_{uu}\, J_{ud}]\tilde{G}_y^{+}$. We can verify that $HF = 0$. Therefore, $J_u = [J_{uu}\, J_{ud}]\tilde{G}_y^{+}\Delta y = H\Delta y = \Delta c$, and thus controlling c ($\Delta c = 0$) leads to $J_u = 0$.

- Proof. Appendix B in: Jäschke and Skogestad, "NCO tracking and self-optimizing control in the context of real-time optimization", *Journal of Process Control*, 1407-1416 (2011)

Example. Nullspace Method for Marathon runner

u = power, d = slope [degrees]

y_1 = hr [beat/min], y_2 = v [m/s]

$F = dy_{opt}/dd = \begin{bmatrix} 0.25 \\ -0.2 \end{bmatrix}$

$H = [h_1 \;\; h_2]$

$HF = 0$ → $h_1 f_1 + h_2 f_2 = 0.25\, h_1 - 0.2\, h_2 = 0$

Choose $h_1 = 1$ → $h_2 = 0.25/0.2 = 1.25$

Conclusion: **c = hr + 1.25 v**

Control **c = constant** → hr increases when v decreases (OK uphill!)

Extension: "Exact local method" (with measurement noise)

$$\min_H \| J_{uu}^{1/2} (HG^y)^{-1} H \underbrace{[FW_d \;\; W_{n^y}]}_{Y} \|_F$$

- General analytical solution ("full" H):

$$\boxed{H = G^{yT} (YY^T)^{-1}}$$

- H is unique, except that it can be premultiplied by any nonsingular matrix.
- No noise (W_{ny}=0): Cannot use above analytic expression because YY^T is then singular, but optimal is clearly HF = 0 (Nullspace method)
    - Assumes enough measurements: #y ≥ #u + #d
    - If "extra" measurements (>) then solution to HF=0 is <u>not unique</u> (but above general solution with noise is unique except for premultiplication)
- No disturbances (W_d= [ ]) + same noise for all measurements (W_{ny}= Y = I): Optimal is $H=G^{yT}$ ("control sensitive measurements")

V. Alstad, S. Skogestad, E.S. Hori, Optimal measurement combinations as controlled variables, Journal of Process Control 19 (1) (2009) 138–148.
Jäschke, J., Cao, Y., & Kariwala, V. (2017). Self-optimizing control–A survey. Annual Reviews in Control, 43, 199–223.

Marathon runner: Exact local method

$$F = \begin{bmatrix} 0.25 \\ -0.2 \end{bmatrix}, W_d = 1, W_{ny} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, G^y = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$Y = [FW_d \quad W_{ny}] = \begin{bmatrix} 0.25 & 1 & 0 \\ -0.2 & 0 & 1 \end{bmatrix}$$

$$H = G^{y^T}(Y\,Y^T)^{-1} \rightarrow H = [0.989 \quad 1.009]$$

Normalized H1 = D*H = [1 1.02]

Conclusion: **c = hr + 1.02 v**

- Before (nullspace method): **c = hr + 1.25 v**
- Note: Gives same as nullspace when W_{ny} is small

Derivation of «exact local method».
3-steps:

1. Express loss in terms of $c-c_{opt}$
2. Express $c-c_{opt}$ in terms of H, d and n^y
3. Find optimal H (min. loss) for expected d and n^y

Expected:
$d = W_d d'$
$n^y = W_n n^{y'}$
Expected magnitude = 1
$\left(\left\| \begin{matrix} d' \\ n^{y'} \end{matrix} \right\|_2 \le 1 \right)$
2-norm

W_d and W_n are diagonal matrices with expected magnitudes for d and n^y

Optimal self-optimizing measurement combination, H

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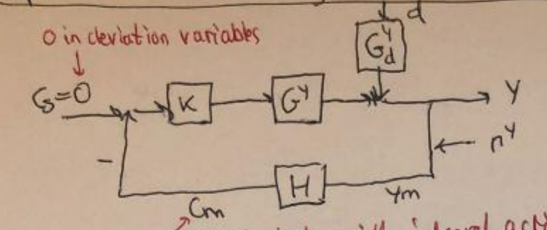
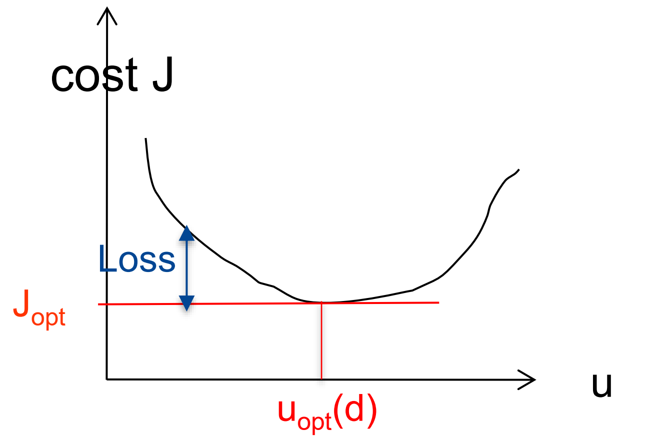
SiS

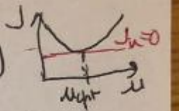


Assume:
-Stedy-state analysis
-Deviation variables (from nominal optimum *)



$c_m = 0$ at steady with integral action in K.

