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Convex reformulations for self-optimizing control optimization problem: Linear Matrix Inequality approach

Mohammad Reza Jafari [a], Mohammad Mehdi Arefi [a,*], Mehdi Panahi [b]

[a] Department of Power and Control Engineering, School of Electrical and Computer Engineering, Shiraz University, Shiraz, Iran
[b] Chemical Engineering Department, Faculty of Engineering, Ferdowsi University of Mashhad, Mashhad, Iran

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ABSTRACT

The purpose of self-optimizing control (SOC) is minimizing the steady-state economic loss of chemical processes in the presence of disturbances and measurement noises by keeping selected controlled variables (CVs) at constant set-points. In self-optimizing control, by defining a desired objective/loss function and selecting the appropriate combination of process measurements, the average loss, the worst-case loss, or both can be minimized. In general, the optimization problem of self-optimizing control is a non-convex problem and there exist some approaches to change it to a convex form by adding another constraint to the optimization problem, using branch and bound algorithm or mixed integer quadratic programming method to solve the SOC problem. Linear Matrix Inequalities (LMIs) are one of the popular and powerful tools to solve convex optimization problems and changing the optimization problems to the LMI form is gaining popularity. In parallel, for some problems that are non-convex and cannot be transformed to the LMI form, Bilinear Matrix Inequalities (BMI) have been developed. In this paper, we present; first a method to change the convex form of SOC problem to the LMI form and second, reformulate the main and non-convex SOC problem to a BMI form and then change it to the LMI form. The proposed methods are then evaluated on three benchmark processes: a binary distillation column, an evaporator, and a Kaibel column. The LMI/BMI methods are implemented using LMI Control Toolbox and PENBMI of YALMIP toolbox of MATLAB® software. Results show that the proposed algorithm outperforms other methods in the case of structured measurement matrix **H**. The main benefit of LMI approach is that the desired structure of matrix **H** can be directly implemented in the optimization method.

1. Introduction

Optimal operation of chemical processes under various conditions such as external disturbance is vital for the economic profitability of the chemical plants. Keeping process operating points near to their designed moving optimal operating points is highly recommended. Large fluctuations from the optimal operating points result in an economic cost and may lead to violation of process operating constraints. Keeping the process close to the optimal operating point needs good operation of control system for both normal conditions and in the presence of disturbance.

One of the typical control structures of chemical processes is a multilayer hierarchical structure of the control system that includes layers with different time scales [1,2]. Typical time scales are depicted in Fig. 1 for different layers of this structure in plantwide control. From the bottom to the top, these layers include [3]: *Control layer* (minutes and seconds), *Local optimization* (hours), *Site-wide optimization* (days), and *Scheduling* (weeks) [4]. In Fig. 1, CV_1 to CV_n refers to the controlled variables that are used to control the process.

The business optimization objectives, such as minimizing loss function or maximizing economic profitability J of plant operation are attained by cascading the objectives from the top layer (scheduling) to the bottom layer (control layer). The interaction and interconnection between different layers of this hierarchical control structure occur through the set points that flow from the upper layer to the bottom layer in a cascade structure. It should be noted that in top layers, setpoints may be updated in an hour but in bottom layers including the control layer, they are updated continuously. One of the most important factors in determining the profitability of the process is control structure selection. Control structure design determines which process variables should be measured and controlled and which variables should be manipulated [1].

The role of controlled variable (CV) selection is highlighted when the operating conditions vary, which can lead to increased profitability and reduction in costs.

* Corresponding author.
E-mail address: arefi@shirazu.ac.ir (M.M. Arefi).

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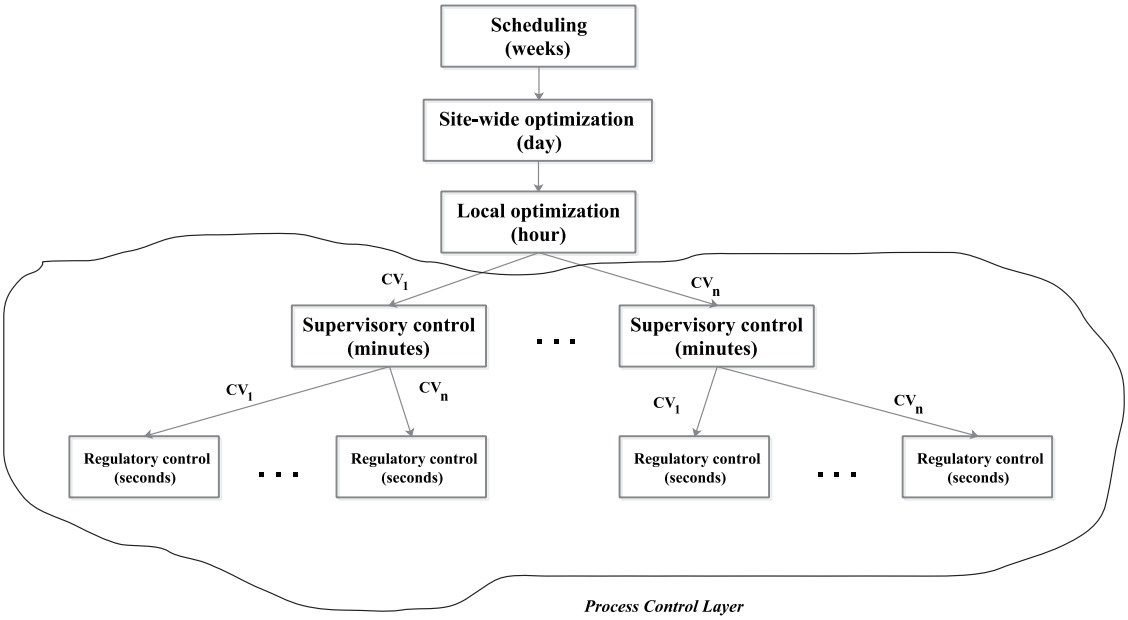

Fig. 1. Plantwide Control system hierarchy diagram of chemical plants.

In the literature on process control structure design, the first step is to formulate the self-optimizing control optimization problem [5]. This method is identified by the selection of self-optimizing CVs based on the chosen objective function. The successful application of self-optimizing control (SOC) needs tools and methods for choosing good CVs. The major difference between SOC and other methods in control structure design [6] is that in SOC, the selection of CVs is done by a specific procedure to systematically minimize the loss function subject to a given cost function. Briefly, the selection procedure in SOC method is started by defining a desired cost function, then by controlling the SOC CVs at their setpoints, the defined cost function should be minimized during the plant operation. Continuous chemical processes usually operate at a steady-state for most of the time, by occurring disturbances, the effect of the transient responses can be neglected if the disturbances stay constant for a long enough time. Therefore, the SOC problem can be stated as an optimization problem in steady-state conditions. Simply, the SOC problem leads to finding function h such that the selected CVs defined by Eq. (1) minimize the defined objective function:

$$c = h(y). \tag{1}$$

Typically, the measurement variable $c = h(y)$ can be defined as any kind of function of measurement variables y, but normally it is selected as a linear function $c = \mathbf{H}y$ where $\mathbf{H}$ is a constant matrix with appropriate dimensions. The primal method for choosing self-optimizing CVs was a brute-force method [5,7]. The idea of brute-force method is to check out the loss for all possible combinations of candidate CVs for any possible combination values of measurement noise and disturbances. The main drawback of brute-force method is that it requires to solve many optimization problems, and it is more serious when there are many process disturbances and controlled variable candidates that makes this approach intractable. Local methods have been developed to minimize the number of checking for different CVs and to disqualify poor CV choices in the step of control structure design. The idea of local methods is to select such a controlled variable set in a way that it has acceptable local behavior around the designed operating points, otherwise it may be omitted instantly. A basic idea in the SOC framework [4] is to minimize the differential loss variable ($L = J - J_{opt}(\mathbf{d})$) instead of directly minimizing the cost J. Sometimes local method is called "*loss minimization*" or the "*minimum loss method*". By selecting a linear combination of measurement variables as controlled variables $c = \mathbf{H}y$, it is implicitly assumed that the steady-state process model is linear and the cost function J is defined in quadratic form. Practically, all of the steady-state optimal operation problems can be approximated by linearizing chemical processes at nominal optimal point. Local method prepares systematic method to select CVs. More precisely, the objective of local method is to find a matrix $\mathbf{H}$ (i.e., a linear combination of controlled variables, $c = \mathbf{H}y$) such that, by controlling these candidate variables indirectly, it leads to minimum loss (L) with acceptable operation of process when disturbances ($\mathbf{d}$) occur in presence of measurement noise $\mathbf{n^y}$. It is also assumed that all active constraints of process has been controlled and kept constant, and the lower-dimensional free/unconstrained subspace have been considered as degrees of freedom.

Originally, the problem of solving SOC optimization problem and finding optimal CVs ($\mathbf{H}$) was believed to be a non-convex optimization problem. Hence, it is generally difficult and computationally expensive to solve it numerically [8] with convex methods. However, in [9] it has been proven that by adding an extra constraint to the original optimization problem, it can be reformulated as a convex and quadratic optimization problem that has linear constraints. In the literature there are other derivations to find optimal measurement combination matrix ($\mathbf{H}$) where the idea is to select the candidate variables to reach an optimal trade-off between measurement noise and rejecting the process disturbances. In [10,11], eigenvalues of a matrix and generalized singular value decomposition (GSVD) method were applied to solve the SOC optimization problem. However, because of the combinatorial nature of the problem, selecting measurement variables individually or linear combinations of a subset of measurements as CVs is more crucial. Practically, it is not required nor desired to use all of available measurements or combinations of them in the selected CVs matrix ($\mathbf{H}$). In [10,12], it has been shown that usually controlling a subset of available measurements can have a performance similar to using all available measurements, so it can lead to a plant control structure design with infinitesimal increase in loss.

