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Brief paper

Integrity of systems under decentralized integral control ☆

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Abstract

A decentralized controller that maintains closed loop stability, when the individual controllers fail or are taken out of service, provides fault tolerance and flexibility in operation. Recently, necessary and sufficient conditions [Gündes, Kabuli, 2001. IEEE Transactions on Automatic Control, 46(2), 296–301] for the existence of a block decentralized controller with integral action for a system to possess integrity against controller failures were proposed. In this paper, these conditions are equivalently expressed using the well-known notions of Niederlinski index (NI) and block relative gain (BRG). The alternate representation implies that under minor assumptions, the available necessary conditions based on NI and BRG are actually both necessary and sufficient. We also show that confirming the existence of a block decentralized controller with integral action such that the system has integrity is NP-hard.
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1. Introduction

This paper deals with reliable stabilization of linear rational stable systems using a block decentralized controller with integral action. A system is said to possess integrity, if there exists a block diagonal controller with integral action in every output channel such that the closed loop stability is maintained when any combination of the individual controllers fails (Campo & Morari, 1994). It is assumed that a controller that fails is immediately taken out of service, i.e. the corresponding entries in the block diagonal controller matrix are replaced by zero. Some researchers have considered the problem of checking whether the closed loop system is reliably stable for a given controller; see Braatz, Morari, and Skogestad (1994) for a review. The focus of this work

is on deriving controller-independent conditions which can establish the existence or non-existence of a controller such that the system possesses integrity.

Because of its practical implications, the integrity problem has been studied widely by researchers, particularly in the area of process control. For fully decentralized control, a well-known result that relates reliable stability with relative gain array (RGA) (Bristol, 1966) is provided by Grosdidier, Morari, and Holt (1985). It is shown that a system has integrity only if all the corresponding relative gains of the steady state gain matrix are positive. Similar to fully decentralized control, a system with specified block pairings has integrity only if the determinant of all the corresponding block relative gains (BRG) (Manousiouthakis, Savage, & Arkun, 1986) of the steady state gain matrix are positive (Grosdidier & Morari (1987)). Grosdidier and Morari (1986) generalized the concept of Niederlinski index (NI) to block pairings to derive similar necessary conditions. Chiu and Arkun (1990) have further suggested that the necessary conditions based on BRG and NI be evaluated for all principal block sub-matrices of the system. These necessary conditions based on BRG and NI are useful for eliminating alternatives for input–output pairings. It is not apparent

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whether the system with the pairings chosen based on these necessary conditions, will have integrity.

Recently, Gündes and Kabuli (2001) presented necessary and sufficient conditions for integrity of the system partitioned into 4 or fewer blocks. In this note, we show that these conditions can be equivalently expressed using NI and, when the individual blocks are square, also using BRG. In general, satisfying the conditions of Gündes and Kabuli (2001) does not guarantee that the decentralized controller will have no unstable poles other than at the origin of the complex plane, as is assumed in the derivation of available necessary conditions based on NI and BRG in Grosdidier and Morari (1986, 1987). When the controller is allowed to have any number of unstable poles, the alternative representation implies that the conditions based on BRG and NI, traditionally believed to be only necessary, are actually both necessary and sufficient. The expressions presented by Gündes and Kabuli (2001) become increasingly complex with the number of blocks. Then, an additional advantage of the alternative representation in terms of NI and BRG is that the extension to the general case, where the system is partitioned into any number of blocks, is simple.

For fully decentralized control, we also show that the necessary and sufficient conditions due to Gündes and Kabuli (2001) are satisfied if and only if (iff) NI calculated based on steady state gain matrix is a $\mathscr{P}$-matrix (Rump, 2003). This observation suggests that verifying the existence of a block decentralized controller with integral action such that the system has integrity is NP-hard unless P = NP (Garey & Johnson, 1979).

2. Necessary and sufficient conditions

In this note, we denote the linear rational stable system as $\mathbf{G}(s)$ and the gain matrix as $\mathbf{G}$. We consider that $\mathbf{G}(s)$ is partitioned into M non-overlapping subsystems such that, $\mathbf{G}_{ii} \in \mathbb{R}^{m_i \times m_j}$; $m_i \leqslant m_j$, $i = 1, 2, \ldots, M$. The matrix containing diagonal blocks of $\mathbf{G}$ is represented as $\tilde{\mathbf{G}}$. The block diagonal controller $\mathbf{K}(s)$ with integral action is expressed as $\mathbf{K}(s) = (1/s) \cdot \mathbf{C}(s)$, where $\mathbf{C}(s) = \operatorname{diag}(\mathbf{C}_{ii}(s))$ and $\mathbf{C}_{ii}(s)$ is a $m_j \times m_i$ dimensional transfer matrix (see Fig. 1). Here, we allow $\mathbf{C}(s)$ to be improper, provided $\mathbf{K}(s)$ is proper. When $\operatorname{rank}(\mathbf{G}_{ii}) = m_i$, the right inverse of $\mathbf{G}_{ii}$ is denoted as $\mathbf{G}_{ii}^{\dagger}$. Note that the existence of $\mathbf{G}_{ii}^{\dagger}$ is necessary for the ith loop to have integral action. We call a square real matrix $\mathbf{A}$ a positive definite matrix (denoted as $\mathbf{A} \succ 0$) if all the eigenvalues of its symmetric part $(\mathbf{A} + \mathbf{A}^{\mathrm{T}})$ are positive (Horn & Johnson, 1985).