Question: What to control, $c = Hy$. What is best H?



Step 1

Steady-state: Economic loss L for given d is caused by $u \neq u_{opt}(d)$

Taylor expansion of loss around optimum:

$$L = J(u,d) - J(u_{opt}(d), d) = \underbrace{J_u^T}_{=0} (u - u_{opt}) + \frac{1}{2}(u-u_{opt})^T \underbrace{J_{uu}}_{\text{Hessian of } J} (u - u_{opt}) + \text{higher-order (neglected)}$$

Gradient $J_u = \frac{\partial J}{\partial u} = 0$ at optimum.

Hessian of J: $J_{uu} = \frac{\partial^2 J}{\partial u^2}$

Here: $c - c_{opt} = H(y - y_{opt}) = HG^y(u - u_{opt})$

$$\Rightarrow (u - u_{opt}) = (HG^y)^{-1}(c - c_{opt})$$

So get

$$L = \frac{1}{2} z^T z \quad \text{where} \quad z = J_{uu}^{1/2}(u - u_{opt}) = \boxed{J_{uu}^{1/2}(HG^y)^{-1}(c - c_{opt})}$$

Step 2

L depends on d and n^y (it's = 0 otherwise)

1. Disturbances change u_{opt} (and y_{opt} and c_{opt}):
Optimal sensitivity $F = \frac{dy_{opt}}{dd} \Rightarrow y_{opt} = Fd \Rightarrow c_{opt} = HFd = HFW_d d'$
deviation variables

Note: HF=0 -> copt=0 (Nullspace method)

2. Noise makes $c \neq 0$:
$c_m = 0 \Rightarrow c_m = Hy_m = H(y + n^y) = 0 \Rightarrow c = Hy = -Hn^y = +HW_n^y n^{y'}$
(integral action in K)
Change from − to + because sign of $n^{y'}$ does not matter.

Note: Signs don't matter for loss = ½ z^T z.

Step 3

So get:

$$z = \underbrace{J_{uu}^{1/2}(HG^y)^{-1} H \overbrace{[FW_d \;\; W_{n^y}]}^{Y}}_{M(H)} \begin{bmatrix} d' \\ n^{y'} \end{bmatrix} \quad (*)$$

Average loss for $N\begin{pmatrix} d' \\ n^{y'} \end{pmatrix} = \{0, 1\}$ (average, variance) is $L_{avg} = \frac{1}{2}\|M\|_F^2$

F = Frobenius norm: $\sqrt{\sum M_{ij}^2}$

Worstcase loss for $\left\|\begin{matrix} d' \\ n^{y'} \end{matrix}\right\|_2 \le 1$ (2-norm of vector) is $L_{wc} = \frac{1}{2}\bar{\sigma}(M)^2$, $\bar{\sigma}$ = singular value

Both cases: Analytical formula for optimal H is $\boxed{H^T = (YY^T)^{-1}G^y}$ exact local method

Note: H is not unique. Can premultiply this H by any non-singular matrix and get same loss (since c_m=0)

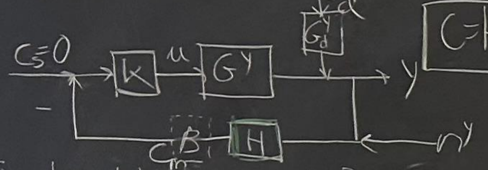
Special case with no noise, $W_{n^y} = 0$ get L=0 with $\boxed{HF = 0}$ nullspace method

From lecture 04 Sep. 2025

Derivation of "exact local method"

Assume $\boxed{\text{Cost} = J\ [\$/s]}$

- Steady state
- Deviation variable (from nom. opt.*)



$\boxed{c = Hy}$

Question: What is best H?

3 steps.

1. Express loss in terms of $c - c_{opt}$.
2. — " — $c - c_{opt}$ — " — H, d, n^y
3. Find optimal H (with loss) for — " —

$$Loss = J - J_{opt} = J(\underset{\text{present } u}{u}, d) - J(u_{opt}(d), d)$$

Step 1 Taylor expansion of loss

$$Loss = J(u,d) - J(u_{opt},d) = \underset{\text{nominal loss}}{0} + \underset{0 \text{ (nominal gradient)}}{J_u^T}(u-u_{opt}) + \frac{1}{2}(u-u_{opt})^T J_{uu}(u-u_{opt}) + \underbrace{\text{higher-order}}_{\approx 0}$$

$$c - c_{opt} = H(y - y_{opt}) = HG^y(u - u_{opt})$$

$$Loss = \frac{1}{2} z^T z, \quad z = J_{uu}^{1/2}(HG^y)^{-1}(c - c_{opt})$$

Step 2. 1. Disturbance change c_{opt}

$$\boxed{c_{opt}} = H y_{opt} = \boxed{HFd}$$

2. Noise/meas. error.

Assume integral action $\Rightarrow c_m = c_s = 0$

$$c_m = Hy_m = H(y + n^y) = 0$$

$$\Rightarrow \boxed{c \overset{\Delta}{=} Hy = -Hn^y}$$

$$Loss = \frac{1}{2} z^T z, \quad z = J_{uu}^{1/2}(HG^y)^{-1}(-Hn^y - HFd)$$

$$= -J_{uu}^{1/2}(HG^y)^{-1}H(Fd + n^y)$$

omit negative sign as we look at $z^T z$ = square.