In the literature, the problem of finding the best subset measurement selection has been solved in two ways: The first method

is tailor-made branch and bound algorithms described in [12–14], and the second one is to formulate the subset measurement selection problem as a mixed integer quadratic optimization problem (MIQP) that utilizes standard MIQP solvers to find the best measurement set [15]. Both of these two methods are based on the convex formulation of the SOC problem which is mentioned in Theorem 1 of [9]. This convex reformulation is based on adding an equality constraint and then simplifying the non-convex problem. It works for full **H** case and some structured cases mentioned in [15]. However, it does not work for decentralized and triangular structures. Although there are some approximation methods for such cases [16], they cannot guarantee finding the CVs set with the minimum loss. For practical applications, it may be needed to impose some limitations on the structure of **H**. That is, it may be required that some elements in **H** be zero.

Sometimes, in order to have good dynamic responses from the process, we may need structural constraints that lead to structured **H**. For example, in the Kaibel column case study, as stated in [17], to get good dynamic response, at least one temperature in the prefractionator should be used in the regulatory layer. Another example is the distillation column case study [18], that triangular **H** is dynamically preferable to avoid large time delays between top stages temperatures and reflux, or we may want to separate different parts of a process from each other like the evaporator case study.

Also, it should be considered that the full **H** case is costly for practical applications in terms of implementation costs, because it needs more instruments to measure selected candidate measurable variables, more cabling and more complicated control system hardware and software and so on. Therefore, it is tolerable to have greater loss than optimal loss, but the implementation cost is reduced.

In [19], a global SOC (gSOC) method has been introduced to find optimal CVs by minimizing average economic loss in the whole space. In this approach similar to local methods, the loss function is formulated as a quadratic form based on the second-order Taylor series expansion at any disturbance scenarios, rather than a single reference point in the local method. Then, the optimal matrix **H** is derived by minimizing the loss function. In [20], by using neural network, the active constraints changing problem is handled and a new method for constrained gSOC (cgSOC) has been proposed. This method finds optimal CVs globally through offline neural network training. Global SOC has been revisited in [21] by applying the nonlinear programming (NLP) framework for gSOC and a sequential solution strategy is proposed. In this method, the polynomial chaos expansion (PCE) is used to enhance the computation speed and in addition, the active set change problem has been handled. To overcome high computational loads, time-consuming convergence, and simulation crashes of flowsheet optimization simulators, in [22] application of surrogate models has been evaluated to simulate the large-scale processes. The multilayer perceptron neural network (MLP-ANN) with radial basis function has been chosen as the surrogate model. It has been shown that the MLP surrogate model has the best performance in predicting the optimal points and in selecting the best self-optimizing CVs. It should be noted that all previous methods have utilized the convex form of SOC introduced in [9].

Another approach to solve the SOC optimization problem, which is introduced in this work is using bilinear matrix inequality (BMI) or linear inequality matrix (LMI) methods. In [9], the nonlinear SOC optimization problem is solved by adding an equality equation to the basic nonlinear problem, which is described in the next section. BMI or LMI approach can help to solve the original nonlinear SOC optimization problem as described in the paper. The focus of this paper is to develop a new approach to solve the nonlinear SOC optimization problem by using BMI or LMI techniques therefore local method or minimum loss method has been considered. As presented in this paper, LMI/BMI techniques can overcome the drawbacks of previous methods in solving SOC problem and it works on all structures of **H**. The most important part of using the LMI/BMI method is to reformulate the main non-convex problem in the LMI/BMI form and then a standard solver can solve the resulting problem. In this paper, this reformulation is presented. The main benefit of this reformulation is that it can find the minimum loss on any structured **H** case such as triangular and block diagonal that other methods such as branch and bound (BAB) and MIQP cannot consider. Moreover, the structure of **H** is directly obtained by solving the optimization problem.

The rest of this paper is organized as follows. In Section 2, preliminary concepts of the self-optimizing control and minimum loss method are introduced. The LMI reformulation of the convex and non-convex forms of the SOC problem is stated in Section 3. The simulation and evaluation of the proposed methods are provided for three popular processes in SOC; an evaporator process, a benchmark binary distillation column that has 41 stages, and a 4-product Kaibel column in Section 4. The conclusion of this work is presented in Section 5.

2. Preliminary concepts

The basic idea behind local methods is to reduce the number of CVs and to omit CV candidates that perform poorly in the control structure design phase. The key concept in the SOC structure is to minimize the loss (Eq. (2)) from the optimal operation point when there are disturbances [4]. The SOC procedure is performed when a tolerable loss is achieved for constant set-points. It should be noted that with implementation of SOC, when disturbances occur, it does not need to reoptimize and find new measurement variables [5].

$$L = J - J_{opt}(\mathbf{d}) \tag{2}$$

More precisely, the aim of SOC is to find CVs such that when held constant, give an acceptable loss rather than finding optimal set-points. In Eq. (2), the loss L is characterized as the difference between the truly optimal value of the defined cost function and the true value of cost function. First of all, the SOC optimization problem must be reformulated based on a loss function, which depends on system matrices.

2.1. SOC problem formulation

Variables have been classified as follows [9,15]:

- $\mathbf{u} \in \mathbb{R}^{n_u}$ as inputs for optimization: they are unconstrained steady-state degrees of freedom (DOF) and should make an independent set.
- $\mathbf{d} \in \mathbb{R}^{n_d}$ as disturbances: it is worth mentioning that process parameter changes can be considered as disturbance.
- $\mathbf{y} \in \mathbb{R}^{n_y}$ are all available measurable variables, which are candidate CVs as well. The manipulated variables (MVs) and measurable inputs ($\mathbf{u}$) can be considered in the measurement set $\mathbf{y}$ too.
- $\mathbf{n}^y$ as measurement noise for $\mathbf{y}$ i.e. $\mathbf{y}_m = \mathbf{y} + \mathbf{n}^y$.
- $\mathbf{c} \in \mathbb{R}^{n_c}$ as selected controlled variables.

As discussed in [8], it is assumed that all of our considerations are of local nature and the set of active inequality constraints does not change with disturbances. Furthermore, SOC uses a linearized model that is expanded around the operating point. Also, we assume that cost function J is smooth or more precisely twice differentiable at the operating point that we are considering.

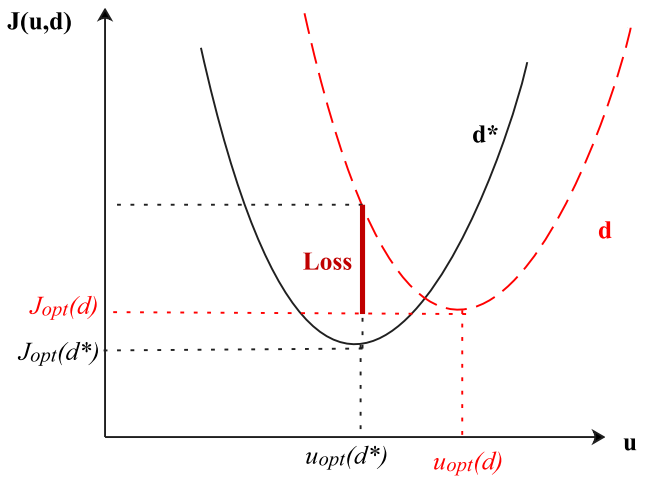

Fig. 2. The loss function as a function of disturbance and input [15].

Process cost/objective function has been defined as a quadratic steady-state function i.e. a second-order Taylor series expansion of the cost function around the nominal point $(\mathbf{u}^*, \mathbf{d}^*)$, where the objective of the SOC optimization problem is to regulate the input $\mathbf{u}$ so that the cost function J is minimized. It should be noted that there is no constraint in this optimization problem and all constraints have been satisfied. The process cost function can be defined as

$$J(\mathbf{u},\mathbf{d}) = J(\mathbf{u}^*,\mathbf{d}^*) + \begin{bmatrix}\mathbf{J}_u^* & \mathbf{J}_d^*\end{bmatrix}\begin{bmatrix}\Delta\mathbf{u}\\ \Delta\mathbf{d}\end{bmatrix} + \frac{1}{2}\begin{bmatrix}\Delta\mathbf{u}\\ \Delta\mathbf{d}\end{bmatrix}^T\begin{bmatrix}\mathbf{J}_{uu}^* & \mathbf{J}_{ud}^*\\ \mathbf{J}_{ud}^{*T} & \mathbf{J}_{dd}^*\end{bmatrix}\begin{bmatrix}\Delta\mathbf{u}\\ \Delta\mathbf{d}\end{bmatrix} \tag{3}$$

In Eq. (3), $\Delta\mathbf{u} = \mathbf{u} - \mathbf{u}^*$ and $\Delta\mathbf{d} = \mathbf{d} - \mathbf{d}^*$ mean deviations from the nominal optimal point $(\mathbf{u}^*, \mathbf{d}^*)$. $\mathbf{J}_u^*$ and $\mathbf{J}_d^*$ are first derivatives of J with respect to $\mathbf{u}$ and $\mathbf{d}$, $\mathbf{J}_{uu}^*, \mathbf{J}_{ud}^*$ and $\mathbf{J}_{dd}^*$ are second derivatives of J with respect to $\mathbf{u}$, $\mathbf{u}$ and $\mathbf{d}$, and $\mathbf{d}$, respectively at $(\mathbf{u}^*, \mathbf{d}^*)$. The nominal point is the current operating point of the process, but the nominal optimal point is when the current operating point of the process is the same as the optimal operating point of the process. For the sake of simplicity, it is assumed that the nominal point and optimal point are the same, and thus $\mathbf{J}_u^* = 0$. Furthermore, it is assumed that process variables have been shifted so that the process optimal point is zero $(\mathbf{u}^*, \mathbf{d}^*) = (0, 0)$ and $\mathbf{y}^* = 0$.

