CONTRIBUTIONS TO THE THEORY OF OPTIMAL CONTROL

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1. Introduction

The purpose of this paper is to give an account of recent research on a classical problem in the theory of control: the design of linear control systems so as to minimize the integral of a quadratic function evaluated along motions of the system. This problem dates back in its modern form to Wiener and Hall at about 1943 ([1], [2]). In spite of its relatively long history, the problem has never been formulated rigorously from a mathematical point of view. Even the most up-to-date expositions of the subject (see, e.g., [3]) are inaccessible to the mathematician due to the lack of precisely stated conditions and results.

The problem is quite broad, and there are many unsettled questions. This paper will be concerned with only the simplest case, the so-called *regulator problem*. For other aspects of the problem, the reader may consult [4]–[8] which, while devoted primarily to questions of theoretical engineering, contain precise mathematical results.

The conventional theory of the regulator problem is based largely on Fourier and Laplace transforms. By contrast, the approach of this paper is direct and uses the well-known theory of ordinary differential equations. We have also drawn on Lyapunov's theory of stability. While our earlier treatments ([5], [6], [8]) followed the point of view of dynamic programing, here we utilize classical tools of the calculus of variations, in particular the Hamilton-Jacobi equation.

The principal contribution of the paper lies in the introduction and exploitation of the concepts of *controllability* and *observability* ([8]), with the aid of which we give, for the first time, a complete theory of the regulator problem. In particular, we prove existence and stability theorems for the regulator problem and study in some detail stability properties of the matrix Riccati equation, which arises as a special case of the Hamilton-Jacobi equation.

A careful discussion of the conceptual aspects of the control problems has been included as an aid to persons not familiar with the field of control. Some mathematical arguments, in particular the review of the calculus of variations, are more leisurely than usual in order to render the paper reasonably self-contained.

2. Notation and terminology

We use standard vector-matrix notation, with the following conventions: small Greek letters are scalars; small Latin letters are vectors, capitals are matrices. The unit matrix is I. Exceptions: i, j, m, n, p are integers; t, E, L, V are scalars; ϕ, ψ, ξ are vectors. The inner product is $[x, y]$. The transpose of a matrix is denoted by the prime. The norm is $\| x \| = [x, x]^{\frac{1}{2}}$. The norm $\| A \|$ of a matrix A is sup $\| Ax \|$ over $\| x \| = 1$. Special conventions: $\| x \|_P^2 = [x, Px]$ where P is

symmetric, nonnegative definite. If A, B are symmetric, $A > B[A \geq B]$ means $A - B$ is positive [nonnegative] definite. The letters t, σ, τ are arbitrary real numbers which always denote the *time*; we use $t_0 \leq t_1 \leq t_2$ to denote fixed, ordered values of t. All scalars, vectors, and matrices are real throughout. For a scalar function $L(u)$ of the vector u, L_u is the gradient vector and L_{uu} is the jacobian matrix.

We shall study the system represented by the equations

$$dx/dt = F(t)x + G(t)u(t) \tag{2.1}$$

$$y(t) = H(t)x(t) \tag{2.2}$$

where: u is an m-vector, x is an n-vector, y is a p-vector; $F(t)$, $G(t)$, as well as $H(t)$ are rectangular matrices continuous in t, either of which may be singular.

In view of the physical motivation of our problem, we adopt the following terminology: Equations (2.1–2) are the *plant* (or *model*); x is the *state* of; $u(t)$ is the *control function* or *input* to; and $y(t)$ is the *output* of the plant. The plant is *constant* if F, G, H are constants. If $u(t) = 0$ or $G(t) = 0$, the plant is *free.*

The behavior of the plant is described by the solution of the differential equation (2.1) which will exist for all t and be unique if, say, $u(t)$ is Lebesgue integrable. As is well known ([9]), the general solution has the form[1]

$$x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi(t, \tau)G(\tau)u(\tau)\, d\tau \tag{2.3}$$

where $\Phi(t, \tau)$, defined for all t, τ, is a fundamental matrix ([9]) of solutions of the free system (2.1), satisfying the additional requirement that

$$\Phi(t, t) = I \quad \text{for all } t \tag{2.4}$$

In view of (2.4), we call Φ the *transition matrix* of (2.1)—a terminology borrowed from the theory of Markov processes ([7], [8], [10]).

The solution (2.3) is conveniently regarded as the *motion* of the state of (2.1); this leads to the notation

$$x(t) = \phi_u(t; x, t_0) \tag{2.5}$$

Read: the motion of (2.1) starting at initial state x at time t_0 and observed at time t, and influenced by the *fixed* control function $u(t)$ defined in the interval $[t_0, t]$. Since (2.5) holds for all t, t_0, we have in particular the identity: $x = \phi_u(t; x, t)$ for all t, x, u. Free motions are denoted by ϕ_f. We observe also that (2.1) has an *equilibrium state* x^* at 0, in other words, a state for which $x^* = \phi_f(t; x^*, t_0)$ for all t, t_0.

3. Statement of problem

In the simplest applications, the object of a control system is the following: *Given any state x of the plant* (2.1) *at any time t_0, "generate" a control function*

[1] The function (2.3) will satisfy the differential equation (2.1) *almost everywhere.*

$u(t)$, defined for $t \geq t_0$ and depending on x, t_0, which causes x to be "transferred" to the equilibrium state 0. In other words, $u(t)$ is chosen so as to assure

$$\lim_{t\to\infty} \phi_u(t; x, t_0) = 0 \tag{3.1}$$

For technological reasons, the function $u(t)$ must be generated from actual measurements of the behavior of the plant. To describe how the control system is to be *physically realized*, one must therefore provide an algorithm for computing the number $u(t_1)$ from the knowledge of $y(t)$ for $t \leq t_1$. This is usually referred to as the *Feedback Principle.*

One may separate the problem of physical realization into two stages:

(A) Computation of the "best approximation" $\hat{x}(t_1)$ of the state $x(t_1)$ from knowledge of $y(t)$ for $t \leq t_1$.

(B) Computation of $u(t_1)$ given $\hat{x}(t_1)$.

In the engineering literature one often makes the simplifying assumption of treating the two problems separately, i.e., simply regarding $\hat{x}(t)$ as though it were $x(t)$. We are concerned here only with Problem (B) and therefore always assume that $x(t)$ is known exactly. Somewhat surprisingly, the theory of Problem (A), which includes as a special case Wiener's theory of the filtering and prediction of time series, turns out to be analogous to the theory of Problem (B) developed in this paper. This assertion follows from the *duality theorem* discovered by the author ([7], [8]); this theorem can be used to show also that the separation of Problems (A) and (B) is indeed legitimate.