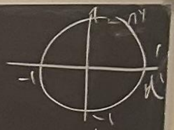
Can be used to analyse: Find loss for given H + given d, n^y

Step 3. Expected disturbance and noise

$d = W_d d'$ — W_d = diagonal matrix of expected disturbance magnitudes

$n^y = W_n n^{y'}$ — W_n = noise magnitudes

Where $\underbrace{\left\| \begin{matrix} d' \\ n^{y'} \end{matrix} \right\|_2 \le 1}_{\text{"} \pm 1 \text{ for } d' \text{ and } n^{y'} \text{"}}$



Loss for this class

$$Z = \underbrace{J_{uu}^{1/2}(HG^y)^{-1}H \overbrace{\left[FW_d \ \ W_n\right]}^{Y}}_{M(H)} \begin{bmatrix} d' \\ n^{y'} \end{bmatrix}$$

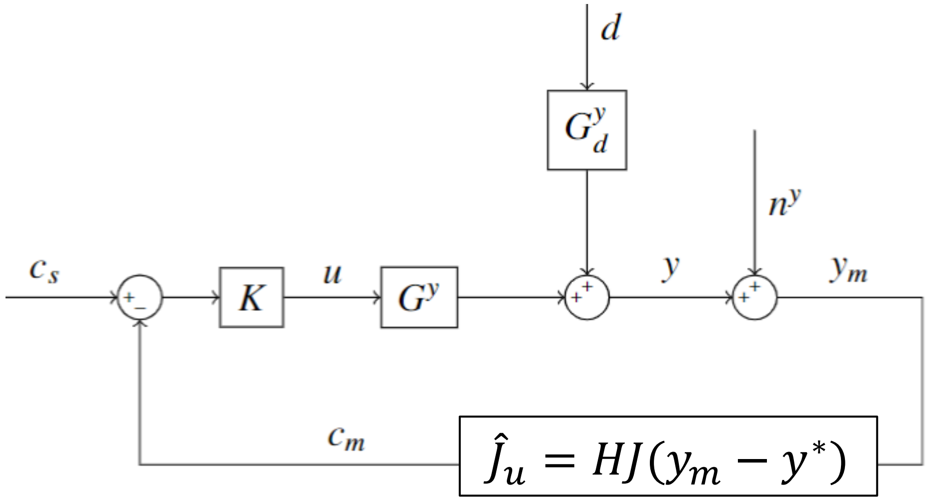
Average loss $= \frac{1}{2} \|M\|_F^2$

Worst-case loss $= \frac{1}{2}(\bar{\sigma}(M))^2$ — $\bar{\sigma}$ = max. singular value

Find H that minimizes loss.

Both cases: $H_{opt} = G^{yT}(YY^T)^{-1}$

Can use H for static gradient estimation. $c_m = \hat{J}_u = H^J(y_m - y^*)$. Very simple and works



From «exact local method» of self-optimizing control ($\tilde{F} \equiv Y$):

$$H^J = J_{uu}\left[G^{yT}\left(\tilde{F}\tilde{F}^T\right)^{-1}G^y\right]^{-1}G^{yT}\left(\tilde{F}\tilde{F}^T\right)^{-1}$$

where $\tilde{F} = [FW_d \quad W_{n^y}]$ and $F = \frac{dy^{opt}}{dd} = G_d^y - G^y J_{uu}^{-1} J_{ud}$.

- So we premultiply the «simple» H to get the right directions
- and add a constant («bias») which may be viewed as the setpoint c_s=Hy*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Optimal measurement-based cost gradient estimate for feedback real-time optimization

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- Bernardino and Skogestad, Optimal measurement-based cost gradient estimate for real-time optimization, Comp. Chem. Engng., 2024

Obtaining F

F is defined as the gain matrix from the disturbances to the optimal measurements → $\Delta y^{opt} = F\,\Delta d$

Brute force method (often the simplest):

- For every disturbance $d_i,\ i = 1, \ldots, n_d$:
  - Perturb the system with $\hat{d}_i = d_i + \Delta d_i$, Δd_i small
  - Reoptimize the system → obtain change in measurements $\Delta y^{opt,i}$
  - Obtain i-th column of F: $F_i = \Delta y^{opt,i} / \Delta d_i$
- Return F

Linearization method for F

F can also be obtained from a linearized state-space model:

$$\Delta y = G^y \Delta u + G_d^y \Delta d$$

$$J_u(u^* + \Delta u, d^* + \Delta d) \approx \overset{=0}{J_u^*} + J_{uu}\Delta u + J_{ud}\Delta d = 0$$
$$\Rightarrow \Delta u^{opt} = -J_{uu}^{-1} J_{ud} \Delta d$$

$$\Delta y^{opt} = G^y \Delta u^{opt} + G_d^y \Delta d = \left(-G^y J_{uu}^{-1} J_{ud} + G_d^y\right) \Delta d$$

$$\boxed{F = -G^y J_{uu}^{-1} J_{ud} + G_d^y}$$

Toy Example.

