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THE INTEGRAL-OF-ERROR-SQUARED CRITERION FOR SERVO MECHANISMS

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SUMMARY

A graphical method is given for obtaining the integral of error-squared for a servo mechanism supplied with a given input signal, and the usefulness of this criterion is discussed. The servo mechanism is characterized by its open-loop response locus, and the input signal by its power spectrum. The method is extended to allow for noise at the input.

(1) INTRODUCTION

It has been suggested[1] that the integral of error-squared,

$$S = \int_0^{\infty} \epsilon^2(t)dt \quad \ldots \ldots \ldots \quad (1)$$

would form a simple and convenient criterion of performance for a servo mechanism. Under certain conditions which are usually fulfilled S can be obtained from information about the error as a function of frequency; in fact,[2]

$$S = \frac{1}{\pi}\int_0^{\infty} |\epsilon(\omega)|^2 d\omega \quad \ldots \ldots \quad (2)$$

Moreover, for a linear system having an open-loop transfer function $G(p)$ we have

$$|\epsilon(\omega)|^2 = \frac{|\theta_i(\omega)|^2}{|1 + G(\omega)|^2} \quad \ldots \ldots \quad (3)$$

where $|\theta_i(\omega)|^2$ is proportional to the quadratic spectrum of the input. Thus the input signal may be characterized by its frequency spectrum alone, and in simple cases S can be evaluated analytically from eqns. (2) and (3).

In those cases where S has been calculated, a step-function has often been used for the input θ_i. It is then found that the system which gives the least value of S has too little damping to be satisfactory in practice. Westcott,[3] following Hall,[1] has suggested that this is because the criterion penalizes a system for failing to respond immediately after a step change of input, which the system is physically incapable of doing. This is equivalent to saying that a servo mechanism cannot respond to arbitrarily high frequencies, and should not be penalized for failing to do so. In place of S, Westcott suggests as a criterion W, where

$$W = \int_0^{\infty} t\epsilon^2(t)dt \quad \ldots \ldots \ldots \quad (4)$$

A graphical method will now be given by which S may be found for a given system supplied with a given input signal. The system may be characterized by its open-loop harmonic-response locus, and the input by its power spectrum. Some examples will be worked out, and then the argument against S as a criterion of performance will be re-examined.

(2) GRAPHICAL METHOD

From eqns. (2) and (3) we have

$$S = \frac{1}{\pi}\int_0^{\infty} \frac{|\theta_i(\omega)|^2}{|1 + G(\omega)|^2} d\omega \quad \ldots \ldots \quad (5)$$

If now we plot $\quad y = |1 + G(\omega)|^{-2} \quad \ldots \ldots \quad (6)$

against $\quad x = \int_{\omega_0}^{\omega} |\theta_i(\omega)|^2 d\omega \quad \ldots \ldots \quad (7)$

we have

$$\int_{\omega=0}^{\infty} y dx = \int_0^{\infty} y \frac{dx}{d\omega} d\omega$$

$$= \int_0^{\infty} \frac{|\theta_i(\omega)|^2}{|1 + G(\omega)|^2} d\omega$$

$$= \pi S \quad \ldots \ldots \ldots \ldots \quad (8)$$

When the signal θ_i has finite energy, ω_0 may be zero, and it is convenient to use the dimensionless quantity P instead of S, where

$$P = \frac{\int_0^{\infty} \frac{|\theta_i(\omega)|^2}{|1 + G(\omega)|^2} d\omega}{\int_0^{\infty} |\theta_i(\omega)|^2 d\omega} \quad \ldots \ldots \quad (9)$$

In Figs. 3 and 4, P is given by the ratio of the area under the curve $y = |1 + G|^{-2}$ to that under the line $y = 1$. For such input signals as an infinitely-long step the integral (7) diverges as ω_0 tends to zero, and the base-line of x is infinitely long, as in Figs. 5 and 6.

The scale of x depends only on the power spectrum of the input and not on the properties of the servo mechanism. The scale of y is invariable. Hence charts can easily be prepared either for a given input characterized by a measured spectrum, or for inputs considered typical under given conditions.

The value of $|1 + G|$ is readily available from the open-loop harmonic-response locus of the system, and is merely the distance from the locus to the point $(-1, 0)$. The frequency response of the system need not be given analytically, but may be measured. The effect upon P of a change in the open-loop gain is particularly easy to find, since it is only necessary to choose a new point to be $(-1, 0)$, change the scale of the response locus appropriately, and plot the new values of $|1 + G|$ against the same scale of x.

Besides giving the value of P (or S), the graphical method indicates the frequencies at which the greatest contributions to P (or S) arise. It therefore shows the frequencies at which an improvement of the system will be most profitable.

