Mathematics of Surge Vessels and Automatic Averaging Control

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In this paper the authors report on a practical application of the quantitative methods which they have described previously (2, 6)[3] in connection with process and control analysis. First, the properties of surge vessels are considered from a functional point of view. The influence on these properties of externally applied controls is next discussed. Proceeding from the simpler to the more involved, control systems of various types are introduced and applied to a vessel. The performance of each of these applied systems is separately examined and illustrated under significant assumed conditions. Considerable attention is given to a definite method of control which involves, as the master instrument, one having a proportional-plus-floating characteristic, and which, it is felt, may justifiably be referred to as "automatic averaging control."

Introduction

An account of the use of "automatic averaging control" as an operating technique in modern continuous processing was given recently in a paper (1) by J. B. McMahon. The present paper is devoted to a quantitative presentation of the mathematics underlying this interesting branch of automatic control.

Dynamically, a surge vessel can be compared both to a shock absorber and to a flywheel. Fluid systems possessing such properties are supposed to absorb or release fluid at such times and in such a manner that violent changes in one or more of a group of related flows need not be accompanied by violent changes in another.

In the case of a surge vessel to which fluid is continuously supplied and from which fluid is continuously withdrawn, all flows pertaining to the vessel may be grouped into two sets—a summed "inflow" and a summed "outflow." When these two flows are exactly equal, the quantity of fluid stored in the vessel remains constant. In general, one of the flows will fluctuate and it will be desired to minimize the effect of such fluctuation on the other flow. For convenience it may be assumed that the inflow is the independently fluctuating quantity and that the outflow varies in some fashion as a result. The reverse circumstance is equally significant, but the two problems are basically analogous and the treatment of one will suffice.

In the case of a tank holding liquid, which for the sake of concreteness will be considered as typical of all possibilities,[4] the level at which the liquid stands is an indication of the quantity stored up in the tank. Thus the three variables, i.e., inflow, level, and outflow, may be taken as completely descriptive of the dynamic state of the system. The behavior of any two of these variables definitely determines the behavior of the third. In uncontrolled surge vessels, the dynamic relationships between inflow and level, level and outflow, and inflow and outflow may have all variety of forms. When a definite relationship of the proper type is enforced between the level and the outflow by the application of automatic control, it will be shown that the efficiency of the surge vessel as a "shock absorber" can be increased to a remarkable extent. In such an application, it should not be considered that the level is being "controlled" in the conventional sense that a predetermined value of level is to be held to within close tolerances, nor indeed that the outflow is to be so controlled. In reality, the true objective of this type of automatic control is to maintain continuously an advantageous relationship between these two variables.

Beginning with an uncontrolled vessel, having only "self-regulation," the application of control is presented in stages leading up to the full automatic-averaging-control installation. Each stage is accompanied by an illustration showing results obtainable in practical cases. Included in each figure is a diagrammatic sketch of the particular physical system considered. In every case the system shown comprises a vessel with a flow line leading to the vessel and a flow line leading from the vessel. Indicating instruments are shown symbolically and are applied to the inflow, level, and outflow. The instrument applied to the inflow serves merely, in each instance, to give a continuous indication of that variable, whereas in some of the cases the level or the outflow or both are controlled as well as measured; this is shown by replacement of the indicator by a controller.

The nature of the relationships among inflow, level, outflow, and time, under cyclic disturbances, makes it appear necessary to resort to the somewhat intricate involvements of classical differential equations in order to develop explicit quantitative expressions for these relationships. However, an investigation into the possibilities offered by the symbolic forms of Heaviside's operational calculus discloses an uncanny applicability to these purposes. Thus, even though the details of the operational methods themselves are beyond the scope of the present paper, such methods have been employed in the analytical development. For the benefit of those interested in the formal mathematics, a condensed description of the operational procedure is given (in italics) in the text under its respective section. The final expressions which give the over-all relationships under cyclic conditions are included in the main body of the text, which is so arranged that complete continuity is not lost by the reader who omits the mathematical development.