Next, we define integrity formally and present the necessary and sufficient conditions of Gündes and Kabuli (2001) for integrity of $\mathbf{G}(s)$.

Definition 1. The system $\mathbf{G}(s)$ is said to have integrity, if there exists a block diagonal controller $\hat{\mathbf{K}}(s) = \mathbf{E}\mathbf{K}(s)$ with

integral action, which stabilizes $\mathbf{G}(s)$ for all $\mathbf{E} \in \mathscr{E}$, where (Campo & Morari, 1994)

$$\mathscr{E} = \{\mathbf{E} = \operatorname{diag}(\varepsilon_i \cdot \mathbf{I}_{m_i}) | \varepsilon_i = \{0, 1\}, i = 1, \ldots, M\}. \tag{1}$$

In the literature, a system possessing integrity has also been referred as *reliably stable with integral action*; see e.g. Gündes and Kabuli (2001). The existence of a block diagonal controller such that $\mathbf{G}(s)$ has *integrity* depends on the chosen input–output pairings. In the remaining discussion, we assume that $\mathbf{G}(s)$ has been permuted such that the subsystems corresponding to the chosen pairings lie along the diagonal blocks of $\mathbf{G}(s)$.

To present the conditions of Gündes and Kabuli (2001), we need the following additional notation. For $j = 2, \ldots, M$, $i = 1, \ldots, j - 1$, define

$$\mathbf{X}_{ij} = \mathbf{G}_{jj} - \mathbf{G}_{ji}\mathbf{G}_{ii}^{\dagger}\mathbf{G}_{ij}. \tag{2}$$

When $M \geqslant 3$, for $k = 1, \ldots, M - 2$ and $\ell, m = k + 1, \ldots, M$, $\ell \neq m$,

$$\mathbf{Y}_{\ell m}^{k} = \mathbf{G}_{\ell m} - \mathbf{G}_{\ell k}\mathbf{G}_{kk}^{\dagger}\mathbf{G}_{km} \tag{3}$$

and for $v = 3, \ldots, M$, $q = 1, \ldots, v - 2$ and $r = q + 1, \ldots, v - 1$,

$$\mathbf{Z}_{rq}^{v} = \mathbf{X}_{qv} - \mathbf{Y}_{vr}^{q}\mathbf{G}_{rr}^{\dagger}(\mathbf{X}_{qr}\mathbf{G}_{rr}^{\dagger})^{-1}\mathbf{Y}_{rv}^{q}. \tag{4}$$

When $M = 4$, define

$$\mathbf{W} = \mathbf{Z}_{24}^{1} - (\mathbf{Y}_{43}^{1} - \mathbf{Y}_{42}^{1}\mathbf{G}_{22}^{\dagger}(\mathbf{X}_{12}\mathbf{G}_{22}^{\dagger})^{-1}\mathbf{Y}_{23}^{1})\mathbf{G}_{33}^{\dagger}$$
$$\times (\mathbf{Z}_{23}^{1}\mathbf{G}_{33}^{\dagger})^{-1}(\mathbf{Y}_{34}^{1} - \mathbf{Y}_{32}^{1}\mathbf{G}_{22}^{\dagger}(\mathbf{X}_{12}\mathbf{G}_{22}^{\dagger})^{-1}\mathbf{Y}_{24}^{1}). \tag{5}$$

Theorem 2 (*Gündes & Kabuli, 2001*). *Let* $\operatorname{rank}(\mathbf{G}_{ii}) = m_i$ *for all* $i = 1, \ldots, M$. *There exists a block diagonal controller with integral action such that* $\mathbf{G}(s)$ *has integrity, if*

$$\det(\mathbf{X}_{ij}\mathbf{G}_{jj}^{\dagger}) > 0, \tag{6}$$

$$\det(\mathbf{Z}_{rq}^{v}\mathbf{G}_{vv}^{\dagger}) > 0, \tag{7}$$

$$\det(\mathbf{W}\mathbf{G}_{44}^{\dagger}) > 0, \tag{8}$$

where $j = 2, \ldots, M$, $i = 1, \ldots, j - 1$ *and* $v = 3, \ldots, M$, $q = 1, \ldots, v - 2$, $r = q + 1, \ldots, v - 1$. *Further, if any* $M - 1$ *controllers are strictly proper, or when* $\mathbf{G}_{ij}$ *or* $\mathbf{G}_{ji}$, $j = 2, \ldots, M$, $i = 1, \ldots, j - 1$ *are strictly proper or when any of these transfer matrices have real blocking zeros,* (6)–(8) *are also necessary.*

The proof of Theorem 2 can be found in Gündes and Kabuli (2001). Some remarks that are relevant to the rest of the discussion in this note are in order.