$J = (u - d)^2$

$n_u = 1$ unconstrained degrees of freedom

$u_{\text{opt}} = d$

Alternative measurements:

$y_1 = 0.1(u - d)$

$y_2 = 20u$

$y_3 = 10u - 5d$

$y_4 = u$

Scaled such that:

$|d| \leq 1$, $|n_i| \leq 1|$, i.e. all y_i's are ± 1

Nominal operating point:

$d = 0 \Rightarrow u_{\text{opt}} = 0, y_{\text{opt}} = 0$

What variable c should we control?

Single measurements

$$L_{wc} = \frac{1}{2}\ \bar{\sigma}(M)^2$$

$$M = J_{uu}^{\frac{1}{2}}(HG^y)^{-1}H\,Y,$$

$$Y = \left[FW_d\ W_{ny}\right], F = -G^y J_{uu}^{-1} J_{ud} + G_d^y$$

. **Exact evaluation of loss:**

$L_{wc,1} = 100$

$L_{wc,2} = 1.0025$

$L_{wc,3} = 0.26$

$L_{wc,4} = 2$

Here: $W_d = 1, W_{ny} = 1,\ J_{uu} = 2, J_{ud} = -2,$

For y_1: $HG^y = 0.1, HG_d^y = -0.1, F = 0, HY = [0\ \ 1], M = \sqrt{2} \cdot 10 \cdot [0\ 1], L_{wc} = \frac{1}{2}\ \bar{\sigma}(M)^2$ =100

For y_2: $HG^y = 20, HG_d^y = 0, F = 20, HY = [20\ \ 1], M = \sqrt{2} \cdot \frac{1}{20} \cdot [20\ 1], L_{wc} = \frac{1}{2}\ \bar{\sigma}(M)^2 = 1.0025$

For y_3: $HG^y = 10, HG_d^y = -5, F = -15, HY = [5\ \ 1], M = \sqrt{2} \cdot \frac{1}{10} \cdot [5\ \ 1], L_{wc} = \frac{1}{2}\ \bar{\sigma}(M)^2 = 0.26$

Reference: I. J. Halvorsen, S. Skogestad, J. Morud and V. Alstad, “Optimal selection of controlled variables”, Industrial & Engineering Chemistry Research, 42 (14), 3273-3284 (2003).

Toy Example. Exact local method. Combine all measurements

$$\boxed{\begin{aligned} &J = (u-d)^2 \\ &n_u = 1 \text{ unconstrained degrees of freedom} \\ &u_{\text{opt}} = d \end{aligned}}$$

Alternative measurements:

$y_1 = 0.1(u-d)$

$y_2 = 20u$

$y_3 = 10u - 5d$

$y_4 = u$

Scaled such that:

$|d| \leq 1$, $|n_i| \leq 1|$, i.e. all y_i's are ± 1

Nominal operating point:

$d = 0 \Rightarrow u_{\text{opt}} = 0, y_{\text{opt}} = 0$

What variable c should we control?

$$Y = [FW_d \; W_{ny}],$$
$$F = -G^y J_{uu}^{-1} J_{ud} + G_d^y$$
$$H = (Y\,Y^T)^{-1}\;G^y$$

Here: $W_d = 1, W_{ny} = I\ (4x4),\ J_{uu} = 2, J_{ud} = -2,$
$G^y = [0.1\ \ 20\ \ 10\ \ 1\,]',\ G_d^y = [-0.1\ \ 0\ \ -5\ \ 0]',$
$F = [0\ \ 20\ \ 5\ \ 1]',$

$$Y = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 20 & 0 & 1 & 0 & 0 \\ 5 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$(Y\,Y^T)^{-1} =$

1.0000	0	0	0
0	0.0632	-0.2342	-0.0468
0	-0.2342	0.9415	-0.0117
0	-0.0468	-0.0117	0.9977

$H = (Y\,Y^T)^{-1}\ G^y\ =\ [0.1000\ \ -1.1241\ \ \ 4.7190\ \ -0.0562]$

Normalized to have 2-norm = 1.