Written contributions on papers published without being read at meetings are invited for consideration with a view to publication.
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(3) EXAMPLES

The servo mechanism which will be considered is that treated by Westcott[3] and shown in Fig. 1. It consists of a Ward Leonard

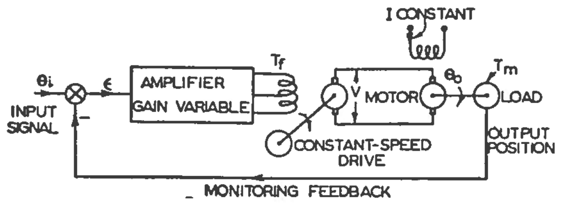

Fig. 1.—Servo system used as an example.

set driving an inertia load, the generator having a split field fed differentially by a valve amplifier of adjustable gain. We have

$$G(\omega) = \frac{K_v}{-j\omega^3 T_m T_f - \omega^2(T_m + T_f) + j\omega} \quad . \quad . \quad (10)$$

where $K_v = K_1K_2$ = Velocity-error constant.
K_1 = Amplifier output voltage per unit error angle, volts/rad.
K_2 = Velocity-constant of the motor, rad/sec/volt.
T_m = Mechanical time-constant of the motor and its load, sec.
T_f = Time-constant of the generator field, sec.

We shall put $T_m = 5$ sec and $T_f = 1$ sec, and shall treat K_v as the design parameter. Westcott has shown analytically that for a step-function input S is least for $K_v = 0 \cdot 33$, while W is least for $K_v = 0 \cdot 21$. This last setting gives a transient response which would be accepted as satisfactory on the basis of practical experience.

The first example uses an input having a power spectrum proportional to $\varepsilon^{-k\omega}$. The form of the spectrum is shown in curve (i) of Fig. 2, with four scales (*a*), (*b*), (*c*) and (*d*), corre-

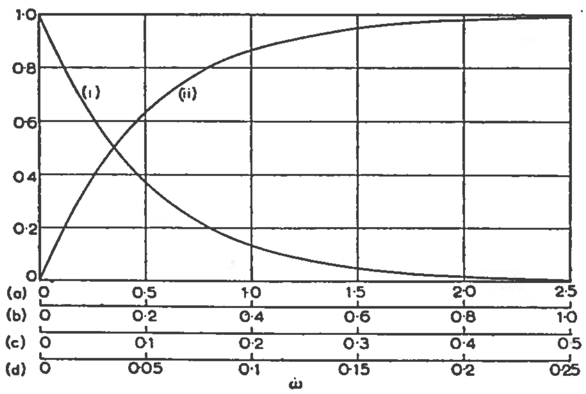

Fig. 2.—Power spectrum of the input signal.

Curve (i) shows the form of the assumed power spectrum, which is proportional to $\varepsilon^{-k\omega}$. The four scales (*a*), (*b*), (*c*) and (*d*) are drawn respectively for k=2, 5, 10 and 20. Curve (ii) gives kx where $x = \int_0^\omega \varepsilon^{-k\omega} d\omega$. The resonant frequency of the servo system is near $\omega = 0 \cdot 25$.

sponding respectively to $k = 2$, 5, 10 and 20. Increase of k from k_1 to k_2 is equivalent in its effect on P to reducing all the time-constants of the servo mechanism in the ratio k_1/k_2. Curve (ii) of Fig. 2 shows kx, where $x = \int_0^\omega \varepsilon^{-k\omega} d\omega$.

Fig. 3 shows the error chart for an input having its power spectrum proportional to $\varepsilon^{-2\omega}$ and curve (*a*) of Fig. 8 shows the

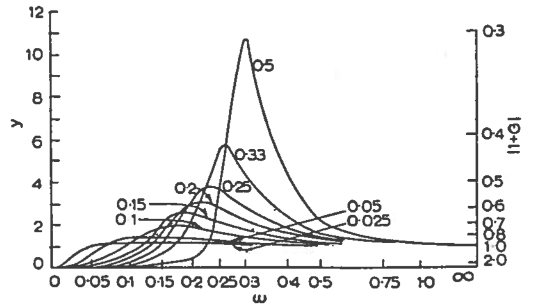

Fig. 3.—Error chart for the system of Fig. 1 when the input signal has a power spectrum proportional to $\varepsilon^{-2\omega}$.

variation of P with K_v. The criterion here calls for very high damping. Fig. 3 represents a case which is likely to occur only if the system is required to give a smoothed representation of the input.[4] In such a case very high damping would, in fact, be used. Although the least value of P is nearly unity, Fig. 3 shows that the low-frequency components of the input are accurately followed.