If the validity of the final expressions can be established either by inspection or by actual usage, it is by no means necessary that the actual user even be concerned with their origin or the manner of their development, except for the personal satisfaction he might derive from a familiarity with the details of the mathematical machinery. Oliver Heaviside himself expressed this attitude in his famous query:

"Shall I refuse my dinner because I do not fully understand the process of digestion?"

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[3] Numbers in parentheses refer to the Bibliography at the end of the paper.

[4] Gas holders, steam accumulators, etc., can be subjected to the same reasoning as is here applied to surge vessels for liquid.

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A major purpose of this paper is that of demonstrating to the practical industrial engineer, such as those who are actually confronted with averaging control problems, the extreme practicability of some of these simplified formulas. The practical, economic value of the formulas cannot, perhaps, be fully appreciated except by numerical substitution. The astonishing character of soundly derived mathematical results was expressed by Heinrich Hertz:

"One cannot escape the feeling that these mathematical formulas have an independent existence and an intelligence of their own; that they are wiser than we are; wiser even than their discoverers; that we get more out of them than was originally put into them."

The formulas which describe the results under cyclic conditions, as presented in the text. contain only those factors which are necessary to a practical determination of the over-all response. They have been simplified by logical assumptions. Much of the complexity and unimportant detail has been eliminated and emphasis given to those factors which are or may be influential in actual industrial applications.

Consideration of sine-wave disturbances leads to the appearance of trigonometric functions in some of the mathematics. It is normal practice in much applied mathematics to express trigonometric angles in radians. Conventional trigonometric tables, however, are compiled in terms of angular degrees. For this reason a departure is taken from normal practice, in that the final forms are made to appear as dimensionless ratios of an "angle whose tangent is something" to an angle of 90°, or as $(\tan^{-1}X)/90°$.

From the simplified general formulas, some exemplary numerical results have been included in the figures. These results pertain only to the particular dimensions assumed for the surge vessel and to the particular nature and magnitude of the assumed disturbances. It is hoped, however, that these tabulations will serve to rationalize the complexities of the general problem.

The following special nomenclature applies for the simplified text as well as for the formal mathematics.

Notation, Definitions, and Units

V = level above an assumed base; feet above bottom of vessel
V_n = normal or "desired" value of V
b = proportional or throttling band of V, ft
r = reset constant, units per min
Q_s = inflow to vessel (total), gpm
Q = outflow from vessel (total), gpm
Q_m = $^1/_2(Q_{\min} + Q_{\max})$ = mid-value of Q
k = $(Q_{\max} - Q_{\min})$ = band in which Q may be varied by controls, gpm
d = diameter of vessel, assumed upright and cylindrical, ft
A = capacity of vessel, gal per ft (= $5.88\,d^2$)
R = resistance to outflow (linear), ft per gpm
R_e = b/k = equivalent "resistance" under control, ft per gpm
t = time, min
h = half-period of oscillation, min
$(X)'$ = first derivative of X
$(X)''$ = second derivative of X
p = d/dt = differential operator

Numerical Values Assumed Constant in All Examples

V_n = 5 ft (mid-value of allowable range of level variation)
$Q_{\min}$ = 100 gpm
$Q_{\max}$ = 300 gpm
Q_m = $^1/_2(Q_{\min} + Q_{\max})$ = 200 gpm
k = $(Q_{\max} - Q_{\min})$ = 200 gpm
d = two values considered = 4.125 and 8.25 ft
A = two values considered = 100 and 400 gal per ft
h = two values considered = 10 and 20 min

Test Disturbances (in Inflow) Applied for All Modes of Control

To represent a wide variety of disturbances, the inflow is assumed to undergo three different sorts of variation, as follows:

So-Called Condition (a)

In a state of perfect balance, the inflow is assumed to change suddenly from a constant value of 200 gpm to a new constant value of 250 gpm

This condition can be expressed mathematically as follows:

$$Q_s = 200 \text{ for } (t < 0),\; Q_s = 250 \text{ for } (t > 0)$$

So-Called Condition (b_1)

The inflow is assumed to be engaged in a permanent sine-wave oscillation about a value of 200 gpm at an amplitude of 50 gpm and with a half-period of 10 min.