2.1.1. Measurement model

The model that is used for the SOC optimization problem is a linear steady-state model as

$$\mathbf{y} = \mathbf{G}^y\mathbf{u} + \mathbf{G}_d^y\mathbf{d} = \tilde{\mathbf{G}}^y\begin{bmatrix}\Delta\mathbf{u}\\ \Delta\mathbf{d}\end{bmatrix} \tag{4}$$

In Eq. (4), $\mathbf{G}^y$ and $\mathbf{G}_d^y$ are gain matrices of process in the steady-state. There are some further assumptions: (1) All active process constraints have been controlled and the vector $\mathbf{u}$ has been used to control the remaining unconstrained subspace. (2) It is assumed that the dimensions of $\mathbf{c}$ and $\mathbf{u}$ are the same. It means that the number of DOF is the same as the number of controlled variables $\mathbf{c}$ ($n_c = dim(\mathbf{c}) = dim(\mathbf{u}) = n_u$) which causes $\mathbf{HG}^y$ to be a square matrix. (3) It is required that the number of independent measurements $\mathbf{y}$ is greater than or equal to the number of DOF $\mathbf{u}$ ($n_y \geq n_u = n_c$). (4) In order to normalize the magnitude of vectors $\mathbf{d}$ and $\mathbf{n}^y$, they can be written as $\mathbf{d} = \mathbf{W}_d\mathbf{d}'$ and $\mathbf{n}^y = \mathbf{W}_{n^y}\mathbf{n}^{y'}$ where $\mathbf{W}_d$ and $\mathbf{W}_{n^y}$ are diagonal matrices with the same dimensions as disturbance and noise respectively and the vectors of $\mathbf{d}'$ and $\mathbf{n}^{y'}$ have unit magnitude.

2.1.2. Problem statement

It is obvious that when the disturbance $\mathbf{d}$ occurs if we have any input $\mathbf{u}$ other than $\mathbf{u}_{opt}(\mathbf{d})$, it will lead to a loss. As illustrated in Fig. 2, it can be seen that there is a loss when disturbance $\mathbf{d}$ happens and keeping $\mathbf{u}$ constant at $\mathbf{u}_{opt}(\mathbf{d}^*)$. By using a feedback control loop to maintain $\mathbf{c}_m$ (the measured CVs) at a constant set point $\mathbf{c}_s = 0$, the following equation is obtained:

$$\mathbf{c}_m = \mathbf{H}(\mathbf{y} + \mathbf{n}^y) = \mathbf{H}\mathbf{y}_m = \mathbf{c}_s = \mathbf{0} \tag{5}$$

Next step is dedicated to calculating the magnitude of loss for a given $\mathbf{H}$ and finding the optimal $\mathbf{H}$ that results in a minimum loss.

Two possibilities for loss can be considered. The first one is the worst-case loss (L_{wc}) when the 2-norm of the augmented vector of normalized disturbance and measurement noise is less than 1 as defined in Eq. (6) and the second one is average or expected loss as defined in Eq. (7). ($\mathbf{E}(.)$, is the expectation operator).

$$\left\|\begin{bmatrix}\mathbf{d}'\\ \mathbf{n}^{y'}\end{bmatrix}\right\|_2 \leq 1 \tag{6}$$

$$L_{avg} = \mathbf{E}(L), \begin{bmatrix}\mathbf{d}'\\ \mathbf{n}^{y'}\end{bmatrix} \in \aleph(0,\ 1) \tag{7}$$

2.1.3. Minimum loss problem solution

In this section, the exact local method or minimum loss problem solution has been derived [8]. For a disturbance $\mathbf{d}$, the optimal input $\mathbf{u}$ can be derived as follows:

$$\mathbf{J}_u(\mathbf{u},\mathbf{d}) = \mathbf{J}_u^*(\mathbf{u}^*,\mathbf{d}^*) + \mathbf{J}_{uu}^*\mathbf{u} + \mathbf{J}_{ud}^*\mathbf{d} \qquad where \qquad (\mathbf{u}^*,\mathbf{d}^*) = (0,0) \tag{8}$$

It is assumed that the input $\mathbf{u}$ changes so that it remains optimal; and consequently the optimal $\mathbf{u}$ can be calculated by the following equations:

$$\mathbf{u} = \mathbf{u}_{opt}(\mathbf{d}), \qquad \mathbf{J}_u(\mathbf{u},\mathbf{d}) = 0 \tag{9}$$

$$\mathbf{u}_{opt} = -\mathbf{J}_{uu}^{*-1}\mathbf{J}_{ud}^*\mathbf{d} \tag{10}$$

In the next step, the cost function $\mathbf{J}$ is approximated by the second order Taylor series expansion around the optimal point $\mathbf{u}_{opt}(\mathbf{d})$, which yields

$$\mathbf{J}(\mathbf{u},\mathbf{d}) = \mathbf{J}(\mathbf{u}_{opt}(\mathbf{d}),\mathbf{d}) + \mathbf{J}_{u,opt}(\mathbf{u} - \mathbf{u}_{opt}(\mathbf{d})) + \frac{1}{2}(\mathbf{u} - \mathbf{u}_{opt}(\mathbf{d}))^T\mathbf{J}_{uu,opt}(\mathbf{u} - \mathbf{u}_{opt}(\mathbf{d})) \tag{11}$$

The Hessian matrices of cost function are assumed to be constant because the cost function has been defined as a quadratic function, i.e. $\mathbf{J}_{uu} = \mathbf{J}_{uu}^*$ and $\mathbf{J}_{ud} = \mathbf{J}_{ud}^*$. In Eq. (11) we have $\mathbf{J}(\mathbf{u}_{opt}(\mathbf{d})) = \mathbf{J}_{opt}(\mathbf{d})$ and $\mathbf{J}_{u,opt} = 0$. Finally, the loss can be written as

$$L(\mathbf{u},\mathbf{d}) = J(\mathbf{u},\mathbf{d}) - J_{opt}(\mathbf{d}) = \frac{1}{2}\mathbf{z}^T\mathbf{z} = \frac{1}{2}\|\mathbf{z}\|_2^2 \tag{12}$$

where the variable $\mathbf{z}$ is defined as

$$\mathbf{z} \triangleq \mathbf{J}_{uu}^{1/2}(\mathbf{u} - \mathbf{u}_{opt}(\mathbf{d})) \tag{13}$$

From Eq. (10) the optimal input can be written as $\mathbf{u}_{opt} = \mathbf{F}_u\mathbf{d}$ where $\mathbf{F}_u = -\mathbf{J}_{uu}^{-1}\mathbf{J}_{ud}$. The optimal sensitivity of the vector $\mathbf{y}$ with respect to $\mathbf{d}$ is defined by $\mathbf{F}$, and can be written as follows:

$$\mathbf{F} = (-\mathbf{G}^y\mathbf{J}_{uu}^{-1}\mathbf{J}_{ud} + \mathbf{G}_d^y) \tag{14}$$

Eq. (14) may not provide a robust way for computing matrix $\mathbf{F}$, while finding it by its definition and $\mathbf{F} = (\frac{d\mathbf{y}_{opt}}{d\mathbf{d}})$ may be more practical. In the following equations, the loss function in Eq. (12)

is written as a function of disturbance $\mathbf{d}$ and measurement noise $\mathbf{n}^y$.

$$\begin{aligned}\mathbf{u}-\mathbf{u}_{opt} &= (\mathbf{HG}^y)^{-1}\mathbf{H}(\mathbf{Fd}+\mathbf{n}^y) = (\mathbf{HG}^y)^{-1}\mathbf{H}(\mathbf{FW}_d\mathbf{d}'+\mathbf{W}_{n^y}\mathbf{n}^{y'})\\ &= (\mathbf{HG}^y)^{-1}\mathbf{HY}\begin{bmatrix}\mathbf{d}'\\ \mathbf{n}^{y'}\end{bmatrix}\end{aligned} \tag{15}$$

According to Eq. (16), the new variable $\mathbf{Y}$ is defined as

$$\mathbf{Y} = \begin{bmatrix}\mathbf{FW}_d & \mathbf{W}_{n^y}\end{bmatrix} \tag{16}$$

The final equation for loss as a function of normalized disturbance and measurement noise can be written as [8]

$$\begin{aligned}\mathbf{L} &= \frac{1}{2}\mathbf{z}^T\mathbf{z} \quad where\\ \mathbf{z} &= \mathbf{J}_{uu}^{1/2}(\mathbf{u}-\mathbf{u}_{opt}) = \mathbf{J}_{uu}^{1/2}(\mathbf{HG}^y)^{-1}\mathbf{HY}\begin{bmatrix}\mathbf{FW}_d & \mathbf{W}_{n^y}\end{bmatrix}\\ &= \mathbf{M}(\mathbf{H})\begin{bmatrix}\mathbf{FW}_d & \mathbf{W}_{n^y}\end{bmatrix}\end{aligned} \tag{17}$$

The magnitude of loss for a pair of normalized disturbance and noise can be defined in two ways. The worst-case loss [8] and average loss [23] for a specified $\mathbf{H}$, can be calculated by using the following theorems.