Curve (*b*) of Fig. 8 is drawn for the input represented by scale (*b*) of Fig. 2. To the accuracy of drawing and measurement used, the minimum value of P has the same value as in the last example and occurs at the same value of K_v. The only advantage derived from the increased relative value of the servo-mechanism resonant frequency seems to be a reduction in the penalty for using too large a value of K_v.

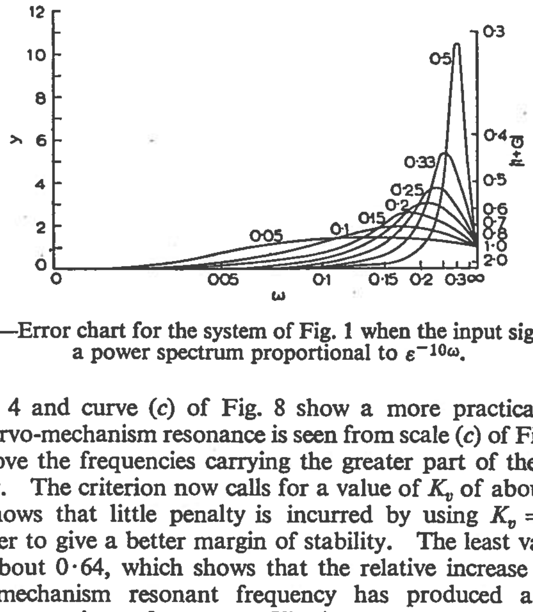

Fig. 4.—Error chart for the system of Fig. 1 when the input signal has a power spectrum proportional to $\varepsilon^{-10\omega}$.

Fig. 4 and curve (*c*) of Fig. 8 show a more practical case. The servo-mechanism resonance is seen from scale (*c*) of Fig. 2 to lie above the frequencies carrying the greater part of the input energy. The criterion now calls for a value of K_v of about $0 \cdot 4$, but shows that little penalty is incurred by using $K_v = 0 \cdot 21$ in order to give a better margin of stability. The least value of P is about $0 \cdot 64$, which shows that the relative increase of the servo-mechanism resonant frequency has produced a great improvement in performance. The important parameter is no longer K_v, as it was in the two previous examples, but it is now the speed of response.

Curve (*d*) of Fig. 8 shows conditions when the servo resonance is at a relatively high frequency. The criterion now calls for very light damping, but it will be shown later that the presence of noise in the input signal modifies this conclusion.

Fig. 5 is drawn for a step-function input, and curve (*e*) of Fig. 8 shows the corresponding values of S. These values are calculated, and are shown to an arbitrary scale since the signal energy is now infinite. Fig. 6 and curve (*f*) of Fig. 8 correspond to an input proportional to $1 + 10\varepsilon^{-t}$. The least value of S occurs for $K_v \simeq 0 \cdot 05$.

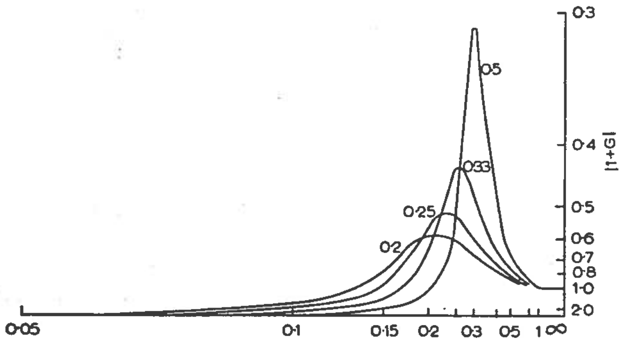

Fig. 5.—Error chart for the system of Fig. 1 when the input signal is a step-function.

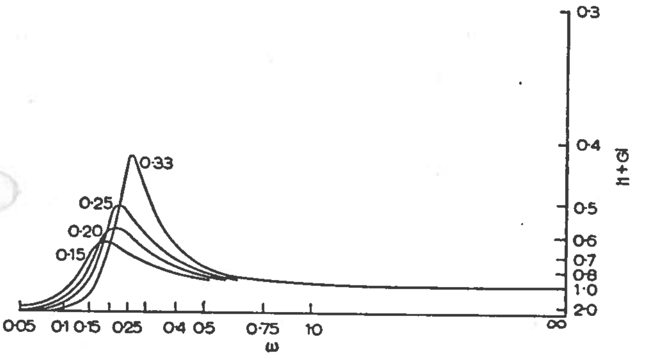

Fig. 6.—Error chart for the system of Fig. 1 when the input signal is proportional to $1 + 10\varepsilon^{-t}$.