This condition can be expressed mathematically as follows:

$$Q_s = 200 + 50 \sin\left[180°\,\frac{t}{10}\right] \text{ for } (-\infty < t < \infty)$$

So-Called Condition (b_2)

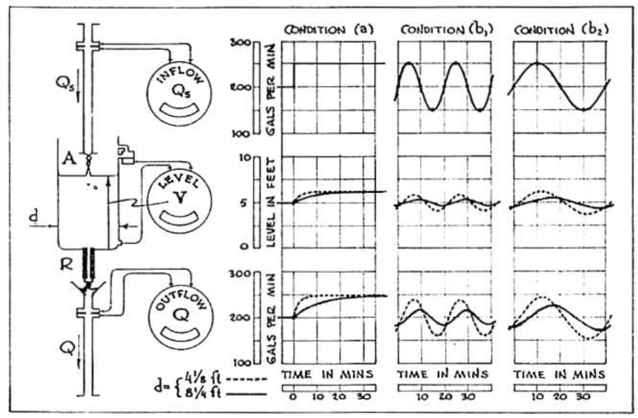
Same as condition (b_1) but with a half period of 20 min.

This condition can be expressed mathematically as follows

$$Q_s = 200 + 50 \sin\left[180°\,\frac{t}{20}\right] \text{ for } (-\infty < t < \infty)$$

Single Resistance-Capacity Unit as Surge Vessel; Self-Regulation

An elementary resistance-capacity system of the sort described



Numerical Solutions		No Explicit Control	$R = 0.025$ feet/(gal/min)	
	$A = 5.88d^2$	Condition (a)	Condition (b_1)	Condition (b_2)
Inflow Q_s gals/min		200 --- $(t<0)$ 250 --- $(t>0)$	$200 + 50\sin\left[180°\frac{t-0}{10}\right]$ (h = 10 mins)	$200 + 50\sin\left[180°\frac{t-0}{20}\right]$ (h = 20 mins)
Level V feet	100 gals/foot ----	$6.25 - 1.25e^{-0.40t}$	$5 + 0.983\sin\left[180°\frac{t-2.12}{10}\right]$	$5 + 1.164\sin\left[180°\frac{t-2.38}{20}\right]$
	400 " ——	$6.25 - 50e^{-0.10t}$	$5 + 0.380\sin\left[180°\frac{t-4.02}{10}\right]$	$5 + 0.671\sin\left[180°\frac{t-6.39}{20}\right]$
Outflow Q gals/min	100 " ----	$250 - 50e^{-0.40t}$	$200 + 39.32\sin\left[180°\frac{t-2.12}{10}\right]$	$200 + 46.54\sin\left[180°\frac{t-2.38}{20}\right]$
	400 " ——	$250 - 50e^{-0.10t}$	$200 + 15.17\sin\left[180°\frac{t-4.02}{10}\right]$	$200 + 26.85\sin\left[180°\frac{t-6.39}{20}\right]$
Time-Boundaries		$(t > 0)$	$(-\infty < t < \infty)$	$(-\infty < t < \infty)$

Fig. 1 Single Resistance-Capacity Unit as Surge Vessel; Self-Regulation

in an earlier paper (2) by one of the authors can be considered in the role of a surge vessel. Fig. 1[5] shows such a system with indicating instruments on inflow, level, and outflow.

[5] In the curves of Figs. 1 to 8, full lines are for one capacity and dotted lines are for one quarter of the capacity (or one half of the diam).