Assuming $x(t)$ is known exactly and taking into account the Feedback Principle, the problem of generating $u(t)$ reduces to specifying the *control law*

$$u(t) = k(x(t), t) \tag{3.2}$$

From the definition of state it is clear that nothing would be gained by letting $u(t)$ depend also on values of the state prior to time t. To assure that (2.1) with (3.2) has a unique solution, it suffices to have $k \in C^1$. If (3.2) does not depend explicitly on t, we say the control law is *constant.*

To arrive at the control law "rationally," we now add the further desideratum that *the integral of a nonnegative function of the state along any motion ϕ_u should be minimized by the choice of $u(t)$.* Stating this requirement with some care, we shall see that it uniquely determines $u(t)$ and hence also the control law (3.2), and even implies (3.1). We call the resulting control system *optimal.*

Let us now state precisely the

(3.3) Optimal Regulator Problem. *Find a control law* (3.2) *for which*

$$V^0(x, t_0, t_1) = \inf_{u(t)} \left\{ \nu(\phi_u(t_1; x, t_0)) + \int_{t_0}^{t_1} L(\phi_u(t; x, t_0)\ u(t), t)\, dt \right\} \tag{3.4}$$

is attained for all x, t_0, and t_1, where ν, L are nonnegative scalar functions of class C^2 in all arguments.

The class of functions $u(t)$ which are admitted to competition in taking the infimum in (3.4) are to be of class D^o (i.e., continuous except at isolated points at which $u(t)$ has finite left- and right-hand limits).

It is well known in the calculus of variations ([11], p. 196) that the condition $L_{uu} \geq 0$ or $L_{uu} \leq 0$ is necessary for the existence of even local extremals. To avoid complications due to the equality sign, we assume from the outset that

$$L_{uu}(x, u, t) > 0 \quad \text{for all } x, u, t \tag{3.5}$$

which is equivalent to assuming that L is strictly convex in u.

In engineering language, one calls t_1 the *terminal time* (it may be infinity!), the integral is the *performance index*, and (at least when it does not depend on u) L is the *error criterion* or more generally the *loss function*. The function ν is added for greater generality. We use the notation $V(x, t_0, t_1; u)$ for the value of the integral in (3.4) for some specified, fixed $u(t)$. The superscript o identifies "optimal".

4. Relations with the calculus of variations

In this section we transcribe some well-known results ([11], Ch. 12) of the local problem of the calculus of variations into a form best suited to our problem. Let us first solve (3.3) in a very special case.

(4.1) LEMMA (Carathéodory). *Let $k(x, t)$ be an m-vector function of class C^1 in x, t. Write $u^o = k(x, t)$. Assume $\nu = 0$ and that the function L in* (3.4) *satisfies the following conditions for all x and all $t_0 \leq t \leq t_1$:*

(a) $L(x, u^o, t) = 0$

(b) $L(x, u, t) > 0$ *for all* $u \neq u^o$

Then the optimal performance index V^o is identically zero for all x and is attained by using the optimal control law given by

$$u^o = k(x(t), t) \tag{4.2}$$

Proof. Let ϕ_{u^o} denote the motion of the plant under control law (4.2); similarly, let ϕ_{u^1} be the motion corresponding to some fixed control function $u^1(t)$. By (b) above and since the integrand of (3.4) is in class D^o in t, it follows that $V(x, t_0, t_1; u^1) > 0$ unless $u^1(t) = u^o(t)$ at every continuity point of $u^1(t)$ in the interval (t_0, t_1). On the other hand, by (a) we see that $V^o(x, t_0, t_1)$ vanishes identically in x.[2] Q. E. D.

Anticipating the final result (4.14), let $V^o(x, t, t_1)$ be an arbitrary scalar function of class C^2 in x, t, (t_1 being a fixed number) and subject also to

$$V^o(x, t_1, t_1) = \nu(x) \tag{4.3}$$

Let us replace L by

[2] It is clear that two optimal controls $u^o(t)$ and $u^1(t)$ can differ only on a set of measure zero.

$$L^*(x, u, t) = L(x, u, t) + V_t^o(x, t, t_1) + [V_x^o(x, t, t_1), F(t)x + G(t)u]$$

The integral of the last two terms along any motion between the limits t_0, t_1 is

$$v(\phi_u(t_1 ; x, t_0)) - V^o(x, t_0, t_1) \tag{4.4}$$

Since the second term in (4.4) does not depend on $u(t)$, it follows that the two variational problems,

$$\inf_{u(t)} \int_{t_0}^{t_1} L^*(\phi_u(t; x, t_0), u(t), t)\, dt \tag{4.5}$$

and (3.3) are equivalent in that they have the same minimizing function $u(t)$ (if such exists).

Now we try to find functions $V^o(x, t, t_1)$ and $k(x, t, t_1)$ for which the hypotheses of the Lemma are satisfied when L is replaced by L^*.

In order that $L^*(x, u, t)$ have a minimum with respect to u at $u = u^o = k(x, t, t_1)$, it is necessary that all first partial derivatives of L^* with respect to u vanish at u^o. This and the condition $L^*(x, u^o, t) = 0$ give

$$G'(t)V_x^o = -L_u(x, u^o, t) \tag{4.6}$$

$$-V_t^o = L(x, u^o, t) + [V_x^o, F(t)x + G(t)u^o] \tag{4.7}$$

These equations are called by Carathéodory the *fundamental equations* of the variational problem (3.3).

From assumption (3.5) it follows at once (by the strict convexity[3] of $L(x, u, t)$ in u) that (4.6) can be solved for u^o; more precisely, there exists a function ψ of class C^1 such that

$$u^o = \psi(x, G'(t)V_x^o(x, t, t_1), t) = k(x, t, t_1) \tag{4.8}$$

which is the desired optimal control law.

To check condition (b) of the Lemma, we write, using (4.6–7),

$$\begin{aligned} L^*(x, u, t) &= L(x, u, t) - L(x, u^o, t) - [u - u^o, L_u(x, u^o, t)] \\ &= E(x, u, u^o, t) \end{aligned} \tag{4.9}$$

which is the well-known *Weierstrass E-function*. It is clear by inspection that E is the quadratic remainder in the Taylor series of L at $u = u^o$. Using the well-known estimate for the remainder, we have

$$E(x, u, u^o, t) = \| u - u^o \|^2_{L_{uu}(x, u+\theta(u^o-u), t)} \qquad (0 \leq \theta \leq 1) \tag{4.10}$$

which is nonnegative and in view of (3.5) vanishes if and only if $u = u^o$.