(4) EFFECT OF NOISE

By an extension of the graphical method given in Section 2 we may take account of noise. If the signal θ_i is applied to the input together with noise θ_{ni}, the output consists of a signal θ_0 together with noise θ_{n0}. The difference between the output and θ_i is

$$\delta = \theta_0 + \theta_{n0} - \theta_i = -\epsilon + \theta_{n0} \quad . \quad . \quad . \quad (11)$$

where ϵ as before means $\theta_i - \theta_0$. We therefore take as our criterion

$$T = \int_0^\infty \delta^2(t)dt$$

$$= \int_0^\infty [\epsilon(t) - \theta_{n0}(t)]^2 dt$$

$$= \int_0^\infty \epsilon^2(t)dt + \int_0^\infty \theta_{n0}^2(t)dt - 2\int_0^\infty \epsilon(t)\theta_{n0}(t)dt \quad . \quad (12)$$

We now suppose that θ_i and θ_{ni} are representative samples of a statistically uniform signal and its accompanying noise. If then the noise is uncorrelated with the signal, and if the samples are sufficiently long, it is physically evident that the last integral in eqn. (12) will be negligible in comparison with the first two. This can be proved from the mathematical definition of correlation,[2] although the proof is not simple.

Thus

$$T = S + \int_0^\infty \theta_{n0}^2(t)dt$$

$$= S + \frac{1}{\pi}\int_0^\infty |\theta_{n0}(\omega)|^2 d\omega$$

$$= S + \frac{1}{\pi}\int_0^\infty \left|\frac{\theta_{ni}(\omega)G(\omega)}{1 + G(\omega)}\right|^2 d\omega \quad . \quad . \quad . \quad (13)$$

and we find the value of the last integral by plotting

$$y = \left|\frac{G(\omega)}{1 + G(\omega)}\right|^2 \quad . \quad . \quad . \quad . \quad . \quad (14)$$

against

$$x = \int_0^\omega |\theta_{ni}(\omega)|^2 d\omega \quad . \quad . \quad . \quad . \quad . \quad (15)$$

As before it is convenient to divide throughout by $\frac{1}{\pi}\int_0^\infty |\theta_i(\omega)|^2 d\omega$, thus obtaining

$$Q = P + \frac{\int_0^\infty \left|\frac{\theta_{ni}(\omega)G(\omega)}{1 + G(\omega)}\right|^2 d\omega}{\int_0^{\omega_1} |\theta_{ni}(\omega)|^2 d\omega} \cdot \frac{\int_0^{\omega_1} |\theta_{ni}(\omega)|^2 d\omega}{\int_0^\infty |\theta_i(\omega)|^2 d\omega} \quad . \quad (16)$$

$$= P + N, \text{ say} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

The reason for writing N in the form given in eqn. (16) is that it is then a product of two dimensionless ratios, one of areas and one of energies, and no difficulty arises over scale factors. Any convenient value can be chosen for ω_1, e.g. that value which makes

$$\int_0^{\omega_1} |\theta_{ni}(\omega)|^2 d\omega = \int_0^\infty |\theta_i(\omega)|^2 d\omega \quad . \quad . \quad . \quad (18)$$

Fig. 7 shows noise charts plotted from eqns. (14) and (15) for the system of Fig. 1 when white noise is applied to the input. The variation of N with K_v is shown by curve (g) of Fig. 8, the

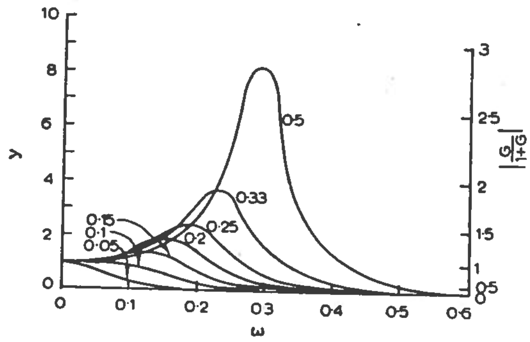

Fig. 7.—Noise chart for the system of Fig. 1 when white noise is applied to the input.

scale of which is such that the noise energy in the band $\omega = 0$ to $\omega = 1$ is equal to the energy in the signals for which P is shown in the same Figure. Curves (c_1) and (c_2) show respectively the effect of adding 10% and 20% of N to curve (c), which relates to Fig. 4. Thus curve (c_2), for example, shows the value of Q for the system of Fig. 1 when the spectrum of the signal is as shown by scale (c) of Fig. 2 and white noise is present with the

