Design of a Self-Optimizing Control System

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This paper examines the problem of building a machine which adjusts itself automatically to control an arbitrary dynamic process. The design of a small computer which acts as such a machine is presented in detail. A complete set of equations describing the machine is derived and listed; engineering features of the computer are discussed briefly. This machine represents a new concept in the development of automatic control systems. It should find widespread application in the automation of complex systems such as aircraft or chemical processes, where present methods would be too expensive or time-consuming to apply.

Introduction

THE art of the design of systems for the automatic control of dynamic processes of many different kinds (such as airplanes, chemical plants, military-weapon systems, and so on) has been reduced gradually to standard engineering practice during the years following World War II. In the simplest possible setting, the problem that the engineer faces in designing such automatic control systems is shown in Fig. 1. It is desired that the output of the process $c(t)$, which may be position, speed, temperature, pressure, flow rate, or the like, be as close as possible at all times to an arbitrarily given input $r(t)$ to the system. In other words, at all instants of time it is desired to keep the error $e(t) = r(t) - c(t)$ as small as possible. Control is accomplished by varying some physical quantity $m(t)$, called the control effort, which affects the output of the process.

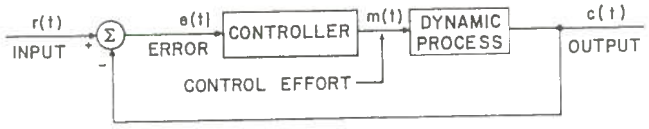

Fig. 1 Block Diagram of Simplest Control Problem

As long as the deviations from an equilibrium value of $r(t)$, $c(t)$, and therefore of $e(t)$ and $m(t)$, are small, the system can be regarded as approximately linear and there is a wealth of theoretical as well as practical information on which engineering design may be based. (When the system is not linear, present-day knowledge supplies only fragmentary suggestions for design; however, nonlinear effects are frequently of secondary importance.) It is generally agreed that the design of high-performance control systems is essentially a problem of matching the dynamic characteristics of a process by those of the controller. Practically speaking, this means that if the dynamic characteristics of the process are known with sufficient accuracy, then the characteristics of a controller necessary to give a certain desired type of performance can be specified. Usually, this amounts to writing down in quantitative terms the differential equations of the controller. Thus the design procedure can be divided roughly into the following distinct stages:

I Measure the dynamic characteristics of the process.
II Specify the desired characteristics of the controller.
III Put together a controller using standard elements (amplifiers, integrators, summers, electric networks, and so on) which has the required dynamic characteristics.

This subdivision of effort in designing a control system is oversimplified, but it will be a convenient starting point for the following discussion.

It has been pointed out by Bergen and Ragazzini (1)[2] that if a high degree of flexibility is desired in design stage (III), it is advantageous to use a sampled-data system. In principle, a sampled-data system is one where the controller is a digital computer. It is probably no exaggeration to say that, because of the great inherent flexibility of a digital computer, *any* desired controller characteristic is practically realizable. The use of a digital computer for the controller reduces stage (III) to a straightforward operation, like that of transcribing a handwritten manuscript by means of a typewriter.

Since the theory of linear control systems is well developed, stages (I–II) also can be made to consist of more-or-less standard procedures. Quick and convenient design even in stage (III) demands or at least suggests a digital computer; so the question arises whether or not stages (I–II) also can be reduced to completely mechanical operations which can be performed by a digital computer. Accordingly, the problem considered in this paper can be stated as follows:

To design a machine which, when inserted in the place of the controller in Fig. 1, will automatically perform steps (I–III), and set itself up as a controller which is optimum in some sense. The design of this machine is to be based on broad principles only. Its operation should require no direct human intervention but merely the measurements of $r(t)$ and $c(t)$.

In other words, such a machine, if it can be built, eliminates the lengthy, tedious, and costly procedure of engineering design—it is only necessary to connect the machine to *any* process. Thus the machine would seemingly eliminate the need for the control-systems engineer, but the latter can be reassured by the fact that the design of the machine itself is a far more ambitious and challenging undertaking than that of conventional control systems.

An even more decisive advantage of the machine over present-day design procedures is the following: In carrying out steps (I–III) it is generally taken for granted that the dynamic characteristics of the process will change only slightly under any operating conditions encountered during the lifetime of the control system. Such *slight* changes are foreseen and are usually counteracted by using feedback. Should the changes become large, the control equipment as originally designed may fail to meet performance specifications. Instances where difficulties of this type are encountered are:

(*a*) Changes of aircraft characteristics with speed.
(*b*) Chemical processes.
(*c*) Any large-scale control operation, where the nature of the system can be affected by uncontrolled and unforeseen factors.

By contrast, the machine can *repeat* steps (I–III) continually and thereby detect and make corrections in accordance with any

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[2] Numbers in parentheses refer to the References at the end of the paper.

changes in the dynamic characteristics of a process which it controls. Such a control system operates always at or near some "optimum," provided only that changes in the dynamic characteristics of the controlled process do not occur very abruptly. It may be said that the machine adapts itself to changes in its surroundings—this may be regarded as an extension of the principle of feedback. The author prefers to call this property of the machine "self-optimization." The word "ultrastability" has been suggested also in a similar context by Ashby (2).

In the stated degree of generality, the problem is certainly not at a stage at present where any clear-cut ("unique") solution can be expected. Therefore this paper does not treat the general problem but presents a specific approach which leads to a practically satisfactory solution. This point is of considerable interest, since some earlier speculations relating to the problem were mostly of theoretical nature, without an attempt to appraise the difficulties (cost, complexity, and so on) of practical implementation (2–5). A machine based on the principles discussed in what follows actually has been built and will be described briefly in a later section.

It should be emphasized that the machine has been designed from a practical engineering point of view, rather than deduced from some law of physics or mathematics. The various single elements in the design of the machine are based on known principles. The choice between alternate possibilities in each stage of the design has been guided by efficiency and cost considerations. It is claimed that the over-all design uniting these principles in one machine is new and represents a major advance in regard to practicality over suggestions contained in the current literature.

General Design Considerations

From the technological point of view, it is clear that the machine discussed in the preceding section must be a computer. There are two possible choices, analog or digital computer. The latter choice is preferable. The reason is this. An analog computer is basically a method of simulating simple dynamic processes as they occur in the physical universe. The machine in question is required to simulate the actions of man, not of nature. This requires much greater flexibility and at the present state of computer technology such flexibility is provided only by digital computers.

The words "digital" and "analog" used here refer to the *external* characteristics of computers. Mathematically speaking, an analog computer performs the operations of analysis, such as differentiation, integration, computing logarithms, and so on, while a digital computer performs only arithmetic operations; namely, addition and multiplication. An analog computer operates on continuous functions (of time), the digital computer deals with discrete numbers. As far as the *internal* construction of these machines is concerned, it may happen that a computer which is called analog by its user contains discrete components (such as very fast counting circuits); and a computer which is called digital by its user may contain continuous components (such as potentiometers). Following these remarks, the computer that is described later may be called externally digital, internally analog.

In a digital computer, mathematical operations must be expressed (using approximations of various types) in numerical form. For instance, a function such as e^x must be computed by means of a series, which involves only repeated addition and multiplication. Another example is measuring the dynamic characteristics (transfer function or impulse response) of a process. Mathematically, this leads to the problem of solving an integral equation for which no satisfactory analog computing technique exists at present. On a digital computer the problem reduces to solving a set of simultaneous algebraic equations which is much simpler than solving an integral equation.

These considerations suggest the first fundamental design requirement:

(*A*) *The machine must be a digital computer.*

Recall now that the machine has a twofold job; namely, design and control. (i) It must measure the dynamic characteristics of the process and then determine the best form of the controller. (ii) It must control the process by providing the required control action $m(t)$. It is naturally desirable to keep these distinct functions independent. Therefore:

(*B*) *The operations necessary for designing a suitable controller must not be allowed to interact with the control action itself.*

It will be seen later that this requirement cannot be satisfied completely; the degree to which it must be relaxed to provide satisfactory operation is one of the unanswered questions at present.