- Even though the off-diagonal blocks of $\mathbf{G}(s)$ may not be strictly proper or may not have real blocking zeros, the controllers can always be designed to be strictly proper. When all controllers are strictly proper, (6)–(8)
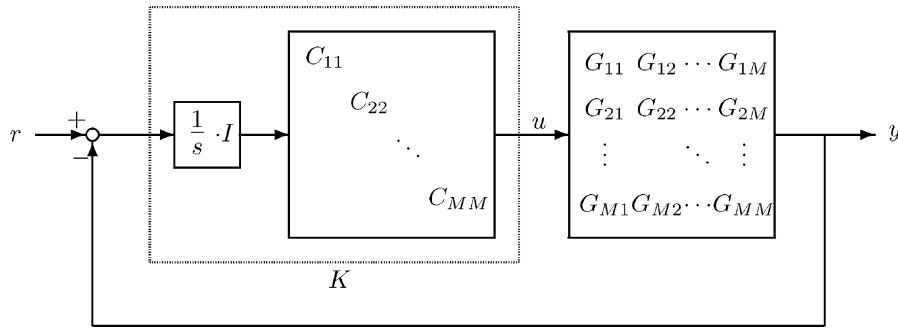
Fig. 1. Closed loop system with integral action controller.

are both necessary and sufficient for existence of a block diagonal controller with integral action such that $\mathbf{G}(s)$ has integrity. We note that a similar assumption is made during the derivation of the necessary conditions based on BRG in Grosdidier and Morari (1987).

- When (6)–(8) are satisfied, existence of a controller with integral action is guaranteed such that the system has integrity. This controller, however, may have additional unstable poles other than at the origin of the complex plane. The existence of pure integral action controllers is implied by the more restrictive conditions: $\mathbf{X}_{ij}\mathbf{G}_{jj}^{\dagger} \succ 0$, $\mathbf{Z}_{rq}^{v}\mathbf{G}_{vv}^{\dagger} \succ 0$ and $\mathbf{WG}_{44}^{\dagger} \succ 0$.

- When the individual blocks are multi-input single-output (MISO), $\mathbf{X}_{ij}\mathbf{G}_{jj}^{\dagger} \succ 0$, $\mathbf{Z}_{rq}^{v}\mathbf{G}_{vv}^{\dagger} \succ 0$ and $\mathbf{WG}_{44}^{\dagger} \succ 0$ are equivalent to (6)–(8). In this case, when (6)–(8) hold, existence of a pure integral action controller is guaranteed such that $\mathbf{G}(s)$ has integrity.

Gündes and Kabuli (2001) also presented a method for controller design such that $\mathbf{G}(s)$ has integrity, when $\mathbf{X}_{ij}\mathbf{G}_{jj}^{\dagger}$, $\mathbf{Z}_{rq}^{v}\mathbf{G}_{vv}^{\dagger}$ and $\mathbf{WG}_{44}^{\dagger}$ are positive definite.

3. Simplified representation

In this section, we show that the conditions in Theorem 2 can be equivalently represented in terms of BRG and NI. For this purpose, we require evaluation of BRG and NI on the principal block sub-matrices of $\mathbf{G}$. We define ψ as the ordered subset of the first M positive integers, consisting of at least 2 elements, and Ψ as the ensemble of all such sets ψ. For example, when $M = 2$, $\Psi = \{(1, 2)\}$ and when $M = 3$, $\Psi = \{(1, 2), (1, 3), (2, 3), (1, 2, 3)\}$. With this representation, $\mathbf{G}_{\psi\psi}$ represents a principal submatrix of $\mathbf{G}$ made up of blocks of $\mathbf{G}$ indexed by ψ, for any $\psi \in \Psi$. Similarly, $[\mathbf{GH}]_{\psi\psi}$ represents a principal submatrix of the product of the matrices $\mathbf{G}$ and $\mathbf{H}$ indexed by ψ. Note that when $\psi = (1, 2, \ldots, M)$, $\mathbf{G}_{\psi\psi}$ represents the matrix $\mathbf{G}$.

Definition 3. Let $\tilde{\mathbf{G}} = \mathrm{diag}(\mathbf{G}_{ii})$, where $\mathbf{G}_{ii} \in \mathbb{R}^{m_i \times m_j}$, $m_i \leqslant m_j$ and $\mathrm{rank}(\mathbf{G}_{ii}) = m_i$ for all $i = 1, \ldots, M$. The Niedrelinski Index (NI) of $\mathbf{G}$ is defined as

$$\mathrm{NI}(\mathbf{G}) = \det(\mathbf{G}\tilde{\mathbf{G}}^{\dagger}). \tag{9}$$

Definition 3 is a generalization of NI defined by Grosdidier and Morari (1986) for systems partitioned into square blocks. The next proposition relates NI with the existence of a controller such that the system has integrity.