$H = [\,0.0206\ \ -0.2317\ \ \ 0.9725\ \ -0.0116]$

Reference: V. Alstad et al. , Journal of Process Control 19 (2009) 138–148

Toy Example: Nullspace method (not unique)

$$c = Hy = \begin{pmatrix} h_1 & h_2 & h_3 & h_4 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = h_1 y_1 + h_2 y_2 + h_3 y_3 + h_4 y_4$$

B1. Nullspace method

Neglect measurement error ($n = 0$):

$$HF = 0$$

Sensitivity matrix

$$\Delta y_{\mathrm{opt}} = F \Delta d; F = \begin{pmatrix} 0 & 20 & 5 & 1 \end{pmatrix}^T$$

To find H that satisfies $HF = 0$ must combine at least two measurements:

$$n_y \geq n_u + n_d = 1 + 1 = 2$$

Toy Example. Nullspace method with 2 measurements

C. Optimal combination

Need two measurements. Best combination is y_2 and y_3:

$$\begin{pmatrix} y_2 \\ y3 \end{pmatrix} = \begin{pmatrix} 20 & 0 \\ 10 & -5 \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix}; \; \underline{\sigma} = 4.45$$

Optimal sensitivity:

$$y_{\text{opt}} = Fd; F = \begin{pmatrix} 20 \\ 5 \end{pmatrix}$$

Optimal combination:

$$HF = 0 \Rightarrow (\, h_1 \quad h_2 \,) \begin{pmatrix} 20 \\ 5 \end{pmatrix} = 0 \Rightarrow 20h_1 + 5h_2 = 0$$

Select $h_1 = 1$. Get $h_2 = -20h_1/5 = -4$, so

$$c_{\text{opt}} = y_2 - 4y_3$$

Check: $c = y_2 - 4y_3 = 20u - 40u + 20d = -20(u - d)$ *(OK!)*

Example where nullspace method «fails»

u= reflux
d=feed rate

J = $(u-d)^2$
y1 = 0.01(u-d)   % temperature product (very small gain!)
y2 = u-0.8d % tempereture inside column
uopt = d
y1opt = 0
y2opt = 0.2 d

Nullspace: H0=[1 0]   % Not good! Use only y1
Exact local method: H=[1 96] % Use y2 instead

```
F =[0 0.2]'
Wd=1*eye(1)
Wn=1*eye(2)
Gy = [0.01 1]'
H0=null(F'); H0=H0'/H0(1)  % nullspace method
Y = [F*Wd Wn],
H1 = Gy' * inv(Y * Y')
H = H1/H1(1)   % exact local method
```

Conclusion: GOOD “SELF-OPTIMIZING” CV = c

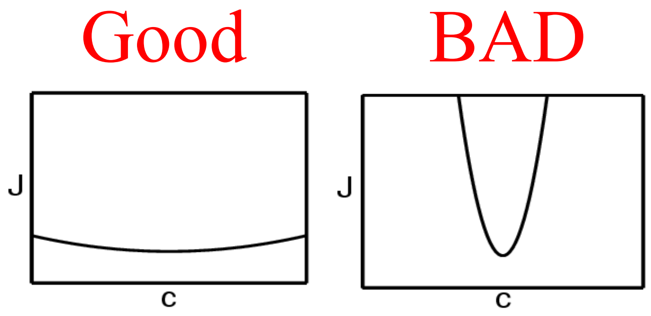
1. Optimal value c_{opt} is constant (independent of disturbance d):

   → Want small optimal sensitivity: $F_c = \frac{\Delta c_{opt}}{\Delta d} = \mathrm{HF}$

2. c is “sensitive” to input u (MV) (to reduce effect of measurement noise)

   → Want large gain $G = HG^y = \frac{\Delta c}{\Delta u}$

   (Equivalently: Optimum should be flat!)



(b) Flat optimum: Implementation easy

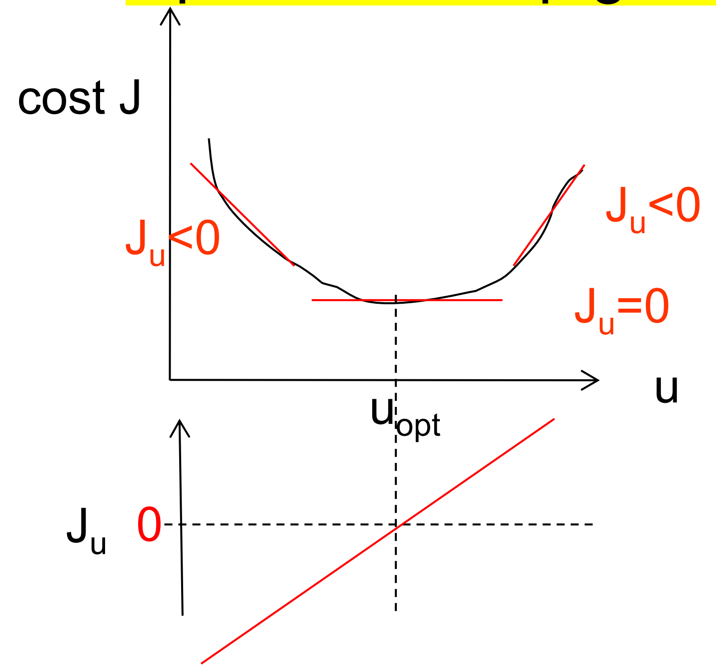
(c) Sharp optimum: Sensitive to implementation erros

Optimal steady-state operation with constraints

$\min_u J(u,d)$

s.t. $g(u,d) \geq 0$ (constraints)

- J = economic cost [$/s]
- Unconstrained case: Optimal to keep gradient $J_u = \partial J/\partial u = 0$