Special Design Considerations

There are several practical requirements, all quite self-evident, which must be satisfied if the machine is to fulfill the expectations presented in the Introduction. All of these are related to design problem (I).

The functioning of the machine must not be critically dependent on obtaining measurements with high accuracy. Determination of the dynamic characteristics of the process is based on knowledge of $m(t)$ and $c(t)$. Since the first of these is actually produced by the machine itself, it may be assumed to be known with arbitrary accuracy; $c(t)$, however, corresponds to some physical quantity such as temperature, flow, and so on, whose determination is always accompanied by errors due to the imperfect operation of measuring equipment. These errors are called *measurement noise*. The standard method of reducing measurement noise is to take a large number of measurements. This leads to the requirement:

(*C*) *The determination of the dynamic characteristics of the process must be based on a large number of measurements so as to minimize the effects of measurement noise.*

As pointed out in the Introduction, one of the potential advantages of such a machine is that it can constantly repeat the entire design procedure and thereby adjust itself in a manner corresponding to any changes in process characteristics. But because of requirement (*C*), the determination of process characteristics requires a large number of measurements, taking a (possibly) long period of time. Since the system characteristics at the end of a series of measurements may be appreciably different from what they were at the beginning of the series of measurements, it is clear that older measurements ("obsolete data") should not be regarded as being as good as more recent measurements. This may be stated as:

(*D*) *Among any two measurements of $c(t)$, the more recent one should be given the higher weight: Measurements of $c(t)$ made infinitely long ago should be given zero weight.*

The cost, size, probability of breakdown, and so on of the machine is roughly proportional to the number of computations it has to perform per unit time. Therefore other things being equal, the number of computations should be as small as possible:

(*E*) *The methods of numerical computation to be used in the machine should be highly efficient.*

This last requirement will make it possible also to choose between alternative methods of computation.

Computation of Transfer Function From Measurements

Sampling. We now examine in detail the problem of measuri
the dynamic characteristics of the process to be controlled.
do this, the functions $m(t)$ and $c(t)$ must be known. Since

cording to requirement (*A*), the machine is to be a digital computer, it is necessary to replace $m(t)$ and $c(t)$, which are continuously varying functions of time, by sequences of numbers which are discretely varying functions of time. This process is known as *sampling*. The most common way of doing this is to perform measurements periodically. Let the sampling instants be $t = kT$, $k = 0, 1, 2, \ldots$, where T is called the sampling period. Then sampling replaces $m(t)$ and $c(t)$ by the sequences of numbers

$$\begin{matrix} m(0), & m(T), & m(2T), \ldots, m(kT), \ldots \\ c(0), & c(T), & c(2T), \ldots, & c(kT), \ldots \end{matrix} \qquad k = 0, 1, \ldots \quad [1]$$

In order to simplify the notation, we frequently will write $m_k = m(kT)$ and $c_k = c(kT)$ from now on. As a result of the sampling process, all experimental information about the functions $m(t)$ and $c(t)$ is contained in the numbers [1]. The sampling process is illustrated in Fig. 2.

The theory of linear control systems in which some of the controlled quantities are subject to sampling (the so-called sampled-data systems) is well developed. For further information, see Ragazzini and Zadeh (6) and Truxal (7).

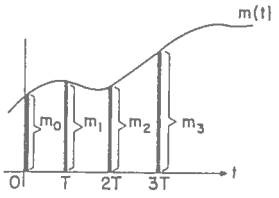

FIG. 2 SAMPLING PROCESS

Step Response of the Process. If the process is *linear, time-invariant, and stable,* it is well known that $c(t)$ is related to $m(t)$ by the convolution integral

$$c(t) = \int_{-\infty}^{t} h(t - u)dm(u) \ldots\ldots\ldots\ldots [2]$$

where $h(t)$ is the step-function response of the process; $h(t) = 0$ when $t < 0$. Once $h(t)$ (or one of its equivalent forms, for instance, its Laplace transform) is known, the dynamic behavior of the process in question is completely characterized. But to find $h(t)$ given $m(t)$ and $c(t)$ by means of Equation [2] requires solving an integral equation which is a very difficult task.

If we consider now the closed-loop system shown in Fig. 1, it is clear that the input $m(t)$ to the process is the output of the self-optimizing controller. Therefore $m(t)$ must depend on the output of a digital computer; in other words, $m(t)$ must be a function of time which is completely determined by its values m_k at the sampling instants. To construct a function $m(t)$ from the series of numbers m_k which has a definite value at every instant of time calls for some method of interpolation. The simplest and practically most frequently used method (6, 7) is to hold the value of $m(t)$ constant after each sampling instant until the next sampling instant. In mathematical notation

$$m(t) = m_k, \qquad kT \leqslant t < (k + 1)T \ldots\ldots\ldots [3]$$

Assuming that $m(t)$ is given by Equation [3], it is easy to show that the convolution integral Equation [2] reduces to the sum

$$c(t) = \sum_{l=-\infty}^{lT<t} h(t - lT)(m_l - m_{l-1}) \ldots\ldots\ldots [4]$$

Noting that $h(kT) = 0$ for all $k < 0$, and considering only sampled values of $c(t)$ and $h(t)$, Equation [4] can be rewritten in the simpler form

$$c_k = \sum_{l=-\infty}^{l=k} (h_{k-l} - h_{k-l-1})m_l = \sum_{l=-\infty}^{l=k} g_{k-l}m_l \ldots\ldots [5]$$

where the g_k's are recognized as the samples of the response of the system to a unit pulse. According to Equation [5], the dynamic behavior of the process is now represented by the sequence of numbers

$$g_0 = h(0), \quad g_1 = h(T) - h(0), \ldots, \quad g_k = h(kT) - h[(k - 1)T], \ldots$$

Moreover, if the input-output sequences [1] are known after some sampling instant, say, $k = 0$, then the numbers g_k can be determined by solving an infinite set of simultaneous linear algebraic equations given by Equation [5]. Since $h_k \to$ const with $k \to \infty$ (otherwise the process would not be stable and therefore Equation [5] would not be valid at all) it can be assumed in practice that $h_k = h_N$ for all $k > N$ if N is sufficiently large. This assumption means that $g_k = 0$ for all $k > N$ so that only a *finite* set of linear algebraic equations has to be solved to get the g_k.

But even with this simplification it would be quite inefficient to represent the process by means of the g_k because this would require a large amount of storage in the digital computer. For instance, if the step response of the process is

$$h(t) = 1 - \exp(-t/\tau)$$

$$g_0 = 0, \qquad g_k = [\exp(T/\tau) - 1] \exp(-kT/\tau), \qquad k \geq 1$$

then approximately $N = 5\tau/T$ numbers are necessary if the error due to neglecting the terms g_k, $k > N$ is to be less than 1 per cent. If fast control is required, the time constant of the closed-loop system must be much less than τ; on the other hand, the response of the closed-loop system on the average cannot take place in less than T seconds. Thus τ/T must be large, which means that a large number of values of g_k must be stored. This and other practical considerations to be discussed later indicate that the numbers g_k do not represent the dynamic characteristics of a process efficiently.

Pulse Transfer Function. A different way to represent a dynamic process is to assume that there is a linear differential equation relating $m(t)$ to $c(t)$. Consequently, m_k and c_k may be assumed to be related by means of a linear difference equation

$$c_k + b_1c_{k-1} + \ldots + b_nc_{k-n} = a_0m_k + a_1m_{k-1} + \ldots + a_qm_{k-q} \ldots\ldots [6]$$

where the a_i and b_i are real constants and b_0 has been set arbitrarily equal to unity. If the differential equation relating $m(t)$ and $c(t)$ is known, the Difference Equation [6] can be derived readily using the theory of sampled-data systems. Such a derivation shows that in general $q = n$. By rearranging Equation [6], it follows that c_k can be expressed in terms of previous inputs and outputs

$$c_k = a_0m_k + a_1m_{k-1} + \ldots + a_nm_{k-n} - b_1c_{k-1} - \ldots - b_nc_{k-n} \ldots\ldots [6a]$$

Usually $a_0 = 0$, since most physical systems do not respond instantaneously. The theoretical difference between Equations [6a] and [4] is that in the latter case in principle all past inputs are needed to determine the present output while in the former case only a finite number of past inputs and outputs is needed. The practical difference is that when the system is known to be governed by a difference equation, much fewer a_i and b_i than g_k are needed to represent the system.