Hence if there is a function V^o satisfying (4.3, 4.6–7) and in addition (3.5)

[3] A scalar function $\alpha(x)$ of a vector x is convex in x if and only if for all x_1, x_2 the function $\beta(\lambda) = \alpha(\lambda x_1 + (1 - \lambda)x_2)$ is convex in λ over the interval $0 \leq \lambda \leq 1$. This will be the case if and only if $d^2\beta/d\lambda^2 = [x_1 - x_2, \alpha_{xx}(x_1 - x_2)] \geq 0$. Hence $\alpha(x)$ is convex if and only if $\alpha_{xx} \geq 0$; similarly, $\alpha(x)$ is strictly convex if and only if $\alpha_{xx} > 0$.

holds, we can apply the Lemma and find that V^o is just the left-hand side of (3.4). The optimal control law is (4.8).

We now define the so-called *conjugate variable* ξ by

$$\xi = V_x^o \tag{4.11}$$

and write $\psi = \psi(x, G'(t)\xi, t)$. We define the *Hamiltonian* as

$$\mathfrak{H}(x, \xi, t) = L(x, \psi, t) + [\xi, F(t)x + G(t)\psi] \tag{4.12}$$

It is easily shown that if L is of class C^2, so is also $\mathfrak{H}$.

Now if $V^o(x, t, t_1)$ is any solution of class C^2 of (4.6–7), it follows by substitution that V^o is a solution of the Hamilton-Jacobi partial differential equation of the first order:

$$V_t^o + \mathfrak{H}(x, V_x^o, t) = 0 \tag{4.13}$$

Conversely, let $V^o(x, t, t_1)$ (t_1 = parameter) be any solution of (4.13) of class C^2. Defining u^o by means of (4.8), it follows that V^o satisfies the fundamental equations (4.6–7).

In summary, we have

(4.14) THEOREM. *If there exists a solution $V^o(x, t, t_1)$ of class C^2 of the Hamilton-Jacobi equation (or, equivalently, of (4.6–7)) which satisfies $V^o(x, t_1, t_1) = v(x)$ and if (3.5) holds, then V^o is the optimal performance index for the regulator problem (3.3), and the corresponding optimal control law is given by (4.8).*

5. Controllability

The purpose of this section is to impose conditions on the plant (2.1) to assure that the problem posed by (3.3) is meaningful in the limit $t_1 = \infty$. Guided by physical intuition, we introduce the

(5.1) DEFINITION. A state x is said to be *controllable at time t_0* if there exists a control function $u^1(t)$, depending on x and t_0 and defined over some finite closed interval $[t_0, t_1]$, such that $\phi_{u^1}(t_1 ; x, t_0) = 0$. If this is true for every state x, we say that the plant is *completely controllable at time t_0* ; if this is true for every t_0, we say simply that the plant is *completely controllable.*

The following equivalent characterization of controllability is useful:

(5.2) PROPOSITION. *A plant is completely controllable at time t* (i) *if and* (ii) *only if the symmetric matrix*

$$W(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_0, t)G(t)G'(t)\Phi'(t_0, t)\, dt \tag{5.3}$$

is positive definite for some $t_1 > t_0$.

Proof. (i) Set

$$u^1(t) = -G'(t)\Phi'(t_0, t)W^{-1}(t_0, t_1)x \tag{5.4}$$

Substitution into (2.3) shows that $\phi_{u^1}(t_1 ; x, t_0) = 0$.

(ii) Suppose there exists some $x \neq 0$ such that $\| x \|^2_{W(t_0,t_1)} = 0$. Define

$$u^2(t) = -G'(t)\Phi'(t_0, t)x$$

which implies that

$$\| x \|^2_{W(t_0,t_1)} = \int_{t_0}^{t_1} \| u^2(t) \|^2 \, dt = 0$$

Since $u^2(t)$ is continuous in t, it is therefore identically zero in the interval $[t_0, t_1]$.

On the other hand, if the plant is completely controllable at t_0, there exists a control function $u^1(t)$ as required by (5.1) which satisfies the relation

$$x = -\int_{t_0}^{t_1} \Phi(t_0, t)G(t)u^1(t) \, dt$$

and therefore

$$\| x \|^2 = -\int_{t_0}^{t_1} [u^1(t), u^2(t)] \, dt = 0$$

contradicting the assumption that $x \neq 0$. Q. E. D.

(5.5) COROLLARY. *A constant plant is completely controllable* (i) *if and* (ii) *only if*

$$\text{rank } [G, FG, \cdots, F^{n-1}G] = n \tag{5.6}$$

(where the square brackets denote a composite matrix of n rows and mn columns) in which case one may choose $t_1 - t_0 > 0$ as small as desired.

Proof. Because of stationarity, controllability does not depend on t_0. Hence take $t_0 = 0$.

(i) By (5.2), it suffices to prove that $W(0, t_1)$ is positive definite no matter how small $t_1 > 0$. Let $g^1, \cdots, g^m$ be the columns of G. If $W(0, t_1)$ is semidefinite, then proceeding as in part (ii) of the proof of (5.2) we conclude that there is a vector $x \neq 0$ such that

$$[x, e^{Ft}g^i] = 0 \quad \text{for all} \quad 0 \leq t \leq t_1 \text{ and } i = 1, \cdots, m$$

Differentiating j times with respect to t, and then setting $t = 0$, we get

$$[x, F^j g^i] = 0 \quad \text{for all} \quad i = 1, \cdots, m \text{ and } j = 0, \cdots, n-1 \tag{5.7}$$

If (5.6) holds, this implies that x is orthogonal to a set of generators of E^n, contradicting the assumption that $x \neq 0$.

(ii) Assume the plant is completely controllable but (5.6) is false. Then there is a vector $x \neq 0$ which satisfies (5.7). By the Cayley-Hamilton theorem

$$[x, e^{Ft}g^i] = \left[x, \left(\sum_{j=0}^{\infty} (Ft)^j/j!\right) g^i\right] = \left[x, \left(\sum_{j=0}^{n-1} \alpha_j (Ft)^j\right) g^i\right], \qquad i = 1, \cdots, m$$

It follows that $\| x \|^2_{W(0,t_1)} = 0$ for all t_1, contradicting the assumption of complete controllability. Q. E. D.