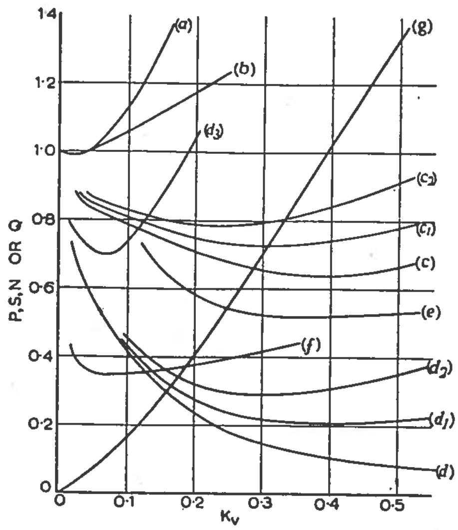

Fig. 8.—Variation of P, S, N or Q with K_v.

The curves all relate to the system of Fig. 1 and show the following:
(*a*) P for signal with power spectrum proportional to $\varepsilon^{-2\omega}$.
(*b*) P for signal with power spectrum proportional to $\varepsilon^{-5\omega}$.
(*c*) P for signal with power spectrum proportional to $\varepsilon^{-10\omega}$.
(*d*) P for signal with power spectrum proportional to $\varepsilon^{-20\omega}$.
(*e*) S for step-function input (scale arbitrary).
(*f*) S for input proportional to $1 + 10\varepsilon^{-t}$ (scale arbitrary).
(*g*) N for white noise at input having energy in the band $\omega = 0$ to $\omega = 1$ equal to that of the input signals for (*a*), (*b*), (*c*) or (*d*).

(c_1) Q for (c) with 10% of (g).
(c_2) Q for (c) with 20% of (g).
(d_1) Q for (d) with 10% of (g).
(d_2) Q for (d) with 20% of (g).
(d_3) Q for (d) with 200% of (g).

signal and has the same energy in the band $\omega = 0$ to $\omega = 5$ as the total energy of the signal. Curves (d_1), (d_2) and (d_3) of Fig. 8 show respectively the effects of adding 10, 20 and 200% of N to curve (d). Curves (c_1), (c_2), (d_1), (d_2) and (d_3) illustrate the fact that the presence of noise requires greater damping.

(5) DISCUSSION OF THE EXAMPLES

From Figs. 3 and 4 it is evident that the important factor in deciding the value of K_v for minimum P is the amount of energy in the signal at frequencies near the servo resonance. The criterion will require high damping if the servo resonance is within a frequency band where the input has appreciable energy. If the servo resonance is not within such a band, the criterion allows lower damping than experience shows to be desirable. The last result is not surprising, since by using P we ignore the effects of noise, and since the practical setting of a servo system must allow for possible changes in the parameters, which with very small damping might lead to instability.

The transition from conditions which demand high damping to those which allow light damping is more sudden than might be expected, and it is accompanied by a sharp decrease in the least value of P. This result may be a consequence of the particular choice of input signal spectrum.

Returning to the criticisms which have been made of the integral of error-squared as a performance criterion, we see from Fig. 5 that, with a step-function input, the criterion allows light damping because there is relatively little energy in the signal at the servo resonant frequency. The high-frequency region where G is negligible contributes to S a quantity which is nearly independent of K_v, and has little effect in locating the minimum of S. This is clearly illustrated also in Fig. 6, where the system is penalized more severely for its failure to respond to a sudden change of input, and yet the criterion requires greater damping owing to the greater energy in the signal near the servo resonant frequency.

It is shown in Appendix 10 that the use of W is equivalent to retaining S as the criterion but altering the input signal. The modified input has, in fact, a smaller proportion of the highest frequencies, but this is of secondary importance. A more important effect of the modification is to introduce into the input a component at each of the (complex) natural frequencies of the system. It is for this reason that the use of W leads to more highly damped systems than are obtained with S.

There seems to be a confusion of purpose in applying S as a criterion with a step-function input. We ought probably to distinguish two sets of conditions which a servo mechanism must satisfy:

(*a*) The system must be stable, and it must have a sufficient margin of stability to cover variations of the parameters which may occur when it is in service. The practical criteria of phase margin, maximum modulus, step-function response, etc., all relate to the margin of stability.* They can all be applied regardless of the signals which the servo mechanism will in fact be called on to follow.

(*b*) The output must fit as closely as possible, according to some accepted performance criterion, to the input. The integral of error-squared is one of many possible measures of this closeness of fit. All such criteria will, in general, give results which depend on the character of the input signal and of any noise which may be present. All will be overridden if they suggest less damping than the stability criteria require.