Theorem 1 (*Worst-case Loss, Eq. 38 of [8]*)**.** *For a given pair of disturbances and noise, the worst-case loss is*

$$L_{wc}(\boldsymbol{H}) = \frac{1}{2}\bar{\sigma}(\boldsymbol{M})^2 \tag{18}$$

In Eq. (18), $\bar{\sigma}(.)$ denotes the maximum singular value, and $\boldsymbol{M}$ is defined as $\boldsymbol{M}(\boldsymbol{H}) = \boldsymbol{J}_{uu}^{1/2}(\boldsymbol{HG}^y)^{-1}\boldsymbol{HY}$.

Theorem 2 (*Average Loss, Proposition 1 of [23]*)**.** *For a given pair of disturbances and noise, the average loss is*

$$L_{avg}(\boldsymbol{H}) = \varepsilon(L) = \frac{1}{2}\|\boldsymbol{M}\|_F^2 \tag{19}$$

In Eq. (19), $\|.\|_F$ denotes the Frobenius norm and $\boldsymbol{M}$ is defined as $\boldsymbol{M}(\boldsymbol{H}) = \boldsymbol{J}_{uu}^{1/2}(\boldsymbol{HG}^y)^{-1}\boldsymbol{HY}$.

Proposition 1 (*Proposition 4 of [23]*)**.** *The matrix $\boldsymbol{H}$ that minimizes the average loss of Eq. (19) can also minimize the worst-case loss of Eq. (18) at the same time.*

According to the prior statement, the problem of SOC is summarized as finding the output measurement combination matrix $\mathbf{H}$ that can minimize the Frobenius norm of $\mathbf{M}$. The following theorem describes how to solve the SOC problem.

Theorem 3 (*Minimum Loss Method, Eq. 24 of [9]*)**.** *For a given pair of disturbances and noise, in order to minimize both the average and worst-case loss, $L_{avg}(\boldsymbol{H})$ and $L_{wc}(\boldsymbol{H})$, find the linear matrix $\boldsymbol{H}$ by solving the following optimization problem:*

$$\min_{\boldsymbol{H}} \left\|\boldsymbol{J}_{uu}^{1/2}(\boldsymbol{HG}^y)^{-1}\boldsymbol{HY}\right\|_F \quad where \quad \boldsymbol{Y} = \begin{bmatrix}\boldsymbol{FW}_d & \boldsymbol{W}_{n^y}\end{bmatrix} \tag{20}$$

As shown in Eq. (20), this optimization problem has a non-convex form. If there is no constraint on the structure of matrix $\mathbf{H}$, the non-convex problem stated in the following theorem can be converted to a convex one.

Theorem 4 (*Convex Reformulation For Full H Case, Eq. 28 of [9]*)**.** *The non-convex problem in Eq. (20) can be converted to a convex one if there are no structural constraints on $\boldsymbol{H}$ (which means that $\boldsymbol{H}$ is a "full" matrix), then it can be stated as a convex quadratic programming problem with linear constraint*

$$\min_{\boldsymbol{H}} \|\boldsymbol{HY}\|_F \quad s.t. \quad \boldsymbol{HG}^y = \boldsymbol{J}_{uu}^{1/2} \tag{21}$$

In this theorem by adding an equality constraint, the non-convex problem is changed to a convex problem. As was mentioned previously, adding equality constraint in Theorem 4 the non-convex problem of Theorem 3 can be reformulated as a convex one. However, adding this constraint can cause a problem in finding $\mathbf{H}$ in some structured $\mathbf{H}$ cases such as decentralized and triangular structures or when there is structured $\mathbf{H}$ with some specified zero elements in $\mathbf{H}$ or structured $\mathbf{H}$ with measurements selection i.e, specified zero elements and some zero columns in $\mathbf{H}$. In these cases, Theorem 4 and both the branch and bound and MIQP methods cannot be used unless the approximation method is applied [16] but it cannot guarantee the minimum loss. Another way to solve both the non-convex and convex SOC problems is using BMI or LMI techniques. In the next section, the SOC problem is solved by BMI and LMI methods. The main benefit of solving the SOC problem by using the BMI/LMI technique is that it reformulates the non-convex problem to the BMI/LMI form and does not add an equality constraint to the main problem, hence, it does not have the drawbacks of previous methods.

3. LMI approach to solve minimum loss problem

Many practical optimization problems in the field of system control can be solved with matrix inequality constraints [24–27]. The main benefit of changing a problem to the optimization problem with LMIs is that it can be solved with interior-point-based methods very efficiently and with any desired accuracy [28–31].

Generally, optimization of BMI problems is computationally expensive and non-deterministic polynomial-time hardness (NP-hard), although different methods have been developed for solving BMI problems [32–37].

BMI method has been applied to solve different control applications such as state-feedback and output-feedback controller design [38–44]. Due to the variety of non-convex problems, designing an efficient general algorithm has remained an open problem although some formulations have been derived for some types of non-convex problems [45]. In [46,47], it has been shown that BMI-constrained optimization problems are NP-hard.

As seen in the previous section, the minimum loss problem method in Theorem 3 is a non-convex problem. The purpose of this paper is two folds. First, the convex formulation of Theorem 4 is solved and then the non-convex problem of Theorem 3 is solved by using LMI/BMI techniques. First, an upper bound is found for the Frobenius norm of Theorem 3. Theorem 5 is our key theorem to solve the SOC problem. Indeed, it changes the problem of minimizing a Frobenius norm to a problem of minimizing a trace of a matrix (Minimizing a trace of a matrix is a general form of problem in BMI/LMI literature).

Theorem 5. *Trace($\boldsymbol{X}$) is an upper bound for the $\|A\|_F$ if matrix $\boldsymbol{X}$ satisfies the constraints of Eq. (22).*

$$\begin{bmatrix}\boldsymbol{I} & \boldsymbol{A}\\ \boldsymbol{A}^T & \boldsymbol{X}\end{bmatrix} \geq 0, \quad \boldsymbol{X} \geq 0, \quad \boldsymbol{X} = \boldsymbol{X}^T \tag{22}$$

Proof. According to the constraints of Eq. (22) and using Schur's lemma [25], it is concluded that if $\begin{bmatrix}\mathbf{I} & \mathbf{A}\\ \mathbf{A}^T & \mathbf{X}\end{bmatrix} \geq 0$ then $\mathbf{A}^T\mathbf{A} \leq \mathbf{X}$ which means $\mathbf{X} - \mathbf{A}^T\mathbf{A} \geq 0$. Since in the assumption of the theorem, matrix $\mathbf{X}$ is symmetric and positive definite, and also $\mathbf{A}^T\mathbf{A}$ is symmetric and positive definite, then $\mathbf{X} - \mathbf{A}^T\mathbf{A}$ is also a symmetric and positive definite matrix, which means that all its eigenvalues are positive. By the definition of matrix trace (sum of matrix eigenvalues), it means that the trace of this matrix is positive.

$Tr(\mathbf{X} - \mathbf{A}^T\mathbf{A}) \geq 0 \longrightarrow Tr(\mathbf{X}) - Tr(\mathbf{A}^T\mathbf{A}) \geq 0$ then

$Tr(\mathbf{A}^T\mathbf{A}) \leq Tr(\mathbf{X})$ (Trace is a linear operator). Since $Tr(\mathbf{A}^T\mathbf{A}) = \|\mathbf{A}\|_F^2$ we can write $\|\mathbf{A}\|_F^2 \leq Tr(\mathbf{X})$ which means that $Tr(\mathbf{X})$ can be an upper bound for the Frobenius norm of $\mathbf{A}$. □

As seen in Theorem 5, the problem of the Frobenius norm minimization can be converted to the LMI/BMI form (see Appendix) and can be solved easily and efficiently by LMI solvers. Therefore, Theorem 4 can be written as an LMI problem as the following.

Theorem 6. *The optimization problem of Theorem 4 can be stated as a Linear Matrix Inequality (LMI) problem as follows:*

$$\min_{H,X} trace(X) \quad s.t. \begin{bmatrix} \boldsymbol{I} & \boldsymbol{HY} \\ (\boldsymbol{HY})^T & \boldsymbol{X} \end{bmatrix} \geq 0, \\ \boldsymbol{X} \geq 0, \quad \boldsymbol{X} = \boldsymbol{X}^T, \quad \boldsymbol{HG}^y = \boldsymbol{J}_{uu}^{1/2} \tag{23}$$

Proof. By using Theorem 5 and applying it to Theorem 4 the optimization problem of Theorem 6 can be concluded directly. □

Since the constraints of the LMI problem in Theorem 6 are linear, this problem is an LMI and convex and has a global optimal solution. LMI problems can be solved with different solvers and one of their benefits is that any element in $\mathbf{H}$ matrix can be forced to zero for solving the problem in the case of structured $\mathbf{H}$. In the following, a reformulation of the general SOC optimization problem of Theorem 3 into the BMI/LMI form is presented. Before that, two helpful lemmas that are needed for the reformulation of the general SOC optimization problem are stated.

Lemma 1 (*Congruence Transformation, Section 4.2.2 of [48]*)**.** *If the matrix $Q \in \mathbb{R}^{n\times n}$ is a positive definite matrix, and matrix $W \in \mathbb{R}^{n\times n}$ is a real and full rank matrix (i.e. rank$(W) = n$), then the following inequality holds*

$$WQW^T \geq 0 \tag{24}$$

Specifically this Lemma says that pre- and post-multiplication of a positive definite matrix by a full rank real matrix and its transpose does not change positive/negative definiteness of primary matrix.