Condition (5.6) has been used as a technical device in several recent papers in the theory of control ([12]–[14]), without reference to the "physical" interpretation (5.1).

(5.8) *Remark.* Let x be the state of the plant at t_0 and y the "desired" state at t_1. It follows easily by a slight extension of the preceding arguments that y is *reachable* from x (i.e., there exists a motion ϕ_{u^1} which meets x at t_0 and y at t_1) if and only if the equation

$$x - \Phi(t_0, t_1)y = W(t_0, t_1)v \tag{5.9}$$

has a solution, in which case

$$u^1(t) = -G'(t)\Phi'(t_0, t)v \tag{5.10}$$

is the appropriate control function.

Moreover, elementary methods of the calculus of variations show (see also [15]) that the minimum *control energy* required to achieve the transfer is

$$\mathcal{E}(x, t_0; y, t_1) = \int_{t_0}^{t_1} \| u^1(t) \|^2 \, dt = \| x - \Phi(t_0, t_1)y \|^2_{W^{-1}(t_0,t_1)} \tag{5.11}$$

Clearly, the required "energy" is zero if and only if the free motion going through x at t_0 intersects y at t_1.

Equation (5.9) may have a solution for some but not all x, y. Then W^{-1} does not exist and it is convenient to replace it with the *generalized inverse* $W\dagger$ in the sense of Penrose ([16], [17]). (See Appendix). With this convention, (5.11) *is the minimum energy required for transferring x as close to y as possible.*

If $W(t_0, t_1)$ is invertible, then (5.9) always has a solution; we see that *a plant is completely controllable at time t_0 if and only if starting from the origin at time t_0 any state x can be reached in a finite length of time by applying an appropriate control function $u(t)$*. In other words, there is a noteworthy "symmetry" between sending x to 0 and sending 0 to x.

(5.12) *Remark.* Using the generalized inverse, we may replace (5.10) by a control law defined in $[t_0, t_1]$:

$$u^1(t) = -G'(t)W\dagger(t, t_1)[x - \Phi(t, t_1)y]$$

Even if the plant is stationary, this control law is not. In fact, a stationary control law can be obtained in this case only by letting $u^1(t)$ be discontinuous ([8]).

The following definition is designed to single out a class of nonstationary plants which are in a sense "quasi-stationary". This will play an important role in the sequel.

(5.13) DEFINITION. A plant is *uniformly completely controllable* if the following relations hold for all t:

(i) $0 < \alpha_0(\sigma)I \leq W(t, t+\sigma) \leq \alpha_1(\sigma)I$

(ii) $0 < \beta_0(\sigma)I \leq \Phi(t+\sigma, t)W(t, t+\sigma)\Phi'(t+\sigma, t) \leq \beta_1(\sigma)I$

where σ is a fixed constant. In other words, one can always transfer x to 0 and 0 to x in a finite length σ of time; moreover, such a transfer can never take place using an arbitrarily small amount (or requiring an arbitrarily large amount) of control energy.

Definition (5.13) has surprisingly far-flung consequences. We mention some of these; the proofs are elementary.

First of all, if (i–ii) hold, then, for all t,

$$\sqrt{\beta_0(\sigma)/\alpha_1(\sigma)} \leq \| \Phi(t+\sigma, t) \| \leq \sqrt{\beta_1(\sigma)/\alpha_0(\sigma)} \tag{5.14}$$

which is equivalent to

$$\sqrt{\alpha_0(\sigma)/\beta_1(\sigma)} \leq \| \Phi(t, t+\sigma) \| \leq \sqrt{\alpha_1(\sigma)/\beta_0(\sigma)} \tag{5.15}$$

(5.16) From formulas (5.3) and (5.14–15) we see that (i–ii) hold also for the constant $\sigma' = 2\sigma$; this implies further that (i–ii) hold for any $\sigma' \geq \sigma$.

(5.17) Using (5.16), we see that (i–ii) actually imply the following stronger bound on the transition matrix:

$$\| \Phi(t, \tau) \| \leq \alpha_3(|\, t - \tau \,|) \quad \text{for all } t, \tau \tag{5.18}$$

(5.19) It is now clear that if any two of the relations (5.18), (i), and (ii) hold, the remaining relation is also true.

The bound (5.18) obviously restricts the class of dynamical systems (2.1). Some such restriction appears to be an unavoidable consequence of any "reasonable" definition of uniform complete controllability. For instance, if only (i) holds, the following peculiar situation may arise. Consider the scalar system:

$$dx/dt = -tx + \sqrt{2(t-1)}e^{-t+1/2}u(t)$$

(defined only for $t \geq 1$). We find easily that

$$\phi(t, \tau) = e^{(\tau^2 - t^2)/2}$$

which does not satisfy (5.18); furthermore,

$$w(t, t+\sigma) = e^{2(\sigma-1)t+(\sigma-1)^2} - e^{-2t+1}$$

and it is clear that w does not satisfy (i) unless $\sigma = 1$, while (ii) is never satisfied. In other words, to transfer x to 0 over an interval of time shorter than 1 may require an arbitrarily large amount of control energy, whereas doing the same job over an interval of time longer than 1 may require only a vanishingly small amount of energy as $t_0 \to \infty$. Transferring 0 to x will require more and more energy as $t_0 \to \infty$.

Finally, let us note a well-known and readily verifiable condition for (5.18) (easily proved using the Gronwall-Bellman lemma):

$$\int_{t_1}^{t_2} \| F(\tau) \| \, d\tau \leq \gamma(t_2 - t_1) \quad \text{for all } t_1, t_2 \tag{5.20}$$

We now seek to characterize a plant according to its "output" properties. This is most conveniently done as follows. Let $t^* = -t$ and $F^*(t^*) = F'(t)$, $G^*(t^*) = H'(t)$, and $H^*(t^*) = G'(t)$. Then

$$\begin{aligned} dx^*/dt^* &= F^*(t^*)x^* + G^*(t^*)u^*(t^*) \\ y^*(t^*) &= H^*(t^*)x^*(t^*) \end{aligned} \tag{5.21}$$

where x^*, u^*, y^* are n, p, and m vectors respectively, is the *dual plant* of (2.1–2). We shall not discuss the significance of this concept in detail (for which see [8]), except for pointing out that (i) the duality relations are reflexive if $t_0^{**} = t_0$; (ii) the transition matrix of (5.21) satisfies the relation

$$\Phi^*(t^*, \tau^*) = \Phi'(\tau, t) \qquad \text{for all } t,\tau \tag{5.22}$$

It is convenient to introduce the:

(5.23) DEFINITION. A plant (2.1–2) is *uniformly completely observable* if its dual is uniformly completely controllable.