Proposition 4. Let $\mathrm{rank}(\mathbf{G}_{ii}) = m_i$ for all $i = 1, \ldots, M$. Then the conclusions of Theorem (2) hold iff

$$\mathrm{NI}(\mathbf{G}_{\psi\psi}) > 0 \quad \forall \psi \in \Psi. \tag{10}$$

Proof. By repeated use of Schur complement lemma (see e.g. (Horn & Johnson, 1985)), it can be shown that

$$\det(\mathbf{X}_{ij}\mathbf{G}_{jj}^{\dagger}) = \det([\mathbf{G}\tilde{\mathbf{G}}^{\dagger}]_{\{i,j\},\{i,j\}}), \tag{11}$$

$$\det(\mathbf{Z}_{rq}^{v}\mathbf{G}_{vv}^{\dagger}) = \frac{\det([\mathbf{G}\tilde{\mathbf{G}}^{\dagger}]_{\{q,r,v\},\{q,r,v\}})}{\det(\mathbf{X}_{qr}\mathbf{G}_{rr}^{\dagger})}, \tag{12}$$

$$\det(\mathbf{WG}_{44}^{\dagger}) = \frac{\det(\mathbf{G}\tilde{\mathbf{G}}^{\dagger})}{\det(\mathbf{Z}_{23}^{1}\mathbf{G}_{33}^{\dagger}) \det(\mathbf{X}_{12}\mathbf{G}_{22}^{\dagger})}, \tag{13}$$

where $j = 2, \ldots, M, i = 1, \ldots, j - 1$ and $v = 3, \ldots, M$, $q = 1, \ldots, v - 2, r = q + 1, \ldots, v - 1$.

Using (11), $\det(\mathbf{X}_{ij}\mathbf{G}_{jj}^{\dagger}) > 0$ iff

$$\det([\mathbf{G}\tilde{\mathbf{G}}^{\dagger}]_{\{i,j\},\{i,j\}}) > 0 \tag{14}$$

for all $j = 2, \ldots, M, i = 1, \ldots, j - 1$. Next, assume that (14) holds. When $M \geqslant 3$, the ordered set $\{r, q\}$ is a subset of $\{i, j\}$. Then, $\det(\mathbf{X}_{qr}\mathbf{G}_{rr}^{\dagger}) > 0$ and using (12), $\det(\mathbf{Z}_{rq}^{v}\mathbf{G}_{vv}^{\dagger}) > 0$, iff

$$\det([\mathbf{G}\tilde{\mathbf{G}}^{\dagger}]_{\{q,r,v\},\{q,r,v\}}) > 0 \tag{15}$$

for all $v = 3, \ldots, M, q = 1, \ldots, v - 2, r = q + 1, \ldots, v - 1$. Similarly, when $M = 4$, $\det(\mathbf{WG}_{44}^{\dagger}) > 0$, iff

$$\det(\mathbf{G}\tilde{\mathbf{G}}^{\dagger}) > 0. \tag{16}$$

Now, the necessity and sufficiency of (10) follows by combining (14)–(16) and noting that $\Psi = \{i, j\} \cup \{q, r, v\}$. $\quad\square$

As M increases, the expressions presented by Gündes and Kabuli (2001) become increasingly complex (cf. (6)–(8)). On the other hand, the extension to the general case is simple (by induction), when the conditions are compactly expressed in terms of NI.

When all the blocks of the partitioned system are square, the conditions of Theorem 2 can also be equivalently expressed in terms of BRG, as shown below.

Definition 5. For $\mathbf{G}(s)$ partitioned into square blocks, let $\mathbf{G}_{ii} \in \mathbb{R}^{m_i \times m_i}$ be non-singular for all $i = 1, \ldots, M$ and let $j = \{1, \ldots, M\}/i$. The BRG of $\mathbf{G}_{ii}$ in $\mathbf{G}$ is defined as (Manousiouthakis et al., 1986)

$$[\mathbf{\Lambda}_B(\mathbf{G})]_{ii} = \mathbf{G}_{ii}\left(\mathbf{G}_{ii} - \mathbf{G}_{ij}\mathbf{G}_{jj}^{-1}\mathbf{G}_{ji}\right)^{-1}. \qquad (17)$$

Proposition 6. *Let* $\mathbf{G}(s)$ *be partitioned into square blocks such that* $\mathbf{G}_{ii} \in \mathbb{R}^{m_i \times m_i}$ *is non-singular for all* $i = 1, \ldots, M$. *Then the conclusions of Theorem 2 or* (10) *hold iff,*

$$\det([\mathbf{\Lambda}_B(\mathbf{G}_{\psi\psi})]_{kk}) > 0 \quad \forall \psi \in \Psi, k = 1, \ldots, |\psi|, \qquad (18)$$

where $|\psi|$ *denotes the cardinality of the set* ψ.