Lemma 2. *Consider two matrices F and P that satisfy convex–concave constraints (Eqs. 7 to 9 of [49]) of Eq. (25), where $(.)^*$ denotes the complex conjugate transpose.*

$$F^*F - P^*P < \gamma I \tag{25}$$

In addition, assume that matrices $F \in \mathbb{C}^{n\times n}$ and $P \in \mathbb{C}^{n\times n}$ are linear with respect to the optimization variables. A constraint in the form of Eq. (25) is called the convex–concave constraint and it can be convexified by using the Taylor series expansion of term P^*P around $P_c \in \mathbb{C}^{n\times n}$ which P_c is an arbitrary known matrix as:

$$P^*P \approx P_c^*P_c + (P - P_c)^*P_c + P_c^*(P - P_c) \tag{26}$$

It should be noted that the right hand side of Eq. (26) is always smaller than the left hand side since

$$P^*P \geq P^*P_c + P_c^*P - P_c^*P_c \tag{27}$$

$$(P - P_c)^*(P - P_c) \geq 0 \tag{28}$$

Theorem 7. *The optimization problem of Theorem 3 can be reformulated as a Bilinear Matrix Inequality (BMI) problem as follows:*

$$\min_{\boldsymbol{H},\boldsymbol{X}} trace(\boldsymbol{X}) \quad s.t. \begin{bmatrix} \boldsymbol{I} & \boldsymbol{J}_{uu}^{1/2}(\boldsymbol{HG})^{-1}\boldsymbol{HY} \\ * & \boldsymbol{X} \end{bmatrix} \geq 0, \quad \boldsymbol{X} \geq 0, \tag{29}$$

(*) in Eq. (29) denotes the complex conjugate transpose of symmetric matrix element.

Proof. By using Theorem 5 and applying it to Theorem 3 the optimization problem of Theorem 7 can be concluded directly. □

As seen in this theorem, the term $(\mathbf{HG})^{-1}\mathbf{HY}$ makes the optimization problem non-convex. Generally solving the non-convex and BMI problems are time consuming and solving them is not straightforward. If it is possible, it would be better to change them to the LMI problem and then solve it by standard solvers. Fortunately the non-convex optimization problem of Theorem 7 can be converted to a convex one by using Taylor series approximation as follows.

Theorem 8. *The optimization problem of Theorem 3 can be reformulated as a Linear Matrix Inequality (LMI) problem by using Taylor series expansion as follows:*

$$\min_{\boldsymbol{H},\boldsymbol{X}} trace(\boldsymbol{X}) \tag{30}$$

$$s.t. \begin{bmatrix} \boldsymbol{P}_c^T\boldsymbol{P}_c + (\boldsymbol{P} - \boldsymbol{P}_c)^T\boldsymbol{P}_c + \boldsymbol{P}_c^T(\boldsymbol{P} - \boldsymbol{P}_c) & \boldsymbol{HY} \\ * & \boldsymbol{X} \end{bmatrix} \geq 0, \quad \boldsymbol{X} \geq 0, \tag{31}$$

where

$$\boldsymbol{P} = \boldsymbol{J}_{uu}^{-\frac{T}{2}}(\boldsymbol{HG})^T \tag{32}$$

and $\mathbf{P}_c \in \mathbb{C}^{n\times n}$ is an arbitrary known matrix. (*) in Eq. (30) denotes the complex conjugate transpose of symmetric matrix element.

It should be noted that any structural constraint $\mathbf{H}$ can be directly imposed by forcing the desired elements of matrix $\mathbf{H}$ to zero.

Proof. Based on Theorem 5, the Eq. (31) which is the constraint of the optimization problem can be written as

$$\begin{bmatrix} \mathbf{I} & \mathbf{J}_{uu}^{1/2}(\mathbf{HG})^{-1}\mathbf{HY} \\ * & \mathbf{X} \end{bmatrix} \geq 0 \tag{33}$$

by applying congruence transformation (Lemma 1) and choosing $\mathbf{W}$ as the matrix stated in Eq. (34) and $\mathbf{Q}$ as a matrix in Eq. (33) we have

$$\mathbf{W} = diag((\mathbf{HG})\mathbf{J}_{uu}^{-1/2} \ , \ \mathbf{I}) \tag{34}$$

$$\mathbf{WQW}^T = \begin{bmatrix} (\mathbf{HG})\mathbf{J}_{uu}^{-1/2}\mathbf{J}_{uu}^{-T/2}(\mathbf{HG})^T & (\mathbf{HG})\mathbf{J}_{uu}^{-1/2}\mathbf{J}_{uu}^{1/2}(\mathbf{HG})^{-1}\mathbf{HY} \\ * & \mathbf{X} \end{bmatrix} \\ = \begin{bmatrix} (\mathbf{HG})\mathbf{J}_{uu}^{-1/2}\mathbf{J}_{uu}^{-T/2}(\mathbf{HG})^T & \mathbf{HY} \\ * & \mathbf{X} \end{bmatrix} \tag{35}$$

By defining matrix $\mathbf{P}$ as stated in Eq. (32) and applying the approximation of Eq. (26) and choosing matrix $\mathbf{P}_c$ as an arbitrary matrix, the constraint of Eq. (31) is derived. As can be seen, since only the matrix $\mathbf{P}$ is a variable, then the optimization constraint is linear and the non-convex optimization problem is changed to a convex one by using the Tylor series approximation and it can be solved by LMI solvers. □

In the proof of Theorem 8 (after applying congruence transformation), on the right-hand side of Eq. (35), the non-convex form is changed to a BMI form. Since matrix $\mathbf{P}_c$ is an arbitrary matrix, for solving the optimization problem and getting acceptable results, an iterative algorithm such as Algorithm 1 can be used to get better results.

It should be noted that the non-convex optimization problem of Theorem 7 can be solved by some BMI solvers such as PENBMI but as will be shown in the simulation results, it takes much more time to achieve the same accuracy.

Algorithm 1 Iterative algorithm of SOC problem optimization based on Theorem 8

INPUT: $\mathbf{G}^y$, $\mathbf{J}_{uu}$, $\mathbf{J}_{ud}$, $\mathbf{G}^d$, $\mathbf{W}_d$, $\mathbf{W}_{n^y}$.
Calculate matrices $\mathbf{F}$ (Eq. (14)) and $\mathbf{Y}$ (Eq. (16))
Initialize matrix $\mathbf{H}$ with random values
while the desired accuracy for loss is not achieved
set $\mathbf{P}_c$ as Eq. (32)
solve LMI problem of Theorem 8
Calculate loss and update $\mathbf{P}_c$
end while if desired accuracy is achieved
OUTPUT: Optimal matrix $\mathbf{H}$ and *Loss*

3.1. The case of structured $\boldsymbol{H}$

In the previous section, the problem of finding the optimal $\mathbf{H}$ in the case that there is no restriction on the element of matrix $\mathbf{H}$ was formulated. In this section, the case of forcing a specified structure on the matrix $\mathbf{H}$ is considered.

In real applications we may want to impose limitations on the structure of $\mathbf{H}$, that is, we may require some elements in $\mathbf{H}$ to be zero. Using a structured $\mathbf{H}$ is beneficial for several cases, for instance, when we do not want to associate measurement variables that are physically located far away from each other, in the case of escape coupling variables that have different behavior or dynamics or have long time-delay, or in the case of merging related measurements with physical meanings for operators. Sometimes, we want to separate different parts of the process from each other like something that happens in the evaporator case study where we want to separate the evaporator from the condenser unit and separator. For implementing these cases, it is required that the matrix $\mathbf{H}$ has a special structure and some elements, columns, or rows of $\mathbf{H}$ must be zero. In addition, it should be noted that as previously mentioned, the full $\mathbf{H}$ case is the worst-case for practical implementation in terms of implementation costs, for the reason that it requires more instruments to measure selected measurable variables, cabling, complicated control system hardware and software, and so on. Therefore, it is strongly desirable to have greater loss than minimum loss but reduce the cost of practical implementation.

In general, as stated in Theorem 3, SOC problem is a nonconvex optimization problem in the decision variable $\mathbf{H}$. In the standard case when $\mathbf{H}$ is a full matrix, it may be reformulated as a convex problem. This follows because the optimal $\mathbf{H}$ is not unique and we have enough extra degrees of freedom in $\mathbf{H}$ to impose a constraint on the matrix $\mathbf{HG}^y$ and we get an equivalent quadratic programming (QP) problem which is convex as stated in Theorem 4. In the structured $\mathbf{H}$ cases, because of all the zeros in $\mathbf{H}$, we do not have enough extra degrees of freedom in $\mathbf{H}$ to satisfy the constraint $\mathbf{HG}^y = \mathbf{J}_{uu}^{1/2}$. It should be noted that the key point for solving the optimization problem of Theorem 4 is that, the matrix $\mathbf{H}$ is not unique and all previous methods have used a transformation like $\mathbf{H}_1 = \mathbf{DH}$ where $\mathbf{D}$ is a nonsingular matrix. If we cannot find a matrix $\mathbf{D}$ for the specified structure, then the optimization problem fails. There are some cases that force us to use structured $\mathbf{H}$ matrix rather than full $\mathbf{H}$ case. In the following, some cases that we should specify a structure on $\mathbf{H}$ has been stated.