It follows easily from (5.23) that the explicit expression for W^* corresponding to (5.3) is

$$\begin{aligned} W^*(t^*, t_0^* + \sigma^*) &= W^*(t_0, t_0 - \sigma^*) \\ &= \int_{t_0-\sigma^*}^{t_0} \Phi'(t, t_0)H'(t)H(t)\Phi(t, t_0)\, dt \end{aligned} \tag{5.24}$$

Using W^* defined by (5.24), we can now state (5.23) explicitly. To avoid any possibility of confusion, the constants α, β, σ occurring in (i–ii) are to be replaced by α^*, β^*, σ^*.

6. Solution of the linear regulator problem

The point of view of the classical calculus of variations outlined in Section 4 is purely "local." At present, there are few global results and just about none in the theory of control. In the "local" (linear) case, however, the ideas of the preceding section lead to (what is hoped to become) a definitive theory of the regulator problem. This is the subject of the remainder of the paper.

To get the linear case of the regulator problem, it is not enough to have a linear model (2.1) for the plant but we need also the assumption:

(A_1) $\qquad L(x, u, t) = \frac{1}{2}\{\| H(t)x \|^2_{Q(t)} + \| u \|^2_{R(t)}\}, \qquad \nu(x) = \frac{1}{2}\| x \|^2_A$

where A is symmetric, nonnegative definite while $Q(t)$, $R(t)$ are symmetric, positive definite and of class C^2 in t.

In view of (A_1), the Hamiltonian function (4.12) is

$$\mathcal{H}(x, \xi, t) = \frac{1}{2}\{\| H(t)x \|^2_{Q(t)} + 2[F(t)x, \xi] - \| G'(t)\xi \|^2_{R^{-1}(t)}\} \tag{6.1}$$

With this choice of $\mathcal{H}$, the function

$$V^0(x, t, t_1) = \tfrac{1}{2}\| x \|^2_{P(t,t_1)} \tag{6.2}$$

(t_1 = parameter) is a solution of the Hamilton-Jacobi equation (4.13) if and only if $P(t, t_1)$ is a solution of the following ordinary nonlinear differential equation of the Riccati type:

$$-\frac{dP}{dt} = F'(t)P + PF(t) - PG(t)R^{-1}(t)G'(t)P + H'(t)Q(t)H(t) \qquad (6.3)$$

It is clear that (6.2) determines P only up to a constant, skew-symmetric matrix (constancy follows from the fact that dP/dt is symmetric). Henceforth, to avoid trivia, we always assume that P is symmetric.

Given any symmetric, nonnegative definitive matrix A, (6.3) has a unique solution $\Pi(t; A, t_1)$ which takes on the value A at $t = t_1$. This solution is known to exist only in some neighborhood of t_1; without further analysis we cannot conclude existence for all t. (Because of the phenomenon of *finite escape time*, for which see [10], Example 3.)

Nonetheless, $\Pi(t; A, t_1)$ does exist for all $t \leq t_1$. We prove this indirectly as follows:

(6.4) EXISTENCE THEOREM. (i) *For all t_1 and all symmetric, nonnegative definite A,* (6.3) *has a unique solution $\Pi(t; A, t_1)$ defined for all $t \leq t_1$.* (ii) *Under Assumption (A_1) the optimal performance index for Problem* (3.3) *is given by*

$$V^o(x, t_0, t_1) = \| x \|^2_{\Pi(t_0; A, t_1)}$$

Moreover, the optimal performance index is attained if and only if the control law is given by

$$u^o(t) = R^{-1}(t)G'(t)\Pi(t; A, t_1)x(t) \qquad (6.5)$$

Proof. If we assume (i), then (ii) follows immediately from (4.14). Therefore, if $\Pi(t; A, t_1)$ exists, it must necessarily satisfy the relation

$$\| x \|^2_{\Pi(t_0; A, t_1)} \leq \int_{t_0}^{t_1} \| H(t)\Phi(t, t_0)\, x \|^2_{Q(t)}\, dt + \| \Phi(t_1, t_0)x \|^2_A$$

$$\leq \alpha(t_1, t_0) \| x \|^2$$

which follows by setting $u(t) \equiv 0$ in (3.4). Since $\alpha(t_1, t_0)$ is finite for all pairs t_1, t_0, it is clear that $\Pi(t; A, t_1)$ (if it exists) is contained in a compact region for all $t \in [t_0, t_1]$. Including this fact in the standard proof of the existence theorem for differential equations proves (i). Q. E. D.

In order to study the case $t_1 \to \infty$, we first define a particular solution of (6.3) which is of central significance for the ensuing development.

(6.6) PROPOSITION. *If the plant is completely controllable, then*

$$\lim_{t_1 \to \infty} \Pi(t; 0, t_1) = \bar{P}(t)$$

(i) *exists for all t and* (ii) *is a solution of* (6.3).

Proof. (i) Suppose the plant is completely controllable at $t = t_0$. Then for every x there exists a control function $u^1(t)$, given by (5.4), which transfers x to 0 at or before $t = t_2(x, t_0)$. We set $u^1(t) = 0$ for $t > t_2$. Then

$$\| x \|_{\Pi(t_0;0,t_1)} = V^0(x, t_0, t_1) \leq V(x, t_0, t_2; u^1) = V(x, t_0, \infty; u^1) \leq \alpha(t_0) \| x \|^2$$

which shows that $\| \Pi(t_0; 0, t_1) \|$ is bounded for all $t_1 \geq t_0$. On the other hand, (3.4) shows that $\| \Pi(t_0; 0, t_1) \|$ is nondecreasing as $t_1 \to \infty$. Hence the desired limit exists for arbitrary $t = t_0$. Q. E. D.

(ii) Using the continuity of solutions of (6.3) with respect to initial conditions, we have

$$\bar{P}(t) = \lim_{t_2 \to \infty} \Pi(t; 0, t_2) = \lim_{t_2 \to \infty} \Pi(t; \Pi(t_1; 0, t_2), t_1)$$

$$= \Pi(t; \lim_{t_2 \to \infty} \Pi(t_1; 0, t_2), t_1) = \Pi(t; \bar{P}(t_1), t_1)$$

which shows that $\bar{P}(t)$ is a solution of (6.3) which is defined for all t. Q. E. D.