Using the notation $z^ic_k = c_{k+i}$ (where i is any integer), it is possible to write down the following basic relationship between the g_k defined by Equation [5] and the a_i and b_i defined by Equation [6]

$$G(z) = \frac{a_1 z^{-1} + \ldots + a_n z^{-n}}{1 + b_1 z^{-1} + \ldots + b_n z^{-n}} = g_1 z^{-1} + g_2 z^{-2} + \ldots + g_k z^{-k} + \ldots \quad [7]$$

where the right-hand term is obtained by the formal expansion of the rational fraction $G(z)$ by long division according to ascending powers of z^{-1}. The first term, g_0, is missing because it was assumed that $a_0 = 0$ which implies that $h_0 = g_0 = 0$. The function $G(z)$ is called the *pulse transfer function* of the process (6, 7). It has the same role in the analysis of linear sampled-data systems as the transfer function (Laplace transform of a differential equation) in the analysis of linear continuous systems.

The number of the a_i and b_i used to represent the process is based also on an assumption as to what the value of n should be. This is a matter of approximation; in other words, n should be chosen sufficiently large so that the a_i and b_i represent the process with some desired accuracy. But the characteristics of the process are not known in advance so that some initial guess must be made about n in setting up the machine. It is, of course, possible in principle to let the machine check the adequacy of this initial guess once experimental data about the process are available. For simplicity, however, the machine discussed in this paper was designed to operate with a fixed choice of n ($n = 2$).

Finally, it should be recalled that use of the numbers g_k is feasible only if the process is stable. No such restriction is inherent in the representation by Equation [6].

To summarize, the first step in the design of the machine is:

(i) *The dynamic characteristics of the process are to be represented in the form of Equation [6], the coefficients of which are to be computed from measurements. The number $n = q$ is assumed arbitrarily. In general, the higher n, the more accurate the representation of the process by the Difference Equation [6].*

Method of Determining Coefficients. According to design requirement (C), the coefficients in Equation [6] must be determined from a large number of measurements. This can be done as follows: Suppose we make a particular guess for the a_i and b_i at the Nth sampling instant. Let us denote these assumed values by $a_i(N)$ and $b_i(N)$, and compute all the past values of c_k using this particular set of coefficients and Equation [6a]. Denoting by $c_k^*(N)$ the values of the output computed in this way, we have

$$c_k^*(N) = -b_1(N)c_{k-1} - b_2(N)c_{k-2} - \ldots - b_n(N)c_{k-n} + a_1(N)m_{k-1} + a_2(N)m_{k-2} + \ldots + a_n(N)m_{k-n} \quad [8]$$

$$k = 0, 1, \ldots, N$$

A convenient measure of how good this choice of coefficients, in the light of past measured data, is the mean squared error

$$\frac{1}{N}\sum_{k=0}^{k=N} \epsilon_k^2(N) = \frac{1}{N}\sum_{k=0}^{k=N} [c_k - c_k^*(N)]^2 \quad [9]$$

where $\epsilon_k^2(N)$ represents the squared error between measured values c_k in the past and the predicted values $c_k^*(N)$ based on a certain choice of coefficients made at the Nth sampling instant; choosing the coefficients $a_i(N)$ and $b_i(N)$ in such a fashion that the mean squared error Equation [9] is a minimum called *least-squares filtering*. In general, any method for determining the $a_i(N)$ and $b_i(N)$ differs from least-squares filtering only in the form of the appropriate expression to be minimized. The advantage of least-squares filtering is that the computations can be carried out fairly simply (see Appendix), which is usually not the case if other types of error expression are used.

In view of design requirement (D), the more recent measurements should receive greater weight than very old ones, since the process dynamics may change with time. To meet this requirement, we proceed as follows: Let $W(t)$ be a continuous, monotonically decreasing function of time such that

$$\left.\begin{aligned} W(0) &= 1 \\ 0 < W(t) < 1,\ 0 < t &< \infty \\ W(\infty) &= 0 \\ \int_0^\infty W(t)dt &< \infty \end{aligned}\right\} \quad [10]$$

A function satisfying such conditions is called a *weighting function*. Writing W_k for $W(kT)$, the final criterion of determining the coefficients may be stated as follows: Choose $a_i(N)$, $b_i(N)$ in such a way that the expression

$$E(N) = \sum_{k=0}^{k=N} \epsilon_k^2(N) W_{N-k} \quad [11]$$

is a minimum. In other words the errors which would have been committed with the present choice of the coefficients $N - k$ sampling periods ago are to be weighted by a number $0 < W_{N-k} < 1$. Practically speaking, this means that the coefficients are calculated by disregarding errors which would have been committed in predicting the output a very long time ago (when the process may have been different) but trying to keep errors in predicting recent outputs small. None of these considerations, however, determines the precise form of the function $W(t)$; this question will be settled later so that an efficient computation procedure is obtained. We now state the second step in the design of the machine:

(ii) *The coefficients a_i and b_i should be determined anew at each sampling instant so as to minimize the weighted mean-square error $E(N)$.*

Numerical Solution of Weighted Least-Squares Filtering Problem. The explicit process necessary to determine the $a_i(N)$ and $b_i(N)$ requires, even after numerous simplifications, lengthy and somewhat involved calculations. These are discussed and recorded in detail in the Appendix. Only a few remarks are given here:

1 It is necessary to compute a number of so-called pseudo-correlation functions in order to write the error expression $E(N)$ in a simple form. These pseudo-correlation functions embody all measurement data up to the Nth sampling instant which is necessary to compute $E(N)$. To compute $E(N + 1)$, it is necessary to modify the pseudo-correlation functions so as to include the data received at the $(N + 1)$st sampling instant. It turns out that this process can be carried out in a simple way only if W_k is the unit pulse response (cf. Equations [5] and [7]) of a linear system governed by a difference equation. Then computation of the pseudo-correlation functions is carried out by passing products of measured values of m_k and c_k through a linear low-pass filter.

2 In order to apply Equation [6] to characterize a process, it is necessary that m_k and c_k be measured with respect to two reference values m_r and c_r such that, if m_r is a constant input to the system, c_r is the output in the steady state. Since the correct choice of such reference levels is not known in general, they must not enter into the computations of the type of Equation [6a]. In practice, the reference levels are usually determined by extraneous considerations such as calibration and range of measuring instruments. One way of avoiding the effect of incorrect reference levels (so-called *bias errors*) is to pass m_k and c_k through identical high-pass filters. After a sufficiently long period of time the bias errors, which are equivalent to a constant input to the filter, will be attenuated by an arbitrarily large factor at the output of an appropriately designed high-pass filter.

After the pseudo-correlation functions have been obtained, the determination of the coefficients reduces to solving a s

simultaneous linear algebraic equations. To do this efficiently, an iteration procedure is used; it turns out that high-pass filtering m_k and c_k (which is equivalent approximately to subtracting the instantaneous mean value of these series of numbers) is a necessary requirement to insure the convergence of the iteration procedure.

The third step in the design is as follows:

(iii) *The calculations necessary for determining the coefficients consist of modifications of the classical least-squares filtering procedure and are given in the Appendix.*

Optimal Adjustment of Controller

Once the pulse-transfer function of the process to be controlled has been obtained, the synthesis of an "optimal" controller as a set of difference equations becomes a routine task (1, 8, 9).

It is not easy to agree, however, on what constitutes optimal control. The design of an optimal controller depends in general on two considerations:

(*a*) The nature of the input and disturbance signals to the system.

(*b*) The performance criterion used.