It may well be that, in many practical cases, the least value of Q occurs for rather less damping than is required by the stability criteria, but Q is little increased by the necessary increase of damping. This would explain why the stability criteria form a useful guide to the best practical setting. Thus it may be only in occasional cases that we need to use Q to fix the damping (Fig. 8). A performance criterion may, on the other hand, determine parameters about which the stability criteria give no information. It may also be useful in deciding whether a proposed alteration of the system is worth making, or in drawing up a specification of performance to be fulfilled by the servo mechanism when supplied with a given input.

When it is used as a specification of performance, the restriction on P can be supplemented by limits on the contribution to P which may arise in given frequency bands. Testing an existing servo mechanism to see whether it conforms to such a specification requires only the measurement of the amplitude of error for harmonic inputs at appropriate frequencies. No measurements of phase are required. The use of Q for purposes of specification requires a little more care since, for a given input signal and given noise, there is an absolute minimum below which Q cannot be reduced.[2]

(6) APPLICATION TO PROCESS CONTROL

The typical problem of a single-loop process-control system is shown in Fig. 9. Here A, B and C represent the fixed transfer

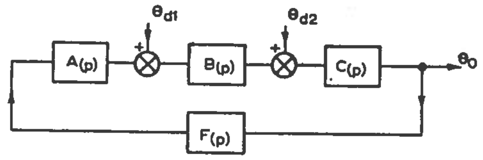

Fig. 9.—Process-control loop with disturbances.
ABCF = G

functions of parts of the process, F is the transfer function of the controller, θ_{d1} and θ_{d2} are disturbances, and θ_i, the set value, remains constant and is therefore made equal to zero. There may be more than two disturbances such as θ_{d1} and θ_{d2}. The object is to reduce the output variations θ_0, and if these may be measured adequately by their mean-square value we may apply the graphical method already given.

* The accepted numerical values used with these criteria may, of course, have some regard to performance under typical conditions, as well as to stability.

We first transfer all the disturbances to the output, obtaining a disturbance θ_d which is equivalent to θ_{d1}, θ_{d2}, etc., and can be calculated from them. It is likely, however, that θ_d will be obtained by measurement, since it is equal to θ_0 when the feedback loop is broken. Thus the quadratic spectrum of θ_d can be obtained by measuring that of θ_0 when $F = 0$.

When the loop is closed we have

$$\int_0^\infty \theta_0^2(t)dt = \frac{1}{\pi}\int_0^\infty |\theta_0(\omega)|^2 d\omega \quad . \quad . \quad . \quad . \quad (19)$$

$$= \frac{1}{\pi}\int_0^\infty \frac{|\theta_d(\omega)|^2}{|1 + G(\omega)|^2}d\omega \quad . \quad . \quad . \quad (20)$$

We therefore take as the criterion of performance

$$D = \frac{\int_0^\infty \frac{|\theta_d(\omega)|^2}{|1 + G(\omega)|^2}d\omega}{\int_0^\infty |\theta_d(\omega)|^2 d\omega} \quad . \quad . \quad . \quad . \quad (21)$$

which is similar in form to P, and may be found in the same way by plotting

$$y = |1 + G(\omega)|^{-2} \quad . \quad . \quad . \quad . \quad . \quad (22)$$

against

$$x = \int_0^\omega |\theta_d(\omega)|^2 d\omega \quad . \quad . \quad . \quad . \quad . \quad (23)$$

We then have to minimize D by adjusting the parameters of F, confirming afterwards that the setting so obtained gives an adequate margin of stability or, if not, making the necessary changes. It will be noticed that this analysis corresponds to that given in Section 2 for a servo mechanism in the absence of noise.

(7) CONCLUSIONS

The graphical method which has been given allows the integral of error-squared to be obtained without too much labour even for moderately complicated systems. It is suggested that when this integral is used as a performance criterion it should be evaluated for the type of input to which the system must respond (including noise, if this is appreciable) and that separate consideration should be given to the margin of stability. Under these conditions the evidence[2] seems to indicate that the criterion will give results in accordance with practical judgment.

(8) ACKNOWLEDGMENTS

The author wishes to express his thanks to Dr. J. H. Westcott for the stimulating discussion which preceded the writing of this paper, and to the Directors of Costain–John Brown Ltd. for permission to publish the paper.