(6.7) EXISTENCE THEOREM. *Assuming* (A_1), $v(x) = 0$, *and* $t_1 = \infty$, *the optimal performance index for Problem* (3.3) *is* $\| x \|^2_{\bar{P}(t)}$ *and the optimal control law is*

$$u^0(t) = R^{-1}(t) G'(t) \bar{P}(t) x(t) \tag{6.8}$$

Proof. Assume throughout the $v(x) = 0$. First we show: *If* $u(t)$ *is determined by the control law* (6.7), *the corresponding performance index is*

$$V(x, t_0, \infty; u^0) = \lim_{t_1 \to \infty} V(x, t_0, t_1; u^0) = \| x \|^2_{\bar{P}(t_0)}$$

We see from (6.4) and (6.6) that

$$V(x, t_0, t_1; u^0) = \| x \|^2_{\bar{P}(t_0)} - \| \phi_{u^0}(t_1; x, t_0) \|^2_{\bar{P}(t_1)} \leq \| x \|^2_{\bar{P}(t_0)}$$

On the other hand,

$$V(x, t_0, t_1; u^0) \geq V^0(x, t_0, t_1) = \| x \|^2_{\Pi(t_0;0,t_1)} \geq \| x \|^2_{\bar{P}(t_0)} - \epsilon$$

where $\epsilon \to 0$ as $t_1 \to \infty$, which proves (6.9). Hence

$$V^0(x, t_0, \infty) \leq V(x, t_0, \infty; u^0)$$

The inequality sign cannot arise. For if $V(x, t_0, \infty; u^0) - V^0(x, t_0, \infty) \geq \eta > 0$, there is some control function u^1 such that $V(x, t_0, \infty; u^0) - V(x, t_0, \infty, u^1) \geq \eta/2$. For t_1 sufficiently large, we then have

$$V(x, t_0, \infty; u^0) = V^0(x, t_0, t_1) + \eta/4 \geq V(x, t_0, t_1; u^1) + \eta/2$$

which is a contradiction and everything is proved. Q. E. D.

In the engineering literature it is often assumed (tacitly and incorrectly) that a system with optimal control law (6.8) is necessarily stable. We now give rigorous sufficient conditions insuring uniform asymptotic stability and point out in the process of proof some trivial but interesting parallels between the calculus of variations and the second method of Lyapunov.

The following definition is standard [10], [18]: The system (2.1) is *uniformly asymptotically stable* if (i) $\| \Phi(t, t_0) \| \leq \alpha$ and (ii) $\| \Phi(t, t_0) \| \to 0$ with $t \to \infty$ uniformly in t_0. It can be shown ([10], Theorem 3] that uniform asymptotic stability in the linear case is equivalent to *exponential asymptotic stability*, which is defined by the condition ($\alpha, \beta > 0$)

$$\| \Phi(t, t_0) \| \leq \alpha \exp [-\beta(t - t_0)] \quad \text{for all } t_0 \text{ and all } t \geq t_0 .$$

(6.10) STABILITY THEOREM. *Consider a plant with control law* (6.8) *which is uniformly completely controllable and uniformly completely observable. In addition to* (A_1), *assume also*

$$(A_2) \qquad Q(t) \geq \alpha_4 I > 0, \qquad R(t) \geq \alpha_5 I > 0$$

$$(A_3) \qquad Q(t) \leq \alpha_6 I, \qquad R(t) \leq \alpha_7 I$$

Then the controlled plant is uniformly asymptotically stable and $V^0(x, t, \infty)$ *is one of its Lyapunov functions.*

Proof. As is well known, it suffices to prove that (a) V^0 is bounded from above and (b) below by increasing functions of $\| x \|$ independent of t, (c) the derivative $\dot{V}^0$ of V^0 along optimal motions of the plant is negative definite [10, 18], and (d) $V^0 \to \infty$ with $\| x \| \to \infty$.

(a) By uniform complete controllability, let $u^1(t)$ be the control function, depending on x, t_0 and defined in $[t_0, t_0 + \sigma]$ (σ = positive constant), which transfers x to 0 at or before $t = t_0 + \sigma$. In accordance with the remarks following (5.13), there is no loss of generality in taking the constants σ and σ^* (occurring in the definition of uniform complete controllability and uniform complete observability) to be the same. Having set $\sigma = \sigma^*$, we let $t_1 = t_0 + \sigma$. If $u^1(t)$ is defined explicitly by means of (5.4), then

$$\begin{aligned} \phi_{u^1}(t; x, t_0) &= \Phi(t, t_0)[I - W(t_0, t)W^{-1}(t_0, t_1)]x \\ &= \Phi(t, t_1)z(t) \end{aligned} \tag{6.11}$$

From the definition of W (see (5.3)), it follows easily[4] that the norm of the bracketed term above is less than or equal to 1. Using (5.18) then gives

$$\| z(t) \| \leq \alpha_8 \| x \|$$

where α_8 depends only on σ and it is therefore constant. By (6.11) and (A_3) we get

$$V^0(x, t_0, \infty) \leq \int_{t_0}^{t_1} \{\alpha_6 \| H(t)\Phi(t, t_1)z(t) \|^2 + \alpha_7 \| u^1(t) \|^2\} \, dt \tag{6.12}$$

[4] We need to show only that if $B > 0$ and $B \geq A \geq 0$, then $\| AB^{-1} \| \leq 1$. Now $\| AB^{-1} \|^2 = \lambda_{\max}(B^{-1}A^{-2}B^{-1}) = \lambda_{\max}(A^2B^{-2})$. By the well-known theorem about simultaneous diagonalization of a positive definite and a symmetric matrix, we have the representation $A^2 = T'\Lambda T$, $B^2 = T'T$, where T is nonsingular and Λ is diagonal. $B \geq A \geq 0$ implies $1 \geq \lambda(\Lambda) \geq 0$. Hence $\lambda_{\max}(A^2B^{-2}) = \lambda_{\max}(\Lambda) \leq 1$.