- Constrained case: KKT-conditions: Active constraints: g=0,

Remaining conconstrained DOFs: $L_u = J_u + \lambda^T g_u = 0$

WITH CONSTRAINTS

Want tight control of active constraints for economic reasons

- Active constraint: $g_A=0$
- Tight control of g_A minimizes «back-off»

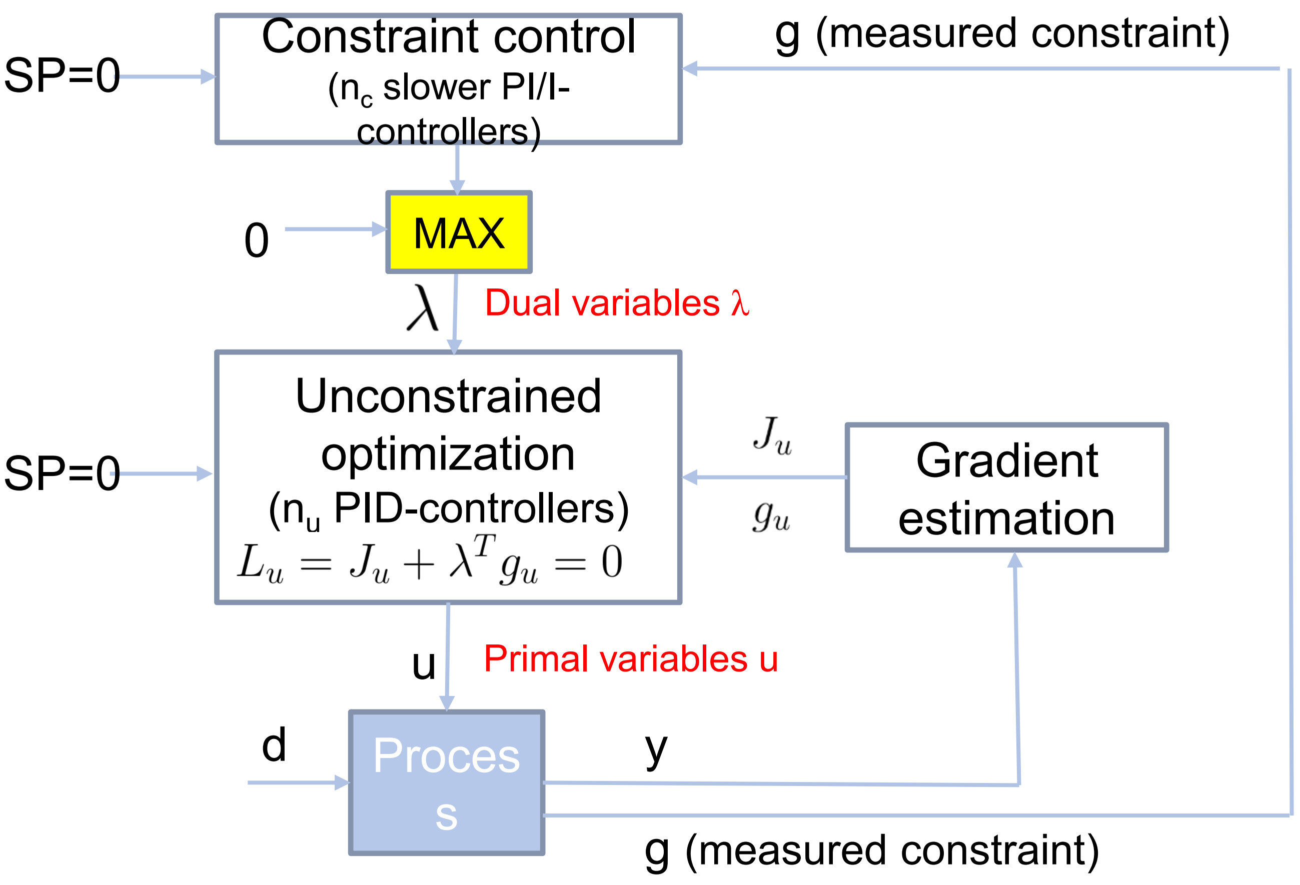
- How can we identify and control active constraints?
- How can we switch constraints?
- How do find the correct gradient when the constraints change?
- How to implement in the control system?
  - We published 3 approaches in JPC in 2024
  - All may use the «unconstrained» gradient estimate presented above:

$$\boxed{\hat{J}_u = HJ(y_m - y^*)}$$

I. Primal-dual control based on KKT conditions: Feedback solution that automatically tracks active constraints by adjusting Lagrange multipliers (= shadow prices = dual variables) λ

$$L_u = J_u + \lambda^T g_u = 0$$

Inequality constraints: $\lambda \geq 0$



Primal-dual feedback control.