For instance, the inputs to the system may consist of step functions of various magnitudes; the performance criterion may be the length of time after the application of the step required by the control system to bring the error within prescribed limits. Or the input may consist of signals which are defined only in the statistical sense, in which case a reasonable performance criterion is the mean squared value of the error signal.

To include in the design of the machine means by which the machine can decide what class of input signals it is subjected to and what type of optimal controller should be used appears to be too ambitious a task at the present time. For this reason, in the practical realization of the machine (see the section Description of Computer), a prearranged method of optimizing the controller was used.

This method was described in a recent note by the author (8). The input signals are to consist of steps. The controller is to be designed in such a fashion that the error resulting from a step input becomes zero in minimum time and remains zero at all values of time thereafter. As a result of these assumptions the optimal controller is described by a difference equation whose coefficients are simple multiples of the coefficients of the pulse transfer function (see Equation [25] in the Appendix.)

We note the last step in the design:

(iv) *The choice of an optimal controller is largely arbitrary, depending on what aspect of system response is to be optimized. The determination of the coefficients in the describing equations of the controller is a routine matter if the coefficients of the pulse-transfer function are known.*

Summary of Machine Organization

Since the describing equations of the self-optimizing controller are somewhat involved, it is helpful to visualize the various computation processes as shown in Fig. 3.

Numbers in brackets indicate equations which characterize the particular operations performed. It should be remembered, of course, that there are many pseudo-correlation functions, coefficients, and so on, to be computed, some of which are indicated only in a schematic fashion.

It is perhaps worth while to emphasize that the closed-loop system consisting of the self-optimizing machine and the process is highly nonlinear. The principal nonlinear operations are:

(*a*) The multiplications before the input to low-pass filters whose outputs are the pseudo-correlation functions.

(*b*) The determination of controller coefficients.

These nonlinear operations have made it necessary to design the self-optimizing machine step by step. There exists at present no general theory for the design of nonlinear control systems of this type.

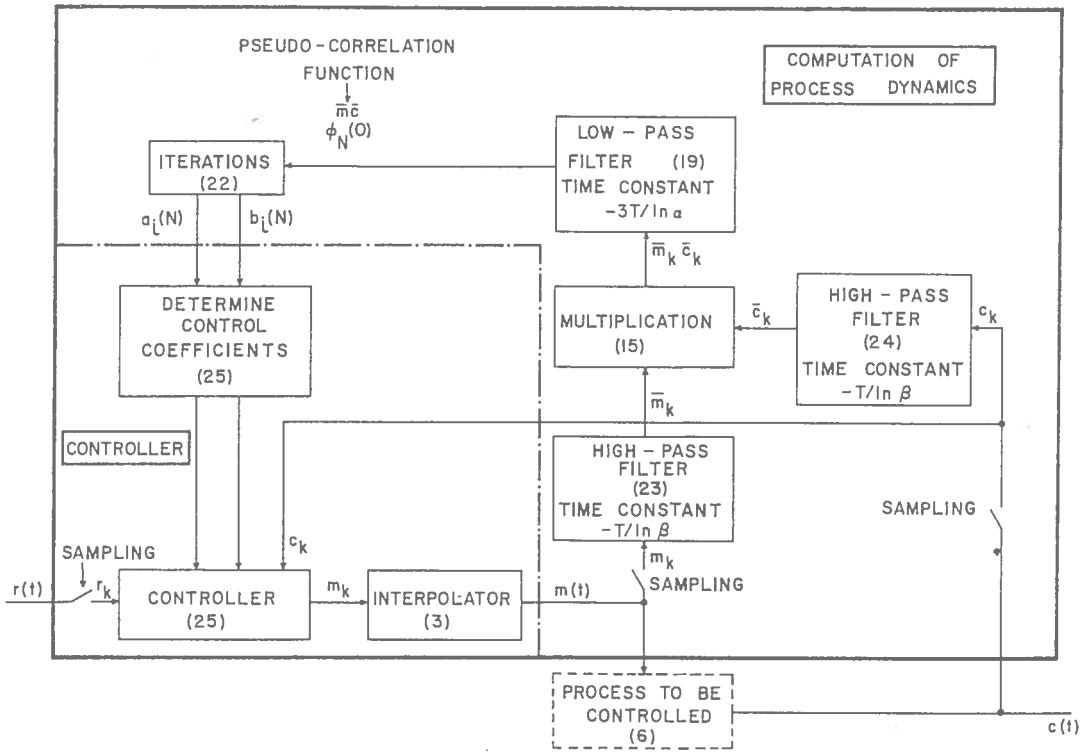

Fig. 3 Block Diagram of Computation Steps for Self-Optimizing Controller

Unsolved Questions

According to the preceding discussion, the operation of the self-optimizing system depends mainly on the accuracy of the computation of the pulse-transfer function from measurement data. Now suppose that the system is under very good control and that the input and disturbances to the system are nearly constant. In that case m_k and c_k will vary only very slightly about their equilibrium values. As a result, the numbers $\bar{m}_k$ and $\bar{c}_k$ (approximately the deviations of m_k and c_k from equilibrium) which are the inputs to the computation process determining the transfer function will be small and of roughly the same order of magnitude as the measurement noise. Under such circumstances, the transfer function cannot be computed very accurately. If the transfer function is not known accurately, then the controller cannot be set up accurately either and the system will not be operating optimally. But then the control will be less good and the deviations from the equilibrium values will increase. This, in turn, will improve the signal-to-noise ratio of the quantities $\bar{m}_k$ and $\bar{c}_k$; the computation of the transfer function will be more accurate, control action more nearly optimal, and so on. This shows that the operation of the system is limited basically by measurement noise. The fluctuations around the equilibrium condition must always be large enough to measure the transfer function with reasonable accuracy even in face of measurement noise. Thus the operation of the system depends on not being entirely at rest; if it were, it is impossible to say anything about the dynamic characteristics of the controlled process. A more precise answer to the problem involved here calls for further study.

Let us now examine qualitatively the effect of the choice α and β (cf. Fig. 3 and Appendix, Equations [19, 23, 24]) on this aspect of system performance. If α is very close to unity, the computation of the pulse-transfer function involves a large number of samples of $\bar{m}_k$ and $\bar{c}_k$ so that even if the system is at rest, i.e., $\bar{m}_k$ and $\bar{c}_k$ are practically zero, the computation of the pulse-transfer function is not affected for a long time, because the system "remembers" results of old measurements. On the other hand, if the process dynamics change rapidly in time, then α should be chosen fairly small because otherwise the computed transfer function will not be the actual transfer function. Thus α is a design parameter whose choice depends somewhat on the nature of a particular situation encountered. There is no reason, of course, why the system cannot adjust α also, but this is a problem beyond the scope of this paper.

The choice of β is guided by similar considerations. If the inputs to the system change slowly then β should be very close to unity for then the low-frequency components in m_k and c_k (slow "drift" about equilibrium point) will be very heavily attenuated. If the system is a more lively one, i.e., m_k and c_k fluctuate appreciably in time due to the effect of inputs or disturbances acting on the system, the β should be chosen smaller to improve the transient response of the high-pass filter. Thus β is another design parameter for the self-optimizing system.

Additional possibilities for improving these aspects of system operation should be considered in future work. More complicated weighting-functions and high-pass filters, suspending the operation of transfer-function computation when signal-to-noise levels become too low, putting in periodic test signals to check the operation of various parts of the computer, and the like, are some topics for future research.

Description of Computer

As soon as the operations discussed in the foregoing sections have been reduced to a set of numerical calculations (see Appendix) the machine has been synthesized in principle. This means that any general-purpose digital computer can be programmed to act as the self-optimizing machine.

In practical applications, however, a general-purpose digital computer is an expensive, bulky, extremely complex, and somewhat awkward piece of equipment. Moreover, the computational capabilities (speed, storage capacity, accuracy) of even the smaller commercially available general-purpose digital computers are considerably in excess of what is demanded in performing the computations listed in the Appendix.