- **Given a subset of measurements.** In this case, for some reason, we will not use some measurements. For example, assume we have 2 inputs and 5 measurements of which we will not use measurements 1 and 3, then the columns 1 and 3 in the matrix $\mathbf{H}$ will be zero. This is a simple case and Theorem 4 is applicable. In this case, $\mathbf{J}_{uu}$ is not needed and simply by deleting corresponding rows in $\mathbf{G}^y$ and $\mathbf{Y}$, the full $\mathbf{H}$ case can be used to solve the SOC problem.
- **Optimal subset of measurements.** In this case, the objective is to select a certain number (n) of measurements which means that $n_y - n$ columns in $\mathbf{H}$ are zero. This case is similar to the previous case, in any variable selection we have a dense $\mathbf{H}$, and Theorem 4 can be used.
- **Best individual measurements for decentralized control.** This is the case that, where we want to select $n = n_c$ measurements. If we want offset free control of $\mathbf{c} = \mathbf{Hy}$, this case is the minimum feasible number of measurements and Theorem 4 may handle it.
- **Restriction on measurements from different process sections.** Consider a process with n_s sections and n_{yk} measurements in section k (k from 1 to s), and for example, we want to select r_k measurements from each section k. This case cannot be dealt with the BAB method, but MIQP can handle it.
- **Adding extra measurements to a given set of measurements.** Consider a process with $n_y = 5$ measurements, where we have decided to use the measurements 2, 3, and in addition, we want to add two other measurements (total 4 measurements). This case cannot be dealt with the BAB method, but MIQP can handle it.

Among different structures for matrix $\mathbf{H}$, triangular and block diagonal cases cannot be handled by BAB and MIQP methods, although there is an approximated MIQP method [16] that was used in our paper for comparison. The point is that, since the previous methods use the equality constraint of $\mathbf{HG}^y = \mathbf{J}_{uu}^{1/2}$, in some cases the suitable matrix $\mathbf{D}$ cannot be found and the optimization problem fails, but the LMI method does not need to satisfy this constraint and there is no significant problem to find the solution.

In the literature, choosing the best subset of measurement variables has been done in three ways: The first method is the tailor-made branch and bound (BAB) algorithms [12–14], and the second approach is to change the loss optimization problem to MIQP and using standard MIQP solvers [15], and the third one is to find the optimal measurement subset by defining a multi-objective optimization function and optimize it by trading-off between steady-state loss and maximizing the cardinality of the $\mathbf{H}$ matrix [50]. All of the previous methods use the convex formulation of Theorem 4 and impose restrictions on the structure of matrix $\mathbf{H}$. It should be noted that the best measurement subset may not be equal for both the worst-case and average loss. In the mentioned methods only the BAB approach can search the total space of subsets and find the best measurement subset to minimize both the worst-case and average loss, while the MIQP method can find the measurement variable subset that minimizes the average loss. However, practically it is not a defect and in most practical cases it is enough to minimize the average loss, because the worst-case may not occur very often. The multi-objective method cannot set a pre-structure for $\mathbf{H}$ and the optimization algorithm finds the best structure while in the MIQP method the structure of $\mathbf{H}$ can be defined. In the proposed method using the LMI formulation, any structure for $\mathbf{H}$ can be defined easily by forcing any desired element of $\mathbf{H}$ to zero and as is shown in the next section the iterative Algorithm 1 can reach the optimum loss very fast. As has been summarized in , for the full $\mathbf{H}$ case, all three methods can solve the SOC problem but for the other two cases PBAB method cannot be used and the MIQP method can only provide an approximate bound for the loss but LMI method can be used without any limitation.

4. Simulation results

In this section, the capability of the proposed theorems and algorithm are verified by applying them to three different processes. The optimization problems were implemented in the

Table 1
Comparison of different methods for structured **H**.

Method	Structured H cases		
	Full **H** with optimal measurement selection (there are some zero columns in **H**)	Block diagonal **H** or Triangular **H**	Structured **H** with measurement selection (with specific zero elements and some zero columns in **H**)
PBAB	√	×	×
MIQP	√	provides an approximate bound	provides an approximate bound
LMI	√	√	√

MATLAB environment [51] using the LMI Control Toolbox and PENBMI of the YALMIP toolbox [52]. The minimum loss optimization problem has been solved based on Theorem 6 for convex form, Theorem 7 for nonconvex form using PENBMI toolbox and finally Theorem 8 and Algorithm 1 for full and structured **H** cases. Simulation hardware and software are Windows 7 Ultimate with *Intel*® (R) Core(TM) Duo Processor *E*8400 (3.00 GHz) using *MATLAB*® R2016a.

4.1. Binary distillation column

The proposed theorems and algorithm are applied to the *column A* distillation column model [18,53]. The feed of the process is a binary mixture. the relative volatility of column feed is 1.5. This binary distillation column has 41 trays which counted from the bottom to the top. The reboiler is considered as stage 1 and the feed is entered on stage 21. This column is a suitable example of a process that has a large number of measurement variables. The reflux (L) of the condenser and boilup (V) of a reboiler are the unconstrained steady-state degrees of freedom (**u**). The column disturbances and their values are: (1) feed flow rate (F) (1 ± 0.2), (2) feed composition (z_F) (0.5 ± 0.1), and (3) fraction of liquid in the feed (q_F) (1 ± 0.1). As composition measurement is difficult and expensive, the composition has been controlled indirectly by controlling stage temperatures. The boiling point difference between the heavy key component (***H***) and the light key component (***L***) is 10 °C. It has been assumed that we have a constant molar flow rate, no vapor hold up, constant component relative volatility, constant pressure, and equilibrium on each stage. By considering the mentioned assumptions, only mass and component balances are applied in the binary column model. Furthermore, the temperatures of each stage are approximated as a linear function of each stage component mole fractions. On each stage of the column, the temperature on stage i (T_i) can be written as a linear function of the liquid composition (x_i) as follows [18]:

$$T_i = 0x_i + 10(1 - x_i) \tag{36}$$

The temperature of 41 distillation column stages are selected as the candidate measurement and they can be measured with an accuracy of ±0.5 °C. It should be noted that the inputs L and V have not been included in measurements. For indirect composition control, the cost function J has been defined as relative composition deviation in the steady-state as

$$J = \left(\frac{x_t^H - x_{t,s}^H}{x_{t,s}^H}\right)^2 + \left(\frac{x_b^L - x_{b,s}^L}{x_{b,s}^L}\right)^2 \tag{37}$$

where x_t^H denotes the top product heavy key component (***H***) composition, and x_b^L denotes the light key component (***L***) composition in bottom product ($x_t^H = x_t^L = 0.01$). The subscript '*s*' refers to the setpoint value [53] (see Fig. 3).

The results of solving the SOC optimization problem based on Theorems 6, 7, and Algorithm 1 have been presented in Table 2. From this Table, it can be seen that the loss value and optimal measurements are the same as PBAB and MIQP methods [12,15]. As depicted in Fig. 4, the proposed algorithm in only two iterations can effectively achieve the optimal loss. In this paper, our focus was on the reformulation of the SOC problem as an LMI form and solving some types of structured **H** cases such as triangular and block diagonal that were not supported by previous methods. The presented method cannot select the optimal combination of measurements for a given number of measurements in its current form. The measurement combination selections are driven from [15,16] and are presented here for the sake of comparison to show that the proposed method can find the minimum loss similar to or even better than the reported methods in the literature for structured **H** case.

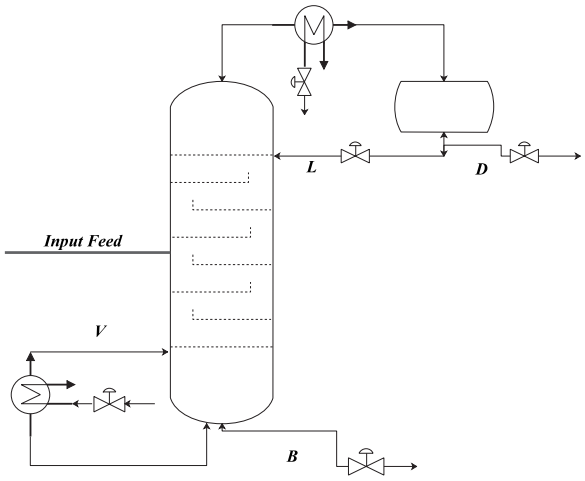

Fig. 3. A typical distillation column with LV configurations.

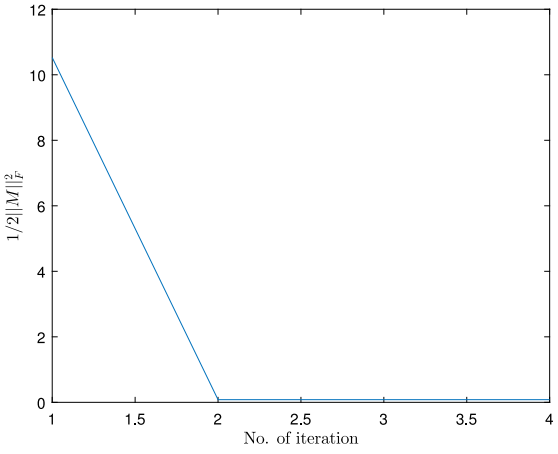

Fig. 4. The loss vs the no. of iterations of Alg. 1 for binary distillation column example.

Table 2
Binary distillation column : Optimal controlled variables, optimal measurements and loss.

No. of meas. (n)	CVs as a function of meas.	Average loss value (Eq. (19))
2	$\mathbf{c} = \begin{bmatrix} T_{12} \\ T_{30} \end{bmatrix}$	0.5478
3	$\mathbf{c} = \begin{bmatrix} 4.6632T_{12} + 0.1374T_{31} \\ -3.3491T_{30} - 3.3916T_{31} \end{bmatrix}$	0.4425
4	$\mathbf{c} = \begin{bmatrix} 1.3261T_{11} + 1.2855T_{12} + T_{31} \\ 0.2231T_{11} + 2.5048T_{30} + 2.5708T_{31} \end{bmatrix}$	0.3437
41	$\mathbf{c} = \begin{bmatrix} f(T_1, \ldots, T_{41}) \\ f(T_1, \ldots, T_{41}) \end{bmatrix}$	0.0813

Table 3
Distillation column case study: Structured H loss comparison for approximated MIQP and LMI methods.