- Makes use of «dual decomposition» of KKT conditions
- Selector on dual variables λ
- Problem: Constraint control using dual variables is on slow time scale (can avoid with override)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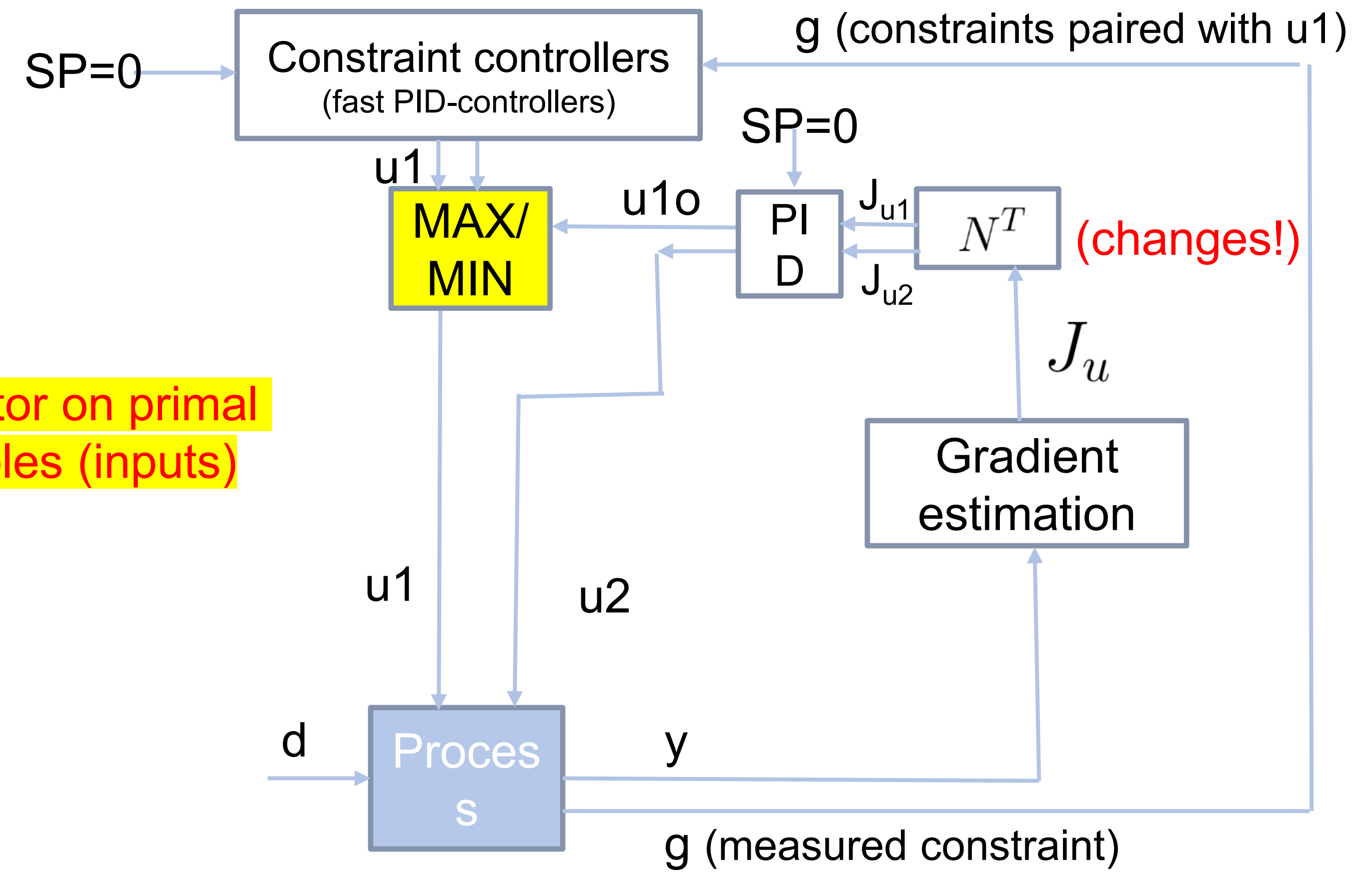
Primal–dual feedback-optimizing control with override for real-time optimization

Risvan Dirza, Sigurd Skogestad

- D. Krishnamoorthy, A distributed feedback-based online process optimization framework for optimal resource sharing, J. Process Control 97 (2021) 72–83,
- R. Dirza and S. Skogestad . Primal–dual feedback-optimizing control with override for real-time optimization. J. Process Control, Vol. 138 (2024), 103208.

II. Region-based feedback solution with «direct» constraint control (for case with more inputs than constraints)



- Selector on primal variables (inputs)

KKT: $L_u = J_u + \lambda^T g_u = 0$

Introduce N: $N^T g_u = 0$

Control
1. Reduced gradient $N^T J_u = 0$
   - «self-optimizing variables»)
2. Active constraints $g_A = 0$.