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(10) APPENDIX

We define

$$W_n[g(p)] = \int_0^\infty t^n \epsilon_g^2(t)dt \quad . \quad . \quad . \quad . \quad (24)$$

where $\epsilon_g(t)$ is the error for an input $g(t)$. Then we have[3]

$$W_n[g(p)] = (-)^n \lim_{\sigma \to 0}\left[\frac{\partial^n}{\partial\sigma^n}\int_0^\infty \epsilon_g^2(t)\varepsilon^{-\sigma t}dt\right] \quad . \quad . \quad . \quad (25)$$

$$= (-)^n \lim_{\sigma \to 0}\left[\frac{\partial^n}{\partial\sigma^n}\frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \epsilon_g(p)\epsilon_g(\sigma - p)dp\right] . \quad (26)$$

provided that the poles of $\epsilon_g(p)$ are all in the left-hand half-plane. Now σ is initially real, but we may regard the integral in eqn. (26) as defining a function $F(\sigma)$ of the complex variable σ. Then for σ within a small region about the origin we can show[5] that $F(\sigma)$ is an analytic function with derivatives of all orders which may be found by differentiating under the sign of integration. It follows that

$$W_n[g(p)] = (-)^n \lim_{\sigma \to 0}\left[\frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \epsilon_g(p)\epsilon_g^{(n)}(\sigma - p)dp\right] \quad . \quad . \quad (27)$$

$$= (-)^n\frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \epsilon_g(p)\epsilon_g^{(n)}(-p)dp \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

$$= (-)^n\frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \epsilon_g(-p)\epsilon_g^{(n)}(p)dp \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

$$= \frac{1}{2}(-)^n\frac{1}{2\pi j}\int_{-j\infty}^{j\infty} [\epsilon_g(p)\epsilon_g^{(n)}(-p) + \epsilon_g(-p)\epsilon_g^{(n)}(p)]dp \quad . \quad (30)$$

$$= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \epsilon_g(p)\epsilon_g(-p)H(p)dp, \text{ say} \quad . \quad . \quad . \quad . \quad . \quad (31)$$

where

$$H(p) = (-)^n\frac{1}{2}\left[\frac{\epsilon_g^{(n)}(p)}{\epsilon_g(p)} + \frac{\epsilon_g^{(n)}(-p)}{\epsilon_g(-p)}\right] \quad . \quad . \quad . \quad (32)$$

But $H(p) = H(-p)$, and we may therefore write

$$H(p) = h(p)h(-p) \quad . \quad . \quad . \quad . \quad . \quad (33)$$

where all the poles of $h(p)$ are in the left-hand half-plane, provided that $\epsilon_g(p)$ has no purely imaginary zeros. In general $h(p)h(-p)$ has poles at the poles and zeros of $\epsilon_g(p)\epsilon_g(-p)$.

Then

$$W_n[g(p)] = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \epsilon_g(p)h(p)\epsilon_g(-p)h(-p)dp \quad . \quad (34)$$

$$= W_0[g(p)h(p)] \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

The function $h(p)$ will not generally be unique, as there will be some choice in the location of its zeros.

It is evident from eqns. (32) and (33) that $|g(\omega)h(\omega)|$ will be less than $|g(\omega)|$ at very high frequencies, but that it may be considerably greater near the resonant frequency of the system.

The function $g(p)h(p)$ has a pole at each complex natural frequency of the system, and if there is a lightly-damped mode the time-function corresponding to $g(p)h(p)$ will be oscillatory with the corresponding frequency.

For example, if

$$g(p) = \frac{1}{p} \quad \text{and} \quad \epsilon_g(p) = \frac{p}{p^2 + \omega p + \omega^2} \qquad (36)$$

we have

$$-\frac{1}{2}\left[\frac{\epsilon_g'(p)}{\epsilon_g(p)} + \frac{\epsilon_g'(-p)}{\epsilon_g(-p)}\right]$$

$$= \frac{\omega(\omega + p)(\omega - p)}{(p^2 + \omega p + \omega^2)(p^2 - \omega p + \omega^2)} \qquad (37)$$

and we may take

$$h(p) = \frac{(\omega + p)\sqrt{\omega}}{p^2 + \omega p + \omega^2} \qquad (38)$$

or

$$h(p) = \frac{(\omega - p)\sqrt{\omega}}{p^2 + \omega p + \omega^2} \qquad (39)$$

The time-function corresponding to $g(p)h(p)$ is then

$$\frac{1}{\sqrt{\omega}}\left[1 - \varepsilon^{-\omega t/2}\left(\cos\frac{\sqrt{(3)}\omega t}{2} - \frac{1}{\sqrt{3}}\sin\frac{\sqrt{(3)}\omega t}{2}\right)\right] \qquad (40)$$

or

$$\frac{1}{\sqrt{\omega}}\left[1 - \varepsilon^{-\omega t/2}\left(\cos\frac{\sqrt{(3)}\omega t}{2} + \sqrt{3}\sin\frac{\sqrt{(3)}\omega t}{2}\right)\right] \qquad (41)$$

Thus for the system of this example the value of $W \equiv W_1$ for a unit step-function input is the same as $S \equiv W_0$ when the input is given by eqn. (40) or (41). By using the standard forms given in Reference 3 it is easy to verify this result, and the common value of W and S is found to be $1/2\omega^2$.