No. Meas.		Structure				
		Full $\mathbf{H}^a$	Block diagonal $\mathbf{H}^a$		Triangular $\mathbf{H}^a$	
			Approximated MIQP [54]	LMI	Approximated MIQP [54]	LMI
2	CV	$c_1 = f(T_{12})$ $c_2 = f(T_{30})$	$c_1 = f(T_{12})$ $c_2 = f(T_{30})$	$c_1 = f(T_{12})$ $c_2 = f(T_{30})$	$c_1 = f(T_{12})$ $c_2 = f(T_{30})$	$c_1 = f(T_{12})$ $c_2 = f(T_{30})$
	Loss	0.548	0.548	0.548	0.548	0.548
3	CV	$c_1 = f(T_{12}, T_{30}, T_{31})$ $c_2 = f(T_{12}, T_{30}, T_{31})$	$c_1 = f(T_{30}, T_{31})$ $c_2 = f(T_{12})$	$c_1 = f(T_{30}, T_{31})$ $c_2 = f(T_{12})$	$c_1 = f(T_{30}, T_{31})$ $c_2 = f(T_{12}, T_{30}, T_{31})$	$c_1 = f(T_{30}, T_{31})$ $c_2 = f(T_{12}, T_{30}, T_{31})$
	Loss	0.443	0.443	0.443	0.464	0.443
4	CV	$c_1 = f(T_{11}, T_{12}, T_{30}, T_{31})$ $c_2 = f(T_{11}, T_{12}, T_{30}, T_{31})$	$c_1 = f(T_{30}, T_{31})$ $c_2 = f(T_{11}, T_{12})$	$c_1 = f(T_{30}, T_{31})$ $c_2 = f(T_{11}, T_{12})$	$c_1 = f(T_{30}, T_{31})$ $c_2 = f(T_{11}, T_{12}, T_{30}, T_{31})$	$c_1 = f(T_{30}, T_{31})$ $c_2 = f(T_{11}, T_{12}, T_{30}, T_{31})$
	Loss	0.344	0.344	0.344	0.353	0.344
41	CV	$c_1 = f(T_1, \ldots, T_{41})$ $c_2 = f(T_1, \ldots, T_{41})$	$c_1 = f(T_{21}, \ldots, T_{41})$ $c_2 = f(T_1, \ldots, T_{20})$	$c_1 = f(T_{21}, \ldots, T_{41})$ $c_2 = f(T_1, \ldots, T_{20})$	$c_1 = f(T_{21}, \ldots, T_{41})$ $c_2 = f(T_1, \ldots, T_{41})$	$c_1 = f(T_{21}, \ldots, T_{41})$ $c_2 = f(T_1, \ldots, T_{41})$
	Loss	0.081	0.105	0.104	0.094	0.088

[a]The coefficient of function f is different for full H, approximated MIQP and LMI methods.

Regarding dynamics, for block-diagonal structure in the distillation column case study, it is desirable to select one combined measurement from the top section of column (stages 21–41) and one from the bottom section (stages 1 to 20). The purpose of the triangular **H** structure is to obtain two CVs, where c_1 is selected from top stages temperatures and c_2 is selected from all stages temperatures. Triangular **H** is dynamically desirable to avoid the large time delays between c_1 and reflux (L) that can arise by including the bottom tray temperatures in c_1.

The performance of the LMI method versus approximated MIQP method [54] in the case of structured **H** has been illustrated in Table 3. As it can be seen in Table 3, it is reasonable to settle for a slightly higher loss in the structured **H** case rather than using a more practically complex full **H**.

4.2. Evaporator process

The evaporator process is a simple and realistic process. The model of the process is a modified version of the basic model [55] as described in [23]. Fig. 5 shows the schematic of this process. The evaporator process has 3 disturbances, 10 candidates for measurement, and 2 inputs (steady-state degrees of freedom) as follows:

$$\mathbf{u} = \begin{bmatrix} F_{200} & F_1 \end{bmatrix}^T \tag{38a}$$

$$\mathbf{y} = \begin{bmatrix} P_2 & T_2 & T_3 & F_2 & F_{100} & T_{201} & F_3 & F_5 & F_{200} & F_1 \end{bmatrix}^T \tag{38b}$$

$$\mathbf{d} = \begin{bmatrix} X_1 & T_1 & T_{200} \end{bmatrix}^T \tag{38c}$$

For this process, the negative profit is selected as the loss function [23] as shown in Eq. (39).

$$J = 600F_{100} + 0.6F_{200} + 1.009(F_2 + F_3) + 0.2F_1 - 4800F_2 \tag{39}$$

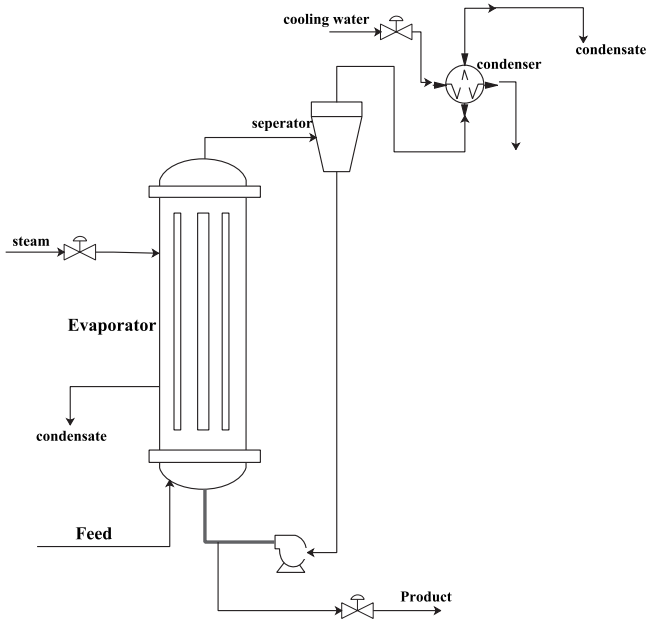

Fig. 5. Evaporator process [15].

Similar to the previous example, the results of solving the SOC optimization problem based on the proposed theorems and algorithm (6, 7, 1) have been shown in Table 4.

As can be seen in Table 4, the three theorems and algorithm (6, 7, 1) give the same results as previous methods [9,15] (see Fig. 6).

Table 4
Evaporator process : Optimal controlled variables, optimal measurements and loss.

No. of meas. (n)	CVs as a function of meas.	Average loss value (Eq. (19))
2	$\mathbf{c} = [F_3 \quad F_{200}]$	56.0260
3	$\mathbf{c} = [F_2 \quad F_{100} \quad F_{200}]$	11.7014
4	$\mathbf{c} = [F_2 \quad T_{201} \quad F_3 \quad F_{200}]$	9.4807
5	$\mathbf{c} = [F_2 \quad F_{100} \quad T_{201} \quad F_3 \quad F_{200}]$	8.0960
6	$\mathbf{c} = [F_2 \quad F_{100} \quad T_{201} \quad F_3 \quad F_5 \quad F_{200}]$	7.7127
7	$\mathbf{c} = [P_2 \quad F_2 \quad F_{100} \quad T_{201} \quad F_3 \quad F_5 \quad F_{200}]$	7.5971
8	$\mathbf{c} = [P_2 \quad T_2 \quad F_2 \quad F_{100} \quad T_{201} \quad F_3 \quad F_5 \quad F_{200}]$	7.5756
9	$\mathbf{c} = [P_2 \quad T_2 \quad F_2 \quad F_{100} \quad T_{201} \quad F_3 \quad F_5 \quad F_{200} \quad F_1]$	7.5617
10	$\mathbf{c} = [P_2 \quad T_2 \quad T_3 \quad F_2 \quad F_{100} \quad T_{201} \quad F_3 \quad F_5 \quad F_{200} \quad F_1]$	7.5499

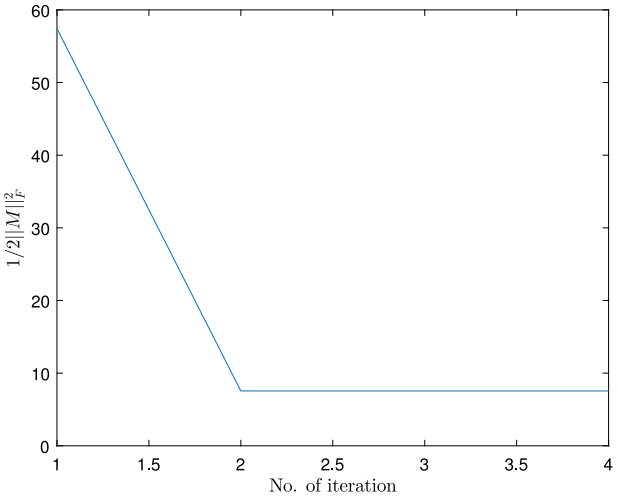

Fig. 6. The loss vs the no. of iterations of Alg. 1 for evaporator example.

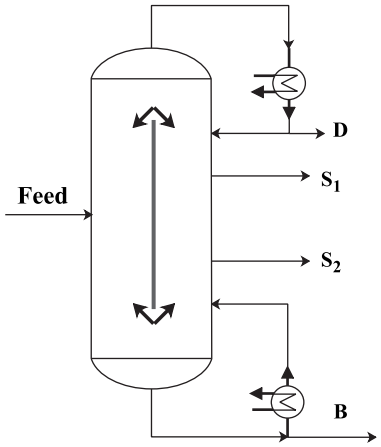

Fig. 7. Kaibel distillation column.