Proof. For equivalence, we show that (18)⇔(10), which in turn implies that the conclusions of Theorem 2 hold, iff (18) holds. Since $\mathrm{NI}(\mathbf{G}_{\{i,j\},\{i,j\}}) = \det([\mathbf{\Lambda}_B(\mathbf{G}_{\{i,j\},\{i,j\}})]_{ii})$, $\mathrm{NI}(\mathbf{G}_{\{i,j\},\{i,j\}}) > 0$, iff

$$\det([\mathbf{\Lambda}_B(\mathbf{G}_{\{i,j\},\{i,j\}})]_{ii}) > 0 \qquad (19)$$

for all $i, j \leqslant M, i \neq j$. Next, assume that (19) holds. When, $M \geqslant 3$ (Chiu & Arkun, 1990)

$$\mathrm{NI}(\mathbf{G}_{\{i,j,k\},\{i,j,k\}}) = \frac{\mathrm{NI}(\mathbf{G}_{\{i,j\},\{i,j\}})}{\det([\mathbf{\Lambda}_B(\mathbf{G}_{\{i,j,k\},\{i,j,k\}})]_{kk})}$$

for all $i, j, k \leqslant M, \ i \neq j \neq k$. Since $\mathrm{NI}(\mathbf{G}_{\{i,j\},\{i,j\}}) > 0$ for all $i, j \leqslant M, i \neq j$, $\mathrm{NI}(\mathbf{G}_{\{i,j,k\},\{i,j,k\}}) > 0$, iff

$$\det([\mathbf{\Lambda}_B(\mathbf{G}_{\{i,j,k\},\{i,j,k\}})]_{ii}) > 0 \qquad (20)$$

for all $i, j, k \leqslant M, i \neq j \neq k$. When, $M = 4$, using (13) and similar arguments as above, $\mathrm{NI}(\mathbf{G}) > 0$ iff

$$\det([\mathbf{\Lambda}_B(\mathbf{G})]_{\ell\ell}) > 0 \qquad (21)$$

for all $i, j, k \leqslant M, i \neq j \neq k, \ell = \{1, \ldots, M\}/\{i, j, k\}$. Now, the necessity and sufficiency of (18) follows by combining (19)–(21). $\square$

To check whether (10) or (18) hold, NI or BRG need to be calculated for all principal sub-matrices of $\mathbf{G}$, which can be formed by combining elements of the diagonal blocks and the corresponding off-diagonal blocks. A similar method was earlier proposed by Chiu and Arkun (1990), where (10) and (18) were shown to be necessary. Chiu and Arkun (1990) assumed that $\mathbf{G}(s)\mathbf{C}(s)$ is strictly proper and $\mathbf{C}(s)$ is stable. If $\mathbf{C}(s)$ is allowed to be unstable, Propositions 4 and 6 imply

that the necessary conditions based on NI and BRG, traditionally believed to be only necessary, are both necessary and sufficient.

For a system partitioned into MISO blocks, satisfying (6)–(8) is equivalent to satisfying $\mathbf{X}_{ij}\mathbf{G}_{jj}^{\dagger} \succ 0$, $\mathbf{Z}_{rq}^{v}\mathbf{G}_{vv}^{\dagger} \succ 0$ and $\mathbf{W}\mathbf{G}_{44}^{\dagger} \succ 0$. In this case, when (10) or (18) hold, the existence of a stable $\mathbf{C}(s)$ is guaranteed such that $\mathbf{G}(s)$ has integrity. In general, there may not exist a stable $\mathbf{C}(s)$ such that $\mathbf{G}(s)$ has integrity, even when (10) or (18) hold. It is worth pointing out that the requirement of $\mathbf{C}(s)$ being unstable is restrictive, as noted by Campo and Morari (1994), but is practically relevant. Derivation of necessary and sufficient conditions for $\mathbf{G}(s)$ to possess integrity such that $\mathbf{C}(s)$ is stable, remains an open problem.

To verify whether (10) holds, NI needs to be evaluated exactly $2^M - (M + 1)$ times, whereas verification of (18) requires that BRG be evaluated many more times. This ambiguity is explained by noting that evaluation of BRG for all principal block sub-matrices of $\mathbf{G}$ is not necessary. For example, when $M = 3$

$$\det([\mathbf{\Lambda}_B(\mathbf{G}_{\{i,j\},\{i,j\}})]_{ii})$$
$$= \frac{\det([\mathbf{\Lambda}_B(\mathbf{G})]_{jj})\det([\mathbf{\Lambda}_B(\mathbf{G}_{\{i,k\},\{i,k\}})]_{ii})}{\det([\mathbf{\Lambda}_B(\mathbf{G})]_{kk})}. \qquad (22)$$

If all the terms on the right-hand side of (22) are positive, $\det([\mathbf{\Lambda}_B(\mathbf{G}_{\{i,j\},\{i,j\}})]_{ii})$ is always positive. The task of finding the set of $2^M - (M + 1)$ non-redundant BRGs requires some book-keeping. In this sense, the use of (10) is advantageous over the use of (18). The usefulness of the results presented in this section is demonstrated next by a numerical example.