Making use of the foregoing and of the elementary inequality

$$\| Ax \|^2 \leq \| A \|^2 \cdot \| x \|^2 = \lambda_{\max}(A'A) \| x \|^2 \leq (\operatorname{tr} A'A) \| x \|^2$$

(valid for any matrix A and any vector x), (6.12) becomes

$$V^o(x, t_0, \infty) \leq \alpha_6\alpha_8(\operatorname{tr} W^*(t_1, t_0)) \| x \|^2 + \alpha_7 \| x \|^2_{W^{-1}(t_0,t_1)}$$

and by uniform complete controllability and observability we have finally

$$V^o(x, t_0, \infty) \leq [n\alpha_6\alpha_8\alpha_1^*(\sigma) + \alpha_7\alpha_0(\sigma)] \| x \|^2 = \alpha_9 \| x \|^2.$$

(b) In view of (6.7), we can define

$$\inf_x V^o(x, t_0, \infty) = \alpha_{10}(t_0) \| x \|^2$$

We show that $\alpha_{10}(t_0) \geq \alpha_{11} > 0$. In fact, in the contrary case we can make ε, defined by

$$\| x \|^2 \varepsilon(x, t) = \int_{t_0}^{\infty} \| u^o(t) \|^2 dt \leq \alpha_5^{-1} \int_{t_0}^{\infty} \| u^o(t) \|^2_{R(t)} dt \leq V^o(x, t_0, \infty)$$

as small as desired by suitable choice of x, t_0. We introduce the abbreviation

$$z(t) = \int_{t_0}^{t_1} \Phi(t_0, t) G(t) u^o(t) dt$$

and note that, by the Schwarz inequality,

$$\| z(t) \|^2 \leq \left(\int_{t_0}^{t_1} \| \Phi(t_0, t) G(t) \|^2 dt \right) \left(\int_{t_0}^{t_1} \| u^0(t) \|^2 dt \right)$$

and by uniform complete controllability,

$$\| z(t) \|^2 \leq n\alpha_1(\sigma) \in (x, t_0) \| x \|^2.$$

Utilizing this estimate, we find with the aid of (A_2):

$$\begin{aligned} V^o(x, t_0, \infty) &\geq \int_{t_0}^{t_1} \alpha_4 \| H(t)\Phi(t, t_0)[x + z(t)] \|^2 dt \\ &\geq \int_{t_0}^{t_1} \alpha_4 \{ \| H(t)\Phi(t, t_0)x \|^2 - \| H(t)\Phi(t, t_0)z(t) \|^2 \} dt \\ &\geq \alpha_4 \{ \| \Phi(t_1, t_0)x \|^2_{W^*(t_1,t_0)} - n\alpha_1(\sigma) \in (x, t_0)[\operatorname{tr} W^*(t_1, t_0)] \| x \|^2 \} \end{aligned}$$

By (5.18) and uniform complete observability, this reduces to

$$\begin{aligned} V^o(x, t_0, \infty) &\geq \alpha_4[\alpha_3^{-2}(\sigma)\alpha_0^*(\sigma) - n^2\alpha_1(\sigma)\alpha_1^*(\sigma) \in (x, t_0)] \| x \|^2 \\ &\geq [\alpha_{13} - \alpha_{14} \in (x, t_0)] \| x \|^2 \end{aligned}$$

which contradicts the assumption that α_{12} (and hence ϵ) can be made arbitrarily small by suitable choice of x, t_0.

(c) Since G and H are allowed to be singular, we cannot prove of course that V is negative definite. However, inspection of the last step of (b) yields the further inequality,

$$V^o(\phi_{u^o}(t_1\,;\,x,\,t_0),\,t_1\,,\,\infty) - V^o(x,\,t_0\,,\,\infty) \leq -[\alpha_{13} - \alpha_{14} \in (x,\,t_0)]\| x \|^2$$

and we have simultaneously also the further inequality

$$V^o(\phi_{u^o}(t_1\,;\,x,\,t_0),\,t_1\,,\,\infty) - V^o(x,\,t_0\,,\,\infty) \leq \alpha_5 \in (x,\,t_0)\| x \|^2$$

which follows immediately by (A_2) and the definition of V^o. Setting

$$\alpha_{15} = \alpha_5\alpha_{13}/(\alpha_5 + \alpha_{14}) > 0$$

we have finally that

$$V^o(\phi_{u^o}(t_0 + \sigma;\,x,\,t_0),\,t_0 + \sigma,\,\infty) - V^o(x,\,t_0\,,\,\infty) \leq -\alpha_{15}\| x \|^2$$

which shows that V^o is strictly decreasing along any interval of time of length σ_2, unless $x = 0$. Taking account of this fact, the proof of Lyapunov's theorem on uniform asymptotic stability ([10]) goes through as usual.

(d) This is trivial in view of $V^o \geq \alpha_{12}\| x \|^2$. Q. E. D.

It is of some interest to observe that if we have merely complete controllability, part (a) does not go through but we have nevertheless proved (nonuniform) asymptotic stability.

7. Stability of the Riccati equation

We now turn again to (6.3) and examine briefly its stability properties. Let $\delta P(t) = P(t) - \bar{P}(t)$ denote the deviation of a given motion $P(t)$ of (6.3) from $\bar{P}(t)$. Substituting into (6.3) shows that

$$d(\delta P)/dt = -\bar{F}'(t)\delta P - \delta P\bar{F}(t) - \delta PG(t)R^{-1}(t)G'(t)\delta P \tag{7.1}$$

where

$$\bar{F}(t) = F(t) - G(t)R^{-1}(t)G'(t)\bar{P}(t)$$

For simplicity, we temporarily drop the argument t in G, H, P, Q, R.

(7.2) Stability Theorem. *Let*

$$\mathfrak{V}(\delta P,\,t) = \tfrac{1}{2}\,\mathrm{tr}\,(\delta P\bar{P}^{-1})^2. \tag{7.3}$$

Then (i) *the derivative of* $\mathfrak{V}$ *along motions of* (7.1) *is*

$$\begin{aligned}&\dot{\mathfrak{V}}(\delta P,\,t)\\ &= tr\,\{(P^{\frac{1}{2}}GR^{-1}G'P^{\frac{1}{2}})\cdot(P^{\frac{1}{2}}\bar{P}^{-1}P^{\frac{1}{2}} - 1)^2 + (\bar{P}^{-\frac{1}{2}}H'QH\bar{P}^{-\frac{1}{2}})(\bar{P}^{-\frac{1}{2}}P\bar{P}^{-\frac{1}{2}} - 1)^2\}\end{aligned} \tag{7.4}$$

provided $P \geq 0$;

(ii) *If* $A \geq 0$, *then under the hypotheses of* (6.10), *all solutions* $\Pi(t;\,A,\,t_1)$ *of* (6.3) *are uniformly asymptotically stable relative to* $\bar{P}(t)$ *as* $t \to -\infty$, *and* $\mathfrak{V}$ *is an appropriate Lyapunov function.*

Proof. (i) This is established by lengthy, elementary calculations. The square root of P exists by assumption and that of $\bar{P}$ by part (b) of (6.10).