For these reasons, a small special-purpose computer was constructed which could be called externally digital and internally analog according to the terminology in the section General Design Considerations. Briefly, this computer is organized as follows:

The computer operates on numbers whose absolute values do not exceed unity. Each number is represented by a 60-cycle-per-sec (cps) voltage. Numbers are stored on multiturn potentiometers, by positioning a given potentiometer by means of a servo arrangement in such a fashion that its output voltage (with unit excitation) is a 60-cps signal of the required magnitude and sign. Numbers are added by feeding corresponding voltages into electronic summing circuits. Two numbers a and b are multiplied by the following well-known method: If output of the potentiometer with unit excitation is b, then the output of the potentiometer with excitation a will be ab. The storage locations and summers can be interconnected in such a fashion that, in any one step of computation, the computer is capable of performing any one of the following types of operations

$$\left.\begin{aligned} & a_1b_1 + a_2b_2 + \ldots + a_7b_7 = x \\ \text{or} \\ & a_1b_1c_1d_1 + a_2b_2c_2 + a_3b_3c_3 = x \\ \text{or} \\ & a_1b_1c_1d_1e_1f_1g_1h_1 = x \end{aligned}\right\} \quad \ldots\ldots\ldots [12]$$

and so on

where each quantity appearing on the left-hand side of Equations [12] is an arbitrary number; x is the desired result of the computation. The fact that several additions and multiplications can be performed simultaneously is very convenient from the standpoint of programming the computer. Usually, each of Equations [12] must be broken up into several parts in programming them on a general-purpose computer.

The front view of the computer, which is roughly of the size of an average filing cabinet, is shown in Fig. 4. Only connections for input-output signals appear on the front panel. The programming of the computer is achieved by inserting wires into a "patch panel" on top of the computer which is shown in Fig. 5. Almost every signal voltage inside the computer is brought out to some contact on the patch panel. This arrangement makes it possible to interconnect the basic components of the computer in any manner desired and also facilitates troubleshooting and maintenance. The disadvantage of a patch-panel type of programming is that the change of program is a time-consuming operation; however, this is of minor significance since the machine is intended to operate with a fixed program in any typical application. The control panel shown in Fig. 5 also contains means for changing the sampling rate and reading numbers into any one of the storage locations in the computer.

The wiring necessary to connect computer components with the patch panel, together with associated relays, timing and checking circuits takes up approximately one third of the volume of the computer. Another one third of the volume is required for the electronic circuits performing summation and multiplication and

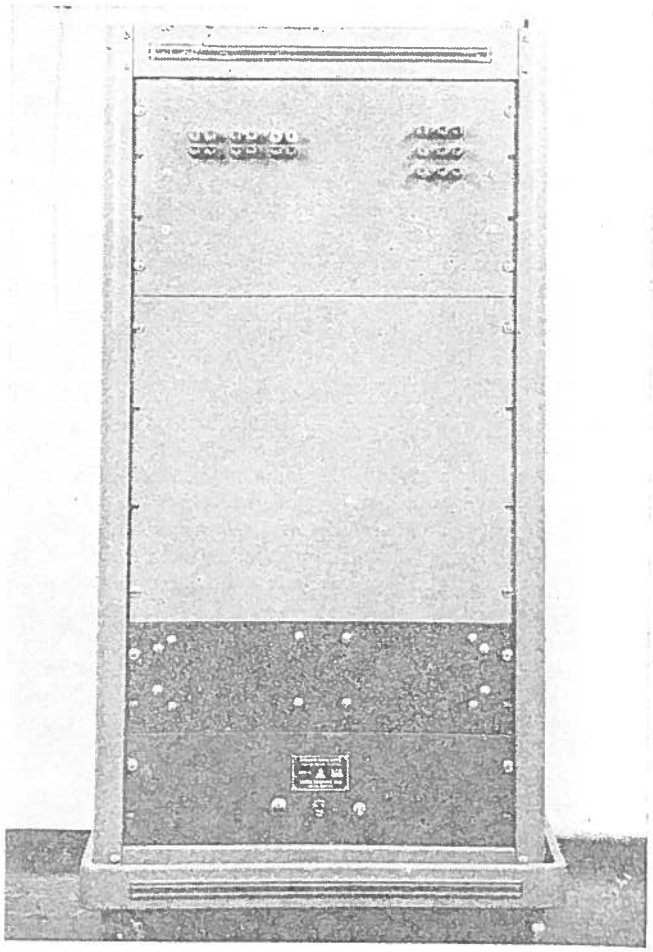

FIG. 4 FRONT VIEW OF COMPUTER

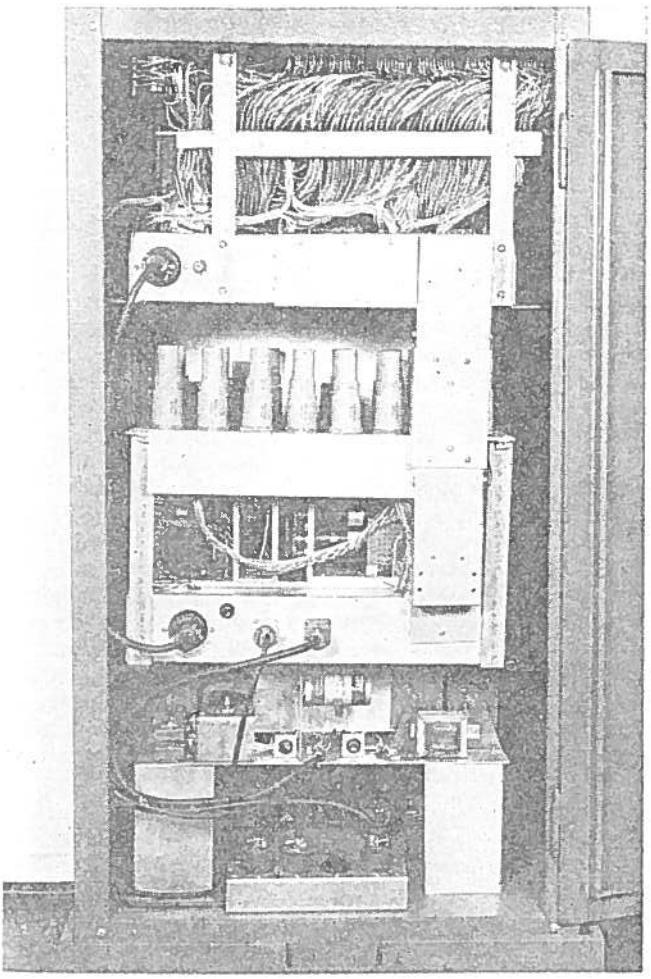

FIG. 6 REAR VIEW OF COMPUTER

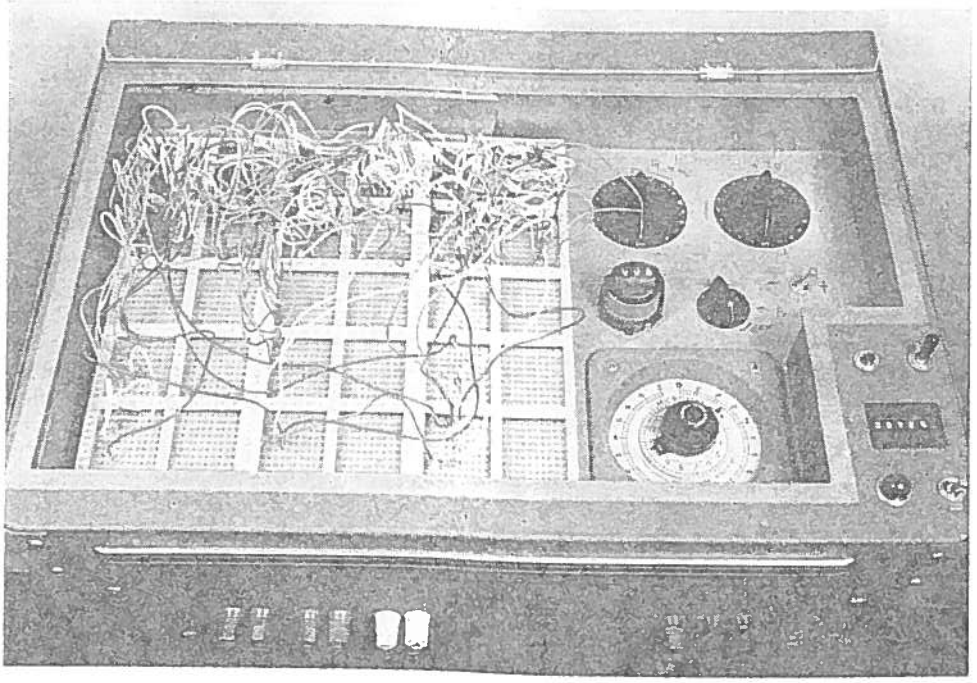

FIG. 5 CONTROL PANEL OF COMPUTER

the storage potentiometers. The remaining one third of space is taken up by power supplies. The internal arrangement of the computer is shown in the rear view of Fig. 6.