Example 7. Consider the following system adapted from Hovd and Skogestad (1992):

$$\mathbf{G}(s) = \frac{(1-s)}{(1+5s)^2}\begin{bmatrix} 1 & -4.19 & -25.96 \\ 6.19 & 1 & -25.96 \\ 1 & 1 & 1 \end{bmatrix}.$$

The objective is to ascertain the integrity of system with pairings selected on the diagonal elements. The $\mathrm{NI}(\mathbf{G})$ and $\mathrm{NI}(\mathbf{G}_{\psi\psi})$ are 26.9 for $\psi = (1, 2), (1, 3)$ and $(2, 3)$. Then, Proposition 4 guarantees that the system has integrity. This result is also confirmed using Proposition 6, where $[\mathbf{\Lambda}_B(\mathbf{G})]_{ii} = 1$ for $i = 1, 2, 3$ and $[\mathbf{\Lambda}_B(\mathbf{G}_{\psi\psi})]_{jj} = 0.037$ for $\psi = (1, 2), (1, 3)$ and $(2, 3)$ for all $j = 1, 2$.

For fully decentralized control, satisfying (10) or (18) guarantees the existence of a pure integral action controller such that $\mathbf{G}(s)$ has integrity. We design a controller of the form $\mathrm{diag}(k_i/s)$ using the algorithm of Gündes and Kabuli (2001), where $k_i = 0.01, 0.002$ and 10^{-5} for $i = 1, 2, 3$, respectively. This controller maintains closed loop stability, when any combination of loops fail. Alternatively, we find using trial and error that the same objective is achieved by the controller $k \cdot \mathbf{I}, k = 0.001$.

4. Computational complexity

In this section, we present some results on computational complexity for establishing the existence of a block diagonal controller such that $\mathbf{G}(s)$ has integrity. It is shown that this problem is NP-hard, unless P = NP (Garey & Johnson, 1979). We introduce the useful notion of $\mathscr{P}$-matrices, which form the basis of the proof for NP-hardness.

Definition 8. A matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is called a $\mathscr{P}$-matrix, if all the principal minors of $\mathbf{A}$ are positive (Rump, 2003).

In the subsequent discussion, we refer to the problem of establishing the existence of a block diagonal controller such that $\mathbf{G}(s)$ has integrity, simply as the *integrity problem*. Note that the integrity problem involves search over all possible partitions of $\mathbf{G}(s)$ and the controller structure is not specified a priori.

Proposition 9. *Let* $\mathrm{rank}(\mathbf{G}_{ii}) = m_i$ *for all* $i = 1, \ldots, M$ *and* $\tilde{\mathbf{G}} = \mathrm{diag}(\mathbf{G}_{ii})$. *If the controller* $\mathbf{K}(s)$ *is restricted to be strictly proper, the integrity problem is* NP-*hard, unless* P = NP.

Proof. For the NP-hardness of the integrity problem, it suffices to show that the integrity problem is NP-hard, when the individual blocks of controller are single-input multi-output (SIMO). Let the class of $n \times n$ real matrices be classified as

- Matrices with at least one negative or zero diagonal elements, $\mathscr{A}_1$;
- Matrices with all positive diagonal elements, $\mathscr{A}_2$.

It readily follows that for any $\mathbf{A} \in \mathscr{A}_1$, the $\mathscr{P}$-matrix problem can be solved in polynomial time through n evaluations. Coxson (1994) has shown that verifying whether a given matrix is $\mathscr{P}$-matrix is co-NP-complete. Then, the $\mathscr{P}$-matrix problem must be co-NP-complete for the set $\mathscr{A}_2$, otherwise the results of Coxson (1994) are contradicted. For any $\mathbf{A}$, since

$$\det(\mathbf{A}\tilde{\mathbf{A}}^{-1}) = \det(\mathbf{A})\det(\tilde{\mathbf{A}}^{-1}) \tag{23}$$

any $\mathbf{A} \in \mathscr{A}_2$ is a $\mathscr{P}$-matrix, iff $\mathbf{A}\tilde{\mathbf{A}}^{-1}$ is a $\mathscr{P}$-matrix.

By reversing the proof of Proposition 4, it follows that $\mathbf{A}\tilde{\mathbf{A}}^{-1}$ is a $\mathscr{P}$-matrix, iff for every transfer matrix $\mathbf{G}(s)$ satisfying $\mathbf{A}\tilde{\mathbf{A}}^{-1} = \mathbf{G}\tilde{\mathbf{G}}^{\dagger}$, there exists a decentralized controller having SIMO blocks and integral action such that $\mathbf{G}(s)$ has integrity. Clearly, the transformations $\mathbf{A}\tilde{\mathbf{A}}^{-1}$ and $\mathbf{G}\tilde{\mathbf{G}}^{\dagger}$ require finite number of operations and can be completed in polynomial time. Thus, the integrity problem is at least as hard as the $\mathscr{P}$-matrix problem and is NP-hard, unless P = NP. $\square$

Based on Proposition 9, it is possible to establish the computational complexity of some more general or special cases of integrity problem, as discussed below:

(1) When the system is partitioned into MISO blocks, satisfying (6)–(8) guarantees the existence of a pure integral action controller such that $\mathbf{G}(s)$ has integrity. In this case, when the controllers are further restricted to have poles at origin only, the integrity problem is also NP-hard.