(ii) Clearly, $\mathfrak{V}$ vanishes if and only if $\delta P = 0$. We recall the fact that for any symmetric, $n \times n$ matrices A, B,

$$\lambda_{\min}(B)\lambda_i(A^2) \leq \lambda_i(ABA) \leq \lambda_{\max}(B)\lambda_i(A^2) \qquad (i = 1, \cdots, n) \quad (7.5)$$

where the λ_i are eigenvalues. This is a consequence of the Fischer-Courant variational description of eigenvalues (for which see [20], p. 115 Theorem 3 and p. 120, Exercise 9). Using (7.5) and the results of (6.10) it follows easily that

$$0 < \alpha \operatorname{tr} (\delta P)^2 \leq \mathfrak{V}(\delta P, t) \leq \beta \operatorname{tr} (\delta P)^2, \qquad \delta P \neq 0$$

Moreover, (7.4) being the trace of a nonnegative definite matrix, $\dot{\mathfrak{V}}$ is clearly nonnegative. By arguments analogous to part (c) of the proof of (6.10), it follows then also that $\mathfrak{V}$ is uniformly decreasing along any motion of (6.3) as as $t \to -\infty$.

(7.6) COROLLARY. *The motion $\bar{P}(t)$ is unstable (as $t \to \infty$).*

Proof. Immediate consequence of part (ii) of the proof of (7.2).

(7.7) *Remark.* If the problem is stationary, i.e., F, G, H, Q, R are constants, $P(t, t_1) = P(t + h, t_1 + h)$ which shows that $dP(t, t_1)/dt_1 = -dP(t, t_1)/dt$. Hence in this case one can compute $P(t) = \text{const.}$ from (6.3) by replacing t by $-t$; for any initial $A \geq 0$, this computation is asymptotically stable in the large.

(7.8) *Remark.* Because of the Corollary, in the nonstationary case (at least one of F, G, H, Q, R not constant), one cannot compute $P(t)$ as $t \to \infty$ from the knowledge of $P(t_0)$.

8. General solution of the Riccati equation

Consider the canonic (Hamiltonian) differential equations associated with (6.1):

$$dx/dt = \mathfrak{H}_\xi(x, \xi, t) = F(t)x - G(t)R^{-1}(t)G'(t)\xi \quad (8.1)$$

$$d\xi/dt = -\mathfrak{H}_x(x, \xi, t) = -H'(t)Q(t)H(t)x - F'(t)\xi \quad (8.2)$$

Let $P(t)$ be a solution of (6.3), defined in some interval $U = (-\infty, t_2)$. In view of (4.11) and (6.2), we assume that the initial conditions of (8.1–2) at time t_1 are related by ($t_1 < t_2$!)

$$\xi(t_1) = V_x^o(x(t_1), t_1) = P(t_1)x(t_1)$$

Then the same relation will hold between solutions of (8.1–2) corresponding to these initial conditions, for all t that $P(t)$ exists:

$$\xi(t) = V_x^o(x(t), t) = P(t)x(t), \qquad t \in U \quad (8.3)$$

We can also verify (8.3) directly by substituting (6.3) into (8.1–2).

Now let $X(t)$, $\Xi(t)$ be a pair of matrix solutions of (8.1–2) satisfying the initial

conditions $X(t_1) = I$, $\Xi(t_1) = P(t_1)$. By (8.3) we have obviously

$$\Xi(t) = P(t)X(t), \qquad t \in U \quad (8.4)$$

which shows that $X(t)$ is a solution of the matrix differential equation

$$dX/dt = [F(t) - G(t)R^{-1}(t)G'(t)P(t)]X, \qquad t \in U$$

Setting $v(x) = \| x \|^2_{P(t_1)}$ in (3.4), we see from (6.4–5) that $X(t)$ is the transition matrix $\Phi^o(t, t_1)$ of the optimally controlled plant corresponding to this choice of v. Since $\Phi^o(t, t_1)$ exists for all $t \in U$, we have

$$P(t) = \Xi(t)\Phi^o(t_1, t), \qquad t \in U \quad (8.5)$$

To obtain an explicit expression for $P(t)$, let

$$\Theta(t, t_1) = \begin{pmatrix} \Theta_{11}(t, t_1) & \Theta_{12}(t, t_1) \\ \Theta_{21}(t, t_1) & \Theta_{22}(t, t_1) \end{pmatrix}$$

be the transition matrix of the system (8.1–2). We get the following formula valid for $t \in U$:

$$P(t) = [\Theta_{21}(t, t_1) + \Theta_{22}(t_1, t)P(t_1)][\Theta_{11}(t, t_1) + \Theta_{12}(t, t_1)P(t_1)]^{-1}$$

This procedure is very well known in the calculus of variations ([21; 11, Ch. 15]) and is being periodically rediscovered ([22], [23]).

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Appendix: The generalized inverse of a matrix

Following Penrose ([16]), the *generalized inverse* of an arbitrary square matrix A is a matrix $A\dagger$ satisfying the relations:

$$\text{(i)}\ AA\dagger A = A, \qquad \text{(ii)}\ A\dagger AA\dagger = A\dagger,$$
$$\text{(iii)}\ (A\dagger A)' = A\dagger A, \qquad \text{(iv)}\ (AA\dagger)' = AA\dagger$$

It can be shown that $A\dagger$ always exists and is uniquely determined by these relations. Examples: (1) If D is diagonal, then the elements of its generalized inverse are

$$d_{ii}^{\dagger} = d_{ii}^{-1} \quad \text{if } d_{ii} \neq 0$$
$$= 0 \text{ otherwise}$$

(2) If A is symmetric, there is an orthogonal transformation T such that $A = T'DT$. Then $A\dagger = T'D\dagger T$.

Consider now the linear equation $Ax = y$. Penrose proves ([17]) that the "best approximate solution" $x^o = A\dagger y$ of this equation has the properties:

(i) $\| Ax - y \| \geq \| Ax^o - y \|$ for all x

(ii) If $\| Ax - y \| = \| Ax^o - y \|$, then $\| x \| \geq \| x^o \|$