The computer described shows that the practical realization of a self-optimizing machine is well within the technological means available at the present time. Actually, the computer described was constructed in 1954/1955. The computer also represents savings in cost and complexity over currently available general purpose digital computers. On the other hand, when self-optimizing control of a large-scale installation is desired, in other words, when there are several dynamic processes to be controlled simultaneously and possibly in an interdependent fashion, then the general-purpose digital computer is much better matched to the problem both in terms of cost and computational capability.

Conclusions

This paper shows the feasibility of mechanizing much of the process by which automatic control systems for standard applications are being designed today. The amount of numerical computations necessary for accomplishing this is relatively modest (after the numerous simplifications discussed) and can be readily implemented in practice at moderate cost.

More importantly, however, the machine described here is an ideal controller since it needs merely to be interconnected with the process to be controlled to achieve optimum control after a short transitory period and hold it thereafter even if the process characteristics change with time. The task of the control engineer of the future will be not to design a specific system, but to improve the principles on which machines of the type described here will operate. Unlike his predecessor, the stock in trade of the new control-systems engineer will not be the graph paper, the slide rule, or even the analog computer but a firm and deep-seated understanding of the fundamental principles, physical and mathematical, on which automatic control is based. The drudgery of computing will be taken over by machines but the challenge of thinking remains.

Acknowledgments

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Appendix

The following is the detailed derivation of the complete set of equations characterizing the self-optimizing controller in the special case when $n = 2$ in the Difference Equation [6]. Using these equations, any digital computer may be programmed to act as a self-optimizing controller. When $n > 2$, the required equations can be obtained similarly.

First of all, instead of performing the computations required to minimize Equation [11] at every sampling instant, they may be performed at every qth (where q is a positive integer) sampling instant. This does not affect the reasoning in the section Method of Determining Coefficients, and results in considerable simplification in the required computations. With this change, the error expression Equation [11] becomes

$$E(N) = \sum_{j=0}^{j=N/q} \epsilon_{qj}^2(N) W_{N-qj} \quad \ldots\ldots\ldots\ldots [13]$$

where $k = qj$ and N is a number divisible by q.

Now assume that $n = 2$ in Equation [6]. Using the recurrence relation Equation [8], $\epsilon_{qj}^2(N)$ can be written as

$$\left.\begin{aligned}
\epsilon_{qj}^2(N) &= [c_{qj} - c_{qj}^*(N)]^2 \\
&= c_{qj}^2 + b_1^2(N)c_{qj-1}^2 + b_2^2(N)c_{qj-2}^2 \\
&\quad + 2b_1(N)c_{qj}c_{qj-1} + 2b_2(N)c_{qj}c_{qj-2} \\
&\qquad + 2b_1(N)b_2(N)c_{qj-1}c_{qj-2} \\
&\quad - 2a_1(N)c_{qj}m_{qj-1} - 2a_2(N)c_{qj}m_{qj-2} \\
&\quad - 2b_1(N)a_1(N)c_{qj-1}m_{qj-1} \\
&\qquad - 2b_1(N)a_2(N)c_{qj-1}m_{qj-2} \\
&\quad - 2b_2(N)a_1(N)c_{qj-2}m_{qj-1} \\
&\qquad - 2b_2(N)a_2(N)c_{qj-2}m_{qj-2} \\
&\quad + a_1^2(N)m_{qj-1}^2 + a_2^2(N)m_{qj-2}^2 \\
&\qquad + 2a_1(N)a_2(N)m_{qj-1}m_{qj-2}
\end{aligned}\right\} \quad \ldots [14]$$

The measured values of c and m occur in Equation [14] always in terms of the type

$$c_{qj-r}c_{qj-s} \qquad c_{qj-r}m_{qj-s} \qquad m_{qj-r}m_{qj-s} \quad \ldots\ldots\ldots [15]$$

where $r, s = 0, 1, 2$. If we now let

$$q = n + 1 = 3$$

then it is clear that factors of the same type will be multiplied by the same coefficients in Equation [14], regardless of the value of j. This property does not arise when $q < 3$. Using the symmetry introduced by the particular choice of q, $E(N)$ can be put in a simpler form by defining the *pseudo-correlation functions*

$$\left.\begin{aligned}
\phi_{N-r}^{cc}(r - s) &= \sum_{j=1}^{j=N/3} c_{3j-r}c_{3j-s}W_{N-3j} \\
\phi_{N-r}^{cm}(r - s) &= \sum_{j=1}^{j=N/3} c_{3j-r}m_{3j-s}W_{N-3j} \\
\phi_{N-r}^{mm}(r - s) &= \sum_{j=1}^{j=N/3} m_{3j-r}m_{3j-s}W_{N-3j}
\end{aligned}\right\} \quad \ldots . [16]$$

With these definitions, $E(N)$ can be written as follows, arranging the terms in the same fashion as in Equation [14]

$$\left.\begin{aligned} E(N) = {} & \phi_N{}^{cc}(0) + b_1{}^2(N)\phi_{N-1}{}^{cc}(0) + b_2{}^2(N)\phi_{N-2}{}^{cc}(0) \\ & + 2b_1(N)\phi_N{}^{cc}(-1) + 2b_2(N)\phi_N{}^{cc}(-2) \\ & \qquad + 2b_1(N)b_2(N)\phi_{N-1}{}^{cc}(-1) \\ & - 2a_1(N)\phi_N{}^{cm}(-1) - 2a_2(N)\phi_N{}^{cm}(-2) \\ & - 2b_1(N)a_1(N)\phi_{N-1}{}^{cm}(0) \\ & \qquad - 2b_1(N)a_2(N)\phi_{N-1}{}^{cm}(-1) \\ & - 2b_2(N)a_1(N)\phi_{N-2}{}^{cm}(1) - 2b_2(N)a_2(N)\phi_{N-2}{}^{cm}(0) \\ & + a_1{}^2(N)\phi_{N-1}{}^{mm}(0) + a_2{}^2(N)\phi_{N-2}{}^{mm}(0) \\ & \qquad + 2a_1(N)a_2(N)\phi_{N-1}{}^{mm}(-1) \end{aligned}\right\} \quad \ldots [17]$$

Remark. The conventional definition of correlation functions is

$$\phi_N{}^{cc}(r) = \frac{1}{N}\sum_{k=0}^{k=N} c_k c_{k+r}$$

To evaluate this function iteratively, as is done in Equation [19] for pseudo-correlation functions, it would be necessary to compute

$$\phi_N{}^{cc}(r) = c_N c_{N+r}/N + (N-1)\phi_{N-1}{}^{cc}(r)/N$$

Since the factor $(N-1)/N$ cannot be calculated accurately enough as $N \to \infty$, such an iterative calculation would be impractical.