(2) When the controller is block decentralized, one only needs to check the positiveness of the minors of the sub-matrices of $\mathbf{G}\tilde{\mathbf{G}}^{\dagger}$ that can be formed by combining elements of different blocks and the corresponding off-block diagonal elements. In this case, if $\det([\mathbf{G}\tilde{\mathbf{G}}^{\dagger}]_{\psi\psi}) > 0$ for all $\psi \in \Psi$, we call $\mathbf{G}\mathbf{G}_{bd}^{\dagger}$ a block $\mathscr{P}$-matrix in the spirit of $\mathscr{P}$-matrices. The worst-case time complexity of an algorithm for the block $\mathscr{P}$-matrix problem is approximately $\mathcal{O}(n^3 2^M)$. Then, for the special case, where the controller structure is specified a priori with M being independent of the system dimensions, the integrity problem lies in class P.

Though the integrity problem is NP-hard it may still be possible to solve the integrity problem in polynomial time for particular instances of the problem. The time complexity of an algorithm evaluating all the principal minors of the given real matrix is approximately $\mathcal{O}(n^3 2^n)$. Tsatsomeros and Li (2000) presented a recursive algorithm that reduces the time complexity to $\mathcal{O}(2^n)$. This algorithm is based on Schur complement lemma and is easily extended for verifying block $\mathscr{P}$-matrices.

Recently, Rump (2003) presented an algorithm, whose time complexity is not necessarily exponential, but can be exponential in the worst case. Rump (2003) has applied this algorithm to a test set of parameterized matrices, whose membership in the class of $\mathscr{P}$-matrices is known beforehand for the given value of the parameter. It is shown that the algorithm can successfully verify whether these matrices having dimensions up to 100×100 are $\mathscr{P}$-matrices in polynomial time. Future work will focus on generalizing Rump's algorithm (Rump, 2003) for verification of block $\mathscr{P}$-matrices. We next present a sufficient condition for verifying whether $\mathbf{G}\tilde{\mathbf{G}}^{\dagger}$ is a $\mathscr{P}$- or block $\mathscr{P}$-matrix.

Proposition 10. *Let* $\mathrm{rank}(\mathbf{G}_{ii}) = m_i$ *for all* $i = 1, \ldots, M$ *and* $\tilde{\mathbf{G}} = \mathrm{diag}(\mathbf{G}_{ii})$. *Define* $\mathbf{E} = (\mathbf{G} - \tilde{\mathbf{G}})\tilde{\mathbf{G}}^{\dagger}$. *Then,* $\mathbf{G}\tilde{\mathbf{G}}^{\dagger}$ *is block* $\mathscr{P}$-*matrix with respect to the structure of* $\tilde{\mathbf{G}}$, *if* $\det(\mathbf{I} + 0.5\mathbf{E}) \neq 0$ *and*

$$\mu_{\Delta}((\mathbf{I} + 0.5\mathbf{E})^{-1}\mathbf{E}) < 2, \tag{24}$$

where μ *is the structured singular value* (Doyle, Wall, & Stein, 1982) *and*

$$\Delta = \{\mathrm{diag}(\delta_i \cdot \mathbf{I}_{m_i}), \delta_i \in \mathbb{C}, |\delta_i| \leqslant 1, i = 1, \ldots, M\}. \tag{25}$$

Proof. Define, $\mathbf{\Delta}_1 = \{\mathrm{diag}(\varepsilon_i \cdot \mathbf{I}_{m_i}), \varepsilon_i = \{0, 1\}, i = 1, \ldots, M\}$. Then, $\mathbf{G}\tilde{\mathbf{G}}^\dagger = \mathbf{I} + \mathbf{E}$, is a block $\mathscr{P}$-matrix iff

$$\det(\mathbf{I} + \mathbf{E}\tilde{\mathbf{\Delta}}_1) > 0 \quad \forall \tilde{\mathbf{\Delta}}_1 \in \mathbf{\Delta}_1. \tag{26}$$

Further, defining $\mathbf{\Delta}_2 = \{\mathrm{diag}(\varepsilon_i \cdot \mathbf{I}_{m_i}), \varepsilon_i \in \mathbb{C}, |\varepsilon_i| \leqslant 1, i = 1, \ldots, M\}$ and noting that $\mathbf{\Delta}_1 \subset \mathbf{\Delta}_2$, (26) holds if $\det(\mathbf{I} + \mathbf{E}\tilde{\mathbf{\Delta}}_2) > 0$ for all $\tilde{\mathbf{\Delta}}_2 \in \mathbf{\Delta}_2$. Since the determinant is a continuous function over convex sets, if $\det(\mathbf{I} + \mathbf{E}\tilde{\mathbf{\Delta}}_2)$ changes sign over the set $\mathbf{\Delta}_2$, there exists some $\tilde{\mathbf{\Delta}}_2 \in \mathbf{\Delta}_2$ such that $\det(\mathbf{I} + \mathbf{E}\tilde{\mathbf{\Delta}}_2) = 0$. Since, $\mathbf{\Delta}_1 \subset \mathbf{\Delta}_2$, (26) holds if

$$\det(\mathbf{I} + \mathbf{E}\tilde{\mathbf{\Delta}}_2) \neq 0 \quad \forall \tilde{\mathbf{\Delta}}_2 \in \mathbf{\Delta}_2 \; \Leftrightarrow \; \mu_{\mathbf{\Delta}_2}(\mathbf{E}) < 1. \tag{27}$$