Table 5 compares the LMI and approximated MIQP [54] methods for different number of measurements in the case of disjoint measurement sets. For the evaporator case study two disjoint measurement subsets can be considered; one for the evaporator, i.e. $\{T_2, F_2, F_{100}, F_3, F_1\}$ and another for the condenser and separator unit i.e. $\{P_2, T_3, T_{201}, F_5, F_{200}\}$. Mainly, for good dynamic responses, it is desirable to separate the evaporator unit from the condenser and separator unit, so that $\mathbf{H}$ has a structure in the form of Eq. (40). As can be seen in Table 5, the LMI method can find a better structure and a lower loss. It should be noted that the block-diagonal structure of $\mathbf{H}$ in the case of 10 measurements is based on the following matrix $\mathbf{H}$:

$$\mathbf{H} = \begin{bmatrix} 0 & h_{12} & 0 & h_{14} & h_{15} & 0 & h_{17} & 0 & 0 & h_{110} \\ h_{21} & 0 & h_{23} & 0 & 0 & h_{26} & 0 & h_{28} & h_{29} & 0 \end{bmatrix} \tag{40}$$

4.3. Kaibel distillation column

A Kaibel distillation column process with vertical partitions [56,57] is a column that can separate a feed mixture of four products into its pure fractions. The feed of the column is a mixture of methanol, ethanol, propanol, and butanol labeled as A, B, C, and D respectively. This column is categorized as a thermally coupled distillation column and potentially has a high energy saving characteristic [57]. The objective function J can be defined to minimize the impurities in the column products as

$$J = D(1 - x_{A,D}) + S_1(1 - x_{B,S_1}) + S_2(1 - x_{D,B}) \tag{41}$$

where the variables D, S_1, S_2, and B are the column distillate, side product 1, side product 2 and bottom flow rate of column per mol/min, respectively. $x_{i,j}$ is the mole fraction of component i in product j. The inputs of the Kaibel column are L, S_1, S_2, R_L. This column has 7 sections and each section has 10 trays. By measuring the temperature of each tray plus the temperature of the reboiler with an accuracy of $\pm 0.1\,^{\circ}$C, we have 71 temperature measurements which are used as the candidate measurement vector ($\mathbf{y}$). The variables of vapor boil up (V), feed flow rate (F), vapor split (R_V), mole fraction of A in feed stream (z_A), mole fraction of B in feed stream (z_B), mole fraction of C in feed stream (z_C), liquid fraction of the feed stream (q_F) are considered as disturbances which vary between 3 ± 0.25, 1 ± 0.25, 0.4 ± 0.1, 0.25 ± 0.05, 0.25 ± 0.05, 0.25 ± 0.05, 0.9 ± 0.05, respectively [17] (see Fig. 7).

Similar to previous examples, results of solving the SOC optimization problem based on the proposed theorems and algorithm (6, 7, 1) have been shown in Table 6.

Simulation results for all examples show that the proposed algorithm (Alg. 1) improves the computation time for solving the optimization problem of Theorems 6 and 7. Results are shown in Table 7. The reason is that solving the LMI problem of Theorem 6 calls for solving an LMI problem with equality constraint which is a strict constraint and thus finding the solution is time-consuming. For solving the optimization problem with Theorem 7, the PENBMI package was used to solve the BMI problem of Theorem 7 and because of the nonlinearity of this problem, it is time-consuming, but Alg. 1 can solve the problem in only two iterations. As can be seen in Tables 2 and 4, the proposed algorithm can find optimal CVs with the same loss as BAB and MIQP methods. It should be noted that this algorithm cannot do

Table 5
Evaporator case study: Structured H loss comparison of approximated MIQP and LMI method.

No. Meas.		Structure		
		Full **H**[a]	Block diagonal **H**	
			Approximated MIQP[a] [54]	LMI[a]
2	CV	$c_1 = f(T_2)$ $c_2 = f(T_{201})$	$c_1 = f(T_2)$ $c_2 = f(T_{201})$	$c_1 = f(T_2)$ $c_2 = f(T_{201})$
	Loss	65.324	65.324	65.324
3	CV	$c_1 = f(T_2, F_5, F200)$ $c_2 = f(T_2, F_5, F200)$	$c_1 = f(T_2)$ $c_2 = f(F_5, F_{200})$	$c_1 = f(T_2)$ $c_2 = f(F_5, F_{200})$
	Loss	56.788	58.655	58.207
4	CV	$c_1 = f(F_2, F_{100}, F_5, F_{200})$ $c_2 = f(F_2, F_{100}, F_5, F_{200})$	$c_1 = f(F_2, F_{100})$ $c_2 = f(F_5, F_{200})$	$c_1 = f(F_2, F100)$ $c_2 = f(F_5, F_{200})$
	Loss	9.954	11.935	11.916
5	CV	$c_1 = f(P_2, F_5, F_{200}, F_2, F_3)$ $c_2 = f(P_2, F_5, F_{200}, F_2, F_3)$	$c_1 = f(P_2, F_5, F_{200})$ $c_2 = f(F_2, F_3)$	$c_1 = f(P_2, F_5, F_{200})$ $c_2 = f(F_2, F_3)$
	Loss	8.000	31.748	20.332
10	CV	$c_1 = f(y)$ $c_2 = f(y)$	$c_1 = f(y)$ $c_2 = f(y)$	$c_1 = f(y)$ $c_2 = f(y)$
	Loss	7.550	9.245	8.8847

[a]The coefficient of function f is different for full H, approximated MIQP and LMI methods.

Table 6
Kaibel column : Optimal controlled variables, optimal measurements and loss.

No. of meas. (n)	CVs as a function of meas.	Average loss value (Eq. (19))
4	$\mathbf{c} = \begin{bmatrix} T_{12} & T_{40} & T_{51} & T_{66} \end{bmatrix}$	11.6589
5	$\mathbf{c} = \begin{bmatrix} T_{12} & T_{51} & T_{62} & T_{65} & T_{66} \end{bmatrix}$	2.9700
6	$\mathbf{c} = \begin{bmatrix} T_{12} & T_{20} & T_{23} & T_{57} & T_{60} & T_{64} \end{bmatrix}$	1.0140
71	$\mathbf{c} = \begin{bmatrix} T_1 & T_2 & \dots & T_{71} \end{bmatrix}$	0.0101

Table 7
Evaporator case study: Computation time of LMI/BMI methods for full H case.

CPU time (s)		
Theorem 6 (Equality constraint)	Theorem 7 (BMI method)	Alg. 1
0.0022	0.0053	0.0010

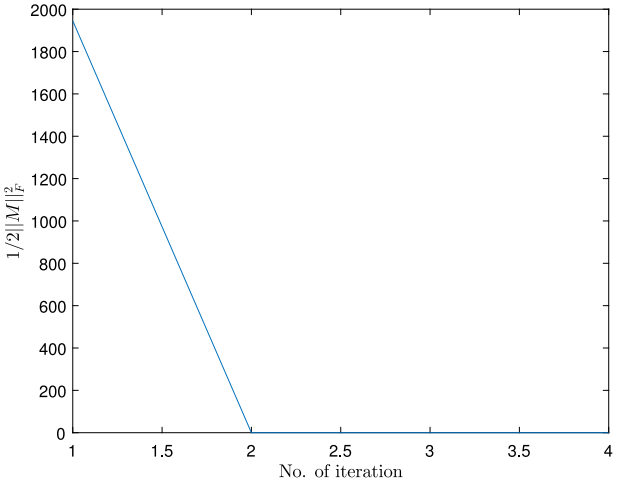

Fig. 8. The loss vs the no. of iterations of Alg. 1 for Kaibel column example.

the measurement selection but due to its capability to define any structure for matrix **H**, we select the same variable as in BAB and MIQP methods to evaluate the performance of the algorithm to calculate the loss. Tables 3 and 5 show the superiority of the LMI method to approximated MIQP method in finding the better loss in block diagonal and triangular cases where the BAB algorithm cannot be used (see Fig. Fig. 8).

5. Conclusion

In this work, LMI techniques have been considered to solve the main SOC optimization problem. The main objective of this paper is to develop a method to solve the non-convex problem of Theorem 3 by LMI techniques. Due to using Tylor series approximation in the proposed algorithm, a globally optimal solution cannot be guaranteed, but simulation results show that the method works and can converge to a local minimum very fast. The main purpose of this work is to change the non-convex problem of SOC to a convex one but not using global optimization solvers for the resulting BMI problem. Instead, the non-convex problem has been solved by PENBMI solver, which it is time consuming. It was shown that the LMI method has a capability to work with matrices, and can be used easily for structured **H** cases by only forcing the desired LMI variables to zero. In addition, it was a general solver for both convex and non-convex problems. According to Theorem 8, the method has the capability of solving optimization problems not only for full **H** case but also for any structured case.

CRediT authorship contribution statement

Mohammad Reza Jafari: Software, Investigation, Writing – original draft, Formal analysis. **Mohammad Mehdi Arefi:** Supervision, Methodology, Writing – original draft, Writing – review & editing. **Mehdi Panahi:** Conceptualization, Methodology, Writing – original draft.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix. LMI form

A linear matrix inequality (LMI) form is as follows [25]

$$F(x) \triangleq F_0 + \sum_{i=1}^{m} x_i F_i > 0 \tag{A.1}$$

where $x \in \mathbb{R}^m$ is the variable and symmetric matrices $F_i = F_i^T \in \mathbb{R}^{n\times n}$ are given matrices. The inequality in Eq. (A.1) denotes that $F(x)$ is a positive definite matrix.

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