DISCUSSION ON "THE MANCHESTER–KIRK O'SHOTTS TELEVISION RADIO-RELAY SYSTEM"*

WESTERN CENTRE, AT CARDIFF, 1ST NOVEMBER, 1954

SOUTHERN CENTRE, AT PORTSMOUTH, 9TH MARCH, 1955

Mr. J. S. Whyte (*at Cardiff*): On the T.D.2 relay system operating in the United States the required amplification at super-frequency is provided by triodes operating between suitable resonant cavities; the output amplifier has a gain of 18 dB when the output power is 0·5 watt. This arrangement has the advantage that an h.t. supply of only a few hundred volts is required compared with the 3 kV stabilized supply demanded by the travelling-wave-valve amplifier and some 60 watts of power required by its focusing coil. What are the authors' views on the relative merits of the two methods? Is it not a fact that the inherent broad-band characteristic of the travelling-wave amplifier, so often quoted as a great advantage, is reduced to a figure not greatly exceeding that of the T.D.2 amplifiers, by the input and output matching difficulties? Would the authors have used the travelling-wave-valve amplifier if suitable triodes had been available in this country when the system was designed?

Is there a routine for checking the automatic-starting arrangements for the Diesel generators, and if so, how frequently do they fail to start without necessarily interrupting traffic?

Mr. G. D. Curtis (*at Cardiff*): It would appear that the use of supervisory equipment for the location of faults in unattended telephone radio-relay stations emanates from the desire to reduce the maintenance manpower required at such stations.

I assume that selected maintenance men in the area are called out in the event of a fault on such a relay station. In view of the complexity and additional fault liability incurred by the use of supervisory fault detection, has there in practice been any saving in the cost of running such stations, or does the interest on the capital cost of the supervisory equipment plus its depreciation total more than the calculated cost of manning such stations during working hours?

The authors have stated that the group-delay/frequency characteristics of the radio system vary with frequency from approximately 50 to 90 millimicrosec. I assume that this delay shows up as a phase displacement varying with frequency.

Would the authors give a comparison between the group-delay characteristics of a radio system and those which one would expect when using coaxial cable?

Mr. W. P. Warren (*at Cardiff*): Could the authors give some indication of the electrical loading of the repeater stations used on the Manchester–Kirk o'Shotts radio-relay system, together with some indication of the relative efficiency? I recall that a load of approximately 200–300 kW is required by a terminal station such as Wenvoe for an output power of only 60 kW.

Is the requirement for standby supplies met at the repeater stations by two alternative Grid sources of supply, or by a standby generator? If standby generators are used, do the Post Office use the type of equipment which is at present on the market and in which a synchronous machine having a large flywheel is normally connected to the supply and is running continuously thereby under normal Grid supply conditions providing a measure of power-factor improvement? On disconnection of the normal supply the inertia of the flywheel allows the machine to operate as a generator until the standby Diesel plant is run up to speed.

The authors have referred to the automatic operation of the repeater stations, and I should be interested to have further information on the method of remote control and indication. Is control by d.c. impulses or by voice-frequency impulses? Are the pilots contained in a cable used solely for this purpose, or are they routed through normal Post Office exchange and repeater circuits? If the latter is the case, have the B.B.C. experienced any outages owing to interruption of the control-circuit pilots?

The authors indicate that if microwave working were also used for telephony circuits a spacing of 47 miles (the longest hop on the Manchester–Kirk o'Shotts route) would probably be too long to avoid fading. If we assume that a distance of 30 miles would be reasonably satisfactory for a microwave system, and we compare this with the repeater spacing of approximately 12 miles for normal multi-core cable carrier working, do the authors think that there would be considerable saving in capital with the adoption of the microwave system as against cable working?

Mr. H. J. R. Townsend (*at Cardiff*): To what extent does the quality of the picture suffer owing to its being "bounced" from point to point along the length of the country?

It is understood that, owing to the shortage of zinc, masts were aluminized and not galvanized. How is the aluminizing standing up to conditions of operation?

* DAWSON, G., HALL, L. L., HODGSON, K. G., MEERS, R. A., and MERRIMAN, J. H. H.: Pa er No 1623 R, January, 1954 (see 101, Part I, p. 93).