The condition (27) is conservative as $\mathbf{I}, -\mathbf{I} \in \mathbf{\Delta}_2$. To reduce conservatism (Braatz et al., 1994), for every $\tilde{\mathbf{\Delta}} \in \mathbf{\Delta}$, $\tilde{\mathbf{\Delta}}_2 \in \mathbf{\Delta}_2$, define $\tilde{\mathbf{\Delta}}_2 = 0.5(\mathbf{I} + \tilde{\mathbf{\Delta}})$. Then

$$\begin{aligned} &\det(\mathbf{I} + \mathbf{E}\tilde{\mathbf{\Delta}}_2) \\ &\quad = \det(\mathbf{I} + 0.5\mathbf{E} + 0.5\mathbf{E}\tilde{\mathbf{\Delta}}) \\ &\quad = \det(\mathbf{I} + 0.5\mathbf{E})\det(\mathbf{I} + 0.5(\mathbf{I} + 0.5\mathbf{E})^{-1}\mathbf{E}\tilde{\mathbf{\Delta}}). \end{aligned}$$

When (24) holds, $\det(\mathbf{I} + 0.5(\mathbf{I} + 0.5\mathbf{E})^{-1}\mathbf{E}\tilde{\mathbf{\Delta}})$ does not change sign over the set $\mathbf{\Delta}$ and $\mathbf{G}\tilde{\mathbf{G}}^\dagger$ is block $\mathscr{P}$-matrix with respect to the structure of $\tilde{\mathbf{G}}$. $\square$

The principal sub-matrices of a positive-definite matrix are also positive-definite (Horn & Johnson, 1985). Thus, when $\mathbf{G}\tilde{\mathbf{G}}^\dagger \succ 0$, $\mathbf{G}\tilde{\mathbf{G}}^\dagger$ is a $\mathscr{P}$- and thus a block $\mathscr{P}$-matrix. Proposition 10 is less conservative than this sufficient condition, as the controller structure is taken into account; however, Proposition 10 is still conservative, as $\tilde{\mathbf{\Delta}}_1$ is a strict subset of $\tilde{\mathbf{\Delta}}$. Further, it is well known that exact calculation of the structured singular value is difficult. There are some upper bounds available which can be calculated tractably using convex optimization and can handle many practical problems of interest; see, e.g. Doyle et al. (1982) and Fu and Barabanov (1997). A practical approach is to check if the upper bound on $\mu_{\mathbf{\Delta}}((\mathbf{I} + 0.5\mathbf{E})^{-1}\mathbf{E}) < 2$ and if not, use the algorithms of Tsatsomeros and Li (2000) for block decentralized control or Rump (2003) for fully decentralized control.

Example 11. To show the advantage of Proposition 10 over the sufficient condition $\mathbf{G}\tilde{\mathbf{G}}^\dagger \succ 0$, we consider $\mathbf{G}(s)$ with

$$\mathbf{G} = \begin{bmatrix} 1 & -\dfrac{\gamma}{\alpha} \\ \alpha & 1 \end{bmatrix}; \quad \alpha, \gamma > 0.$$

For fully decentralized control with pairing on diagonal elements, $\mathrm{NI}(\mathbf{G}) = 1 + \gamma$ and the system has integrity for all allowable values of α and γ. For 2×2 systems, we note that the upper bound on the structured singular value obtained using the D-scaling method is exact (Doyle et al., 1982). Using this method and some lengthy but straightforward

algebraic manipulations, it can be shown that

$$\mu_{\mathbf{\Delta}}((\mathbf{I} + 0.5\mathbf{E})^{-1}\mathbf{E}) = 2\sqrt{\frac{\gamma}{4 + \gamma}}$$

which satisfies (24) for all allowable finite values of α and γ. This example demonstrates that Proposition 10 is not always conservative. On the other hand, the eigenvalues of $(\mathbf{G}\tilde{\mathbf{G}}^\dagger + (\mathbf{G}\tilde{\mathbf{G}}^\dagger)^{\mathrm{T}})$ are $2 \pm (\alpha - \frac{\gamma}{\alpha})$ and the sufficient condition $\mathbf{G}\tilde{\mathbf{G}}^\dagger \succ 0$ is satisfied only when $|(\alpha - \frac{\gamma}{\alpha})| < 2$. For example, when $\gamma = 1$, $\mathbf{G}\tilde{\mathbf{G}}^\dagger \succ 0$ only for $0.416 \leqslant \alpha \leqslant 2.416$. Clearly, this is highly conservative, as the integrity of the system is independent of α.

5. Conclusions

When the controller is allowed to have unstable poles other than at origin, it is shown that the conditions for the integrity problem based on NI and BRG, generally believed to be necessary, are both necessary and sufficient. It is also shown that solving the integrity problem, i.e. establishing the existence of a block diagonal controller with integral action such that the system has integrity, is NP-hard. This result implies that no computationally easy algorithm exists for solving the integrity problem and the engineer needs to be content with conditions that are easily computable but are either necessary or sufficient, but not both.

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