The pseudo-correlation functions can be evaluated iteratively as follows: Suppose that, in addition to meeting Conditions [10], the weighting function W_k is a sequence of numbers such as the g_k given by Equation [7]. Then it follows that the pseudo-correlation functions can be regarded as the output of a linear system governed by a difference equation, whose input consists of products such as Equation [15]. In particular, if we let

$$W^{3j} = \alpha^j \qquad (0 < \alpha < 1) \ldots\ldots\ldots\ldots [18]$$

then every pseudo-correlation function satisfies a first-order difference equation of the type

$$\phi_{3j-r}{}^{cm}(r-s) - \alpha\phi_{3(j-1)-r}{}^{cm}(r-s) = c_{3j-r}m_{sj-s} \ldots [19]$$

According to Equation [17] the determination of the coefficients of the pulse-transfer function requires first that all input-output data (the measured values of c and m) be consolidated into the pseudo-correlation functions. Because of the recurrence relation Equation [19], the computation of the latter is quite simple, since to get the pseudo-correlation functions at the Nth sampling instant requires only the knowledge of the same functions at the end of the $(N-3)$th sampling instant, plus the values of c_{N-2}, c_{N-1}, c_N, m_{N-2}, m_{N-1}. Once the new pseudo-correlation functions have been computed, the data measured during the preceding three sampling periods can be discarded and the system is ready to receive new data. Thus the use of the pseudo-correlation functions and the choice of a suitable weighting function greatly simplifies the implementation of mean-square filtering.

In order that $E(N)$ be a minimum with respect to the a_i and b_i, it is *necessary* that the partial derivatives

$$\frac{\partial E(N)}{\partial a_i} = 0 \qquad \frac{\partial E(N)}{\partial b_i} = 0 \quad (i = 1, 2, \ldots, n) \ldots [20]$$

vanish. The proof that these conditions are also *sufficient* to insure the existence of a minimum of $E(N)$ is quite difficult. Refer to Milne (11) for discussion of a closely related problem.

The Conditions [20] lead to four linear equations in the coefficients $a_1(N)$, $a_2(N)$, $b_1(N)$, $b_2(N)$ as follows

$$\left.\begin{aligned} & a_1(N)\phi_{N-1}{}^{mm}(0) + a_2(N)\phi_{N-1}{}^{mm}(-1) - b_1(N)\phi_{N-1}{}^{cm}(0) \\ & \qquad - b_2(N)\phi_{N-2}{}^{cm}(1) = \phi_N{}^{cm}(-1) \\ & a_1(N)\phi_{N-1}{}^{mm}(-1) + a_2(N)\phi_{N-2}{}^{mm}(0) \\ & \quad - b_1(N)\phi_{N-1}{}^{cm}(-1) - b_2(N)\phi_{N-2}{}^{cm}(0) = \phi_N{}^{cm}(-2) \\ & -a_1(N)\phi_{N-1}{}^{cm}(0) - a_2(N)\phi_{N-1}{}^{cm}(-1) \\ & \quad + b_1(N)\phi_{N-1}{}^{cc}(0) + b_2(N)\phi_{N-1}{}^{cc}(-1) = -\phi_N{}^{cc}(-1) \\ & -a_1(N)\phi_{N-2}{}^{cm}(1) - a_2(N)\phi_{N-2}{}^{cm}(0) \\ & \quad + b_1(N)\phi_{N-1}{}^{cc}(-1) + b_2(N)\phi_{N-2}{}^{cc}(0) = -\phi^{cc}(-2) \end{aligned}\right\} \quad \ldots [21]$$

Any method for solving linear simultaneous equations can be used for finding the a_i and b_i from Equation [21]. However, the standard elimination methods (which, incidentally, are much more efficient than solving Equation [21] by Cramer's rule) require a rather large amount of storage and somewhat lengthy computations. These disadvantages become increasingly worse as n increases. However, an exact computation of a solution of Equation [21] is very wasteful in that, if a solution of Equation [21] at the $(N-3)$th sampling instant is available, then that solution is also an excellent guess for the solution of Equation [21] at the Nth sampling instant since the correlation function can have changed only slightly, unless a very small value of α is used. This suggests an *iteration* procedure for solving Equation [21], of which the simplest is the so-called Gauss-Seidel method (10).

Applying the Gauss-Seidel method to Equation [21] leads to the equations

$$a_1(N) = \frac{-a_2(N-3)\phi_{N-1}{}^{mm}(-1) + b_1(N-3)\phi_{N-1}{}^{cm}(0) + b_2(N-3)\phi_{N-2}{}^{cm}(1) + \phi_N{}^{cm}(-1)}{\phi_{N-1}{}^{mm}(0)} \ldots [22a]$$

$$a_2(N) = \frac{-a_1(N)\phi_{N-1}{}^{mm}(-1) + b_1(N-3)\phi_{N-1}{}^{cm}(-1) + b_2(N-3)\phi_{N-2}{}^{cm}(0) + \phi_N{}^{cm}(-2)}{\phi_{N-2}{}^{mm}(0)} \ldots [22b]$$

$$b_1(N) = \frac{a_1(N)\phi_{N-1}{}^{cm}(0) + a_2(N)\phi_{N-1}{}^{cm}(-1) - b_2(N-3)\phi_{N-1}{}^{cc}(-1) - \phi_N{}^{cc}(-1)}{\phi_{N-1}{}^{cc}(0)} \ldots [22c]$$

$$b_2(N) = \frac{a_1(N)\phi_{N-1}{}^{cm}(1) + a_2(N)\phi_{N-2}{}^{cm}(0) - b_1(N)\phi_{N-1}{}^{cc}(-1) - \phi_N{}^{cc}(-2)}{\phi_{N-2}{}^{cc}(0)} \ldots [22d]$$

If desired, the cycle of iterations just written down can be repeated to obtain better accuracy.

A necessary and sufficient condition for the convergence of the iteration Equations [22] is that the diagonal coefficients in Equations [21], i.e., $\phi_{N-1}{}^{mm}(0)$, $\phi_{N-2}{}^{mm}(0)$, $\phi_{N-1}{}^{cc}(0)$, $\phi_{N-2}{}^{cc}(0)$ should be larger in absolute value than any of the other coefficients in the same equation. To insure rapid convergence, it is highly desirable that the diagonal coefficients be as large as possible compared to the off-diagonal coefficients.

A glance at Equation [19] shows that the pseudo-correlation

functions just mentioned are always the sum of positive numbers because the right-hand side of Equation [19] is always positive, being a square. To make the pseudo-correlation functions corresponding to the off-diagonal elements in Equations [21] smaller in absolute value than the diagonal elements, the right-hand side of Equation [19] for these functions must be alternatively positive and negative. This can be achieved by subtracting from each c_k and m_k the average (mean) values of these quantities over a long period of time. Unless this is done, c_k and m_k might vary only slightly about a large average value in which case all the correlation functions will be approximately equal and the iteration Equation [22] will not converge fast enough, if at all.

To estimate the mean of a time series in a very reliable way is not an easy problem. In the present case, however, sophisticated statistical methods are not required because the precise knowledge of the mean is not important. The simplest procedure then is to put both c_k and m_k through identical high-pass filters which remove the slowly varying components (i.e., the mean) of these quantities. When the mean is constant in time, it is equal to the zero frequency component of the signal. The simplest high-pass filter on numerical data is represented by the difference equation

$$c_k - c_{k-1} = \bar{c}_k - \beta\bar{c}_{k-1} \qquad (0 < \beta < 1) \ldots\ldots [23]$$

where $\bar{c}_k$ is approximately equal to c_k − mean (c_k). The closer β is to 1, the better the removal of the mean if the latter is constant. On the other hand, if the mean varies β should be somewhat smaller for best results. A similar equation holds for $\bar{m}_k$

$$m_k - m_{k-1} = \bar{m}_k - \beta\bar{m}_{k-1} \qquad (0 < \beta < 1) \ldots\ldots [24]$$

A simple substitution in Equation [6] shows that $\bar{c}_k$ and $\bar{m}_k$ are related by the same difference equation as c_k and m_k. This is because if two quantities are linearly related, the relationship remains undisturbed if both quantities are put through identical linear filters. Thus the removal of the mean represented by Equations [23] and [24] does not affect the computation of the pulse-transfer function of the process to be controlled, except for greatly improving the convergence of the iteration process Equations [22]. Hence all pseudo-correlation functions should be computed using the $\bar{c}_k$ and $\bar{m}_k$.