Problem: Simple switching requires at least as many MVs (u) as constraints

- Jaschke and Skogestad, «Optimal controlled variables for¨polynomial systems». S., J. Process Control, 2012
- D. Krishnamoorthy and S. Skogestad, «Online Process Optimization with Active Constraint Set Changes using Simple Control Structure», I&EC Res., 2019
- Bernardino and Skogestad, Decentralized control using selectors for optimal steady-state operation with changing active constraints, J. Process Control, Vol. 137, 2024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Decentralized control using selectors for optimal steady-state operation with changing active constraints
Lucas Ferreira Bernardino, Sigurd Skogestad

III. Region-based MPC with switching of cost function (for general case)

Standard MPC with fixed CVs: Not optimal

Proposed: With changing cost (switched CVs)

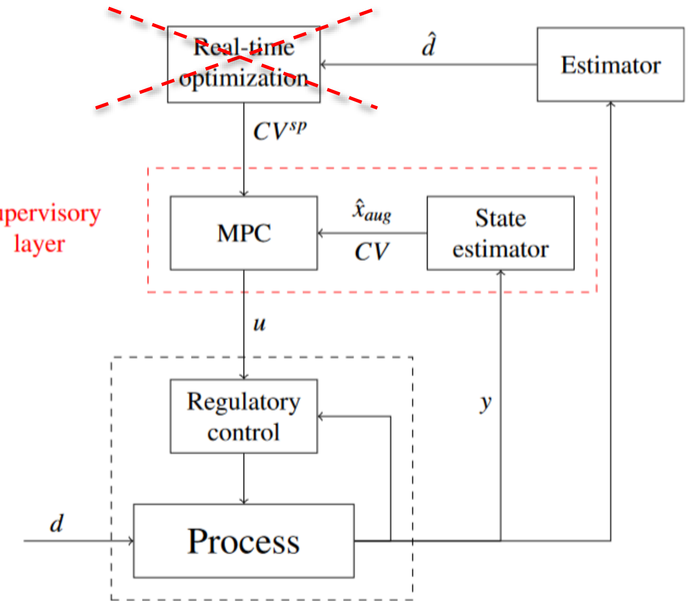

Figure 1: Typical hierarchical control structure with standard setpoint-tracking MPC in the supervisory layer. The cost function for the RTO layer is J^{ec} and the cost function for the MPC layer is J^{MPC}. With no RTO layer (and thus constant setpoints CV^{sp}), this structure is not economically optimal when there are changes in the active constraints. For smaller applications, the state estimator may be used also as the RTO estimator.

$$J^{MPC} = \sum_{k=1}^{N} \|CV_k - CV^{sp}\|_Q^2 + \|\Delta u_k\|_R^2$$

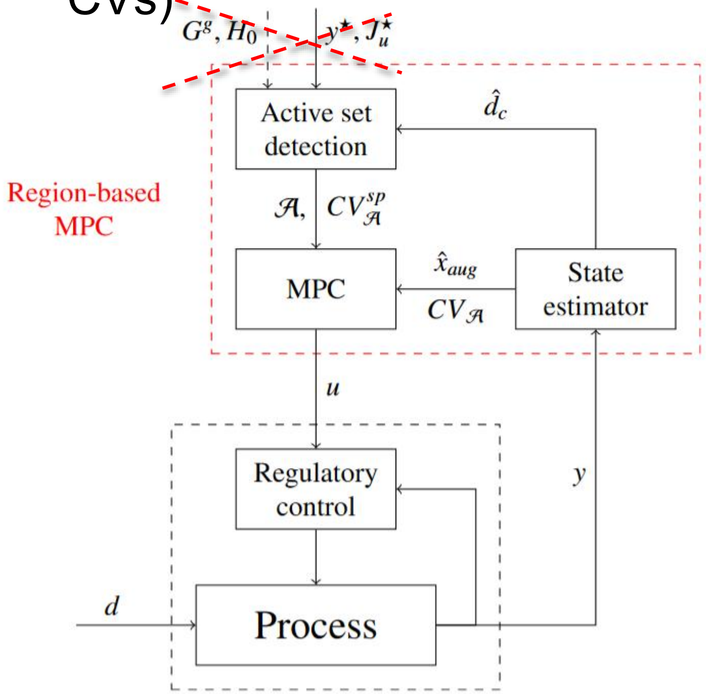

Figure 2: Proposed region-based MPC structure with active set detection and change in controlled variables. The possible updates from an upper RTO layer (y^*, J_u^* etc.) are not considered in the present work. Even with no RTO layer (and thus with constant setpoints $CV_{\mathcal{A}}^{sp}$, see (14) and (15), in each active constraint region), this structure is potentially economically optimal when there are changes in the active constraints.

$$J_{\mathcal{A}}^{MPC} = \sum_{k=1}^{N} \|CV_{\mathcal{A}} - CV_{\mathcal{A}}^{sp}\|_{Q_{\mathcal{A}}}^2 + \|\Delta u_k\|_{R_{\mathcal{A}}}^2$$

$$CV_{\mathcal{A}} = \begin{bmatrix} g_{\mathcal{A}} \\ c_{\mathcal{A}} \end{bmatrix} = \begin{bmatrix} g_{\mathcal{A}} \\ N_{\mathcal{A}}^T H_0 y \end{bmatrix} \qquad (14)$$

$$H_0 = \begin{bmatrix} J_{uu} & J_{ud} \end{bmatrix} \begin{bmatrix} G^y & G_d^y \end{bmatrix}^\dagger$$


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- Bernardino and Skogestad, Optimal switching of MPC cost function for changing active constraints. J. Proc. Control, 2024