It remains to show how the equations of the controller can be obtained from the knowledge of the coefficients of the pulse-transfer function. As mentioned earlier, the controller is to be digital. Using a method of synthesis due to the author (8), which yields the optimum design if the closed-loop system is to respond to a unit step input in minimal time without overshoot (for a given fixed sampling period T), the numbers necessary to specify the controller are very simply related to the coefficients of the pulse-transfer function of the process which is to be controlled. In fact, the difference equation specifying the controller is

$$[a_1(N) + a_2(N)]m_k - a_1(N)m_{k-1} - a_2(N)m_{k-2} = e_k + b_1(N)e_{k-1} + b_2(N)e_{k-2} \ldots\ldots [25]$$

where $$e_k = r_k - c_k$$

Equation [25] is valid for $N + 1 \leq k \leq N + 3$, after which a new set of coefficients must be used from the next determination of the pulse-transfer function. It should be noted that Equation [25] holds only if the (continuous) transfer function of the process is approximately $H(s) = K/(s + a)(s + b)$ with $a, b, K > 0$. If, for instance, $a = 0$, the form of Equation [25] is different. For methods of synthesizing digital controllers which are optimal in some other sense, see references (1, 9).

For convenience, the time sequence of computations to be performed during a cycle of $q = 3$ sampling periods is listed as follows.

$k = N - 2$

(1) Compute m_{N-2} using [25]
(2) Compute $\bar{c}_{N-2}$ using [23]
(3) Compute $\bar{m}_{N-2}$ using [24]
(4) Compute $\phi_{N-2}^{\bar{m}\bar{m}}(0)$, $\phi_{N-2}^{\bar{c}\bar{c}}(0)$, $\phi_{N-2}^{\bar{c}\bar{m}}(0)$ using Equation [19]

$k = N - 1$

(1) Compute m_{N-1} using [25]
(2) Compute $\bar{c}_{N-1}$ using [23]
(3) Compute $\bar{m}_{N-1}$ using [24]
(4) Compute $\phi_{N-1}^{\bar{m}\bar{m}}(0)$, $\phi_{N-1}^{\bar{m}\bar{m}}(-1)$, $\phi_{N-1}^{\bar{c}\bar{c}}(0)$, $\phi_{N-1}^{\bar{c}\bar{c}}(-1)$ $\phi_{N-1}^{\bar{c}\bar{m}}(0)$, $\phi_{N-1}^{\bar{c}\bar{m}}(-1)$, $\phi_{N-2}^{\bar{c}\bar{m}}(1)$ using Equation [19]

$k = N$

(1) Compute m_N using [25]
(2) Compute $\bar{c}_N$ using [23]
(3) Compute $\bar{m}_N$ using [24]
(4) Compute $\phi_N^{\bar{c}\bar{c}}(-1)$, $\phi_N^{\bar{c}\bar{c}}(-2)$, $\phi_N^{\bar{c}\bar{m}}(-1)$, $\phi_N^{\bar{c}\bar{m}}(-2)$ using Equation [19]
(5) Compute $a_1(N)$, $a_2(N)$, $b_1(N)$, $b_2(N)$ using [22]

Discussion

RANE L. CURL.[3] The author has presented with skill his proposal for a self-optimizing control system. He has also covered most of the limitations in both the theory and design of his machine. I will only mention perhaps one or two points that come to mind.

On the first stage of the author's procedure, *measure the dynamic characteristics of the process*, a difficulty would be met in most real processes of the regulatory type. The proposed method of determining the system characteristics is subject to error when the existence of an error signal is due to load disturbances entering between the *control effort* and the *output*. This error may be of two types. The first is from poor "response" information in the presence of noise, and is inherent in any method which does not use process response information over a very long time. The desire to make the self-optimizing machine respond to changes in process dynamics is anathema to obtaining a good measure of the transfer function in the presence of noise. The second type of error is inherent in *all* methods which determine process dynamics while the process is on closed loop control. The noise circulates in the loop and there exists a correlation between the *noise component* of the output $c(t)$, and the control effort $m(t)$.

The importance of the regulatory type of controller and the difficulty of obtaining good process dynamics when it is in use suggests a reason additional to that of the author as to why technological unemployment of control engineers will not result from this machine.

The author's use of $n = 2$, while a strict limitation, was, as the author correctly stated, a matter of convenience and not an inherent limitation. It would be of interest if the author would comment on the behavior of the machine described in his paper when used with systems having incompatible transfer functions, i.e., for processes for which Equation [25] does not represent the optimum controller.

The well known "optimizing" controller for adjusting a set point in order to maximize yield, profit, etc., introduces its own disturbance as a "tracer" on system performance. This is another possibility, in some cases, to computation suspension at low signal to noise ratios as in the author's machine.

I agree with the author that this machine does "represent. . . an advance. . . .in practicality over suggestions. . . .in the current literature." But I ask last the primary unanswered question: Does it work?

[3] Shell Development Company, Emeryville, Calif.

Author's Closure

Before taking up in detail the questions raised in Dr. Curl's discussion, the author wishes to answer his last and most important point, "Does it work?" The answer is, "Yes."

Dr. Curl's remarks on difficulties of determining the process transfer function amplify some of the matters discussed in the section, Unsolved Questions. As in any method of measurement based on statistical principles, the determination of the process dynamics depends on obtaining a large number of data with stationary statistical properties so that the effect of unwanted influences acting on the system can be averaged out. If the load disturbances have a nonzero mean value, then their effect on the plant appears as a shift in the operating point. The computation procedure determines the linear system dynamics for small deviations about this "phantom" operating point. Since the computation of the transfer function can take account of slow changes, shifts in the mean value of the load disturbances do not affect the operation of the system, provided that these shifts occur slowly relative to the sampling period. The accuracy of computation of the transfer function depends on the effective signal-to-noise ratio, that is, on the ratio of the mean-square value of the control effort $m(t)$ required under normal operating conditions to the mean-square value of the combined effect of load disturbances and measurement noise. When this ratio is too small, it may be improved by introducing special "test signals" into the plant, or the operation of the transfer-function computation may be temporarily suspended until the signal-to-noise ratio is improved.

The effect of circulating noise determines the maximum accuracy achievable by a self-optimizing system and can, in general, only be determined experimentally. If the effect is too large, more accurate instrumentation must be used. It should be borne in mind also that since the controller of a self-optimizing system is closely matched to the dynamics of the plant, any errors due to circulating noise can be rapidly corrected. In other words, measurement noise is not amplified by the system.

By way of illustration, it may be pointed out that measurements performed by the author using high-accuracy measuring equipment support the foregoing remarks. The computation of the transfer function of a crude third-order electrical analog (3 capacitors in cheap electronic circuity without voltage regulation) yielded the following experimental results, over about 500 sampling points:

Largest time constant $\tau_1 \cong$ constant $\pm$ 0.1 per cent
Next time constant $\tau_1/3 \cong$ constant $\pm$ 1.0 per cent
Smallest time constant $\tau_1/10 \cong$ constant $\pm$ 10 per cent

The high accuracy with which the dominant time constant τ_1 can be determined is quite remarkable. The variation is only slightly worse than the errors introduced by the measuring process. On the other hand, the large error in the determination of the smallest time constant is due to the combined effect of amplifier noise, temperature transients, and so forth. From this measurement, it may be concluded that the system may be regarded as effectively second-order. Indeed, the system could be controlled quite satisfactorily with a sampled-data controller with a fixed, second-order program. Conclusive results concerning the performance of the self-optimizing controller in an actual plant installation cannot be given here.

In conclusion, the author does not share Dr. Curl's pessimism that the presence of noise problems makes a self-optimizing system impractical. Probably the most serious practical difficulty barring better process control at the present time is the unavailability of accurate data on process dynamics. This difficulty can be circumvented in many cases by use of a self-optimizing controller. The author may not be unduly optimistic in expressing his feeling that (disregarding economic considerations) sufficient theoretical and technological know-how exists already to bring practical process control close to the best performance achievable in the light of the limitations imposed by physical measuring equipment.