Following the development in the earlier reference we may write

$$(V)' = (Q_s - Q)/A \ldots\ldots\ldots\ldots\ldots\ldots [1]$$

$$V = RQ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots [2]$$

The two following basic equations are obtainable by familiar methods from Equations [1] and [2]

$$AR(V)' + V = RQ_s \ldots\ldots\ldots\ldots\ldots [3]$$

$$AR(Q)' + Q = Q_s \ldots\ldots\ldots\ldots\ldots [4]$$

Solutions, similar to those in the earlier paper (2), for the response of the level and the outflow, when the inflow is changed suddenly from a constant value of 200 gpm to a new constant value of 250 gpm, are shown by the curves under condition (*a*) of Fig. 1. The numerical equations given in the same figure, for the same assumed conditions, express the deviation of the level V from the normal value of $V_n = 5$ ft and the deviation of the outflow Q from $Q_m = 200$ gpm.

* * *

Operational methods can also be used for solutions of this sort and are especially useful when oscillatory disturbances are to be dealt with. Operational or symbolic calculus has been placed on a rigorous foundation and a number of excellent texts (3, 4, 5) are available which describe its application. From Equations [3] and [4], the following equivalent operational expressions are directly derived

$$V = \frac{R}{1 + ARp} \cdot Q_s \ldots\ldots\ldots\ldots\ldots [5]$$

$$Q = \frac{1}{1 + ARp} \cdot Q_s \ldots\ldots\ldots\ldots\ldots [6]$$

For a single sudden change in Q_s, simple exponential solutions can be obtained directly from Equations [5] and [6] as well as from Equations [3] and [4].

For steady sine-wave oscillations in the inflow, the amplitude and phase of the resulting oscillations of level and outflow are obtainable by replacing p in the operators with the imaginary angular velocity $(i\pi/h)$. Briefly, if the operator then becomes $(u + iv)$, the relative amplitude is given by $\sqrt{u^2 + v^2}$ and the phase angle by $\tan^{-1}(v/u)$, while the true[6] lag in time units is $-(h/\pi)\tan^{-1}(v/u)$. Thus for steady oscillations in the inflow the amplitude and lag response of the level and outflow can be obtained from Equations [5] and [6] and are summarized as follows:

$$\frac{\text{Ampl. of } V}{\text{Ampl. of } Q_s} = \frac{R}{\sqrt{1 + G^2}}$$

where

$$G = \pi AR/h$$

$$\text{Lag of } V \text{ versus } Q_s = (h/\pi)\tan^{-1}(G)$$

$$\frac{\text{Ampl. of } Q}{\text{Ampl. of } Q_s} = \frac{1}{\sqrt{1 + G^2}}$$

$$\text{Lag of } Q \text{ versus } Q_s = \text{same as for } V$$

* * *

The equations expressing the values of V and Q under cyclic disturbances of the inflow must contain harmonic functions of time. These can be brought into the equations as sine functions of angular degrees. General forms for the equations of V and Q under the cyclic conditions (b_1) and (b_2) may be written

[6] Time lag, as such, should not be given significance except in the case of sinusoidal oscillations, as here, or in the case of a pure time delay or distance-velocity lag (2).

$$V = V_n + A_v \sin\left[180^\circ \frac{t - T_v}{h}\right] \ldots\ldots\ldots [7]$$

$$Q = Q_m + A_q \sin\left[180^\circ \frac{t - T_q}{h}\right] \ldots\ldots\ldots [8]$$

The expressions for use under cyclic conditions, which were developed as previously shown by operational methods, may be used to supply the following formulas for the new constants appearing in Equations [7] and [8].

$$A_v = \text{Ampl. of } V = \frac{R \times (\text{Ampl. of } Q_s)}{\sqrt{1 + G^2}}$$

$$= \text{level variation in feet}$$

$$A_q = \text{Ampl. of } Q = \frac{(\text{Ampl. of } Q_s)}{\sqrt{1 + G^2}}$$

$$= \text{outflow variation in gpm}$$

$$T_v = T_q = \frac{h}{2}\,\frac{\tan^{-1}(G)}{90^\circ}$$ = time in minutes by which cycles of V and of Q lag behind the cycles of Q_s

The constant G depends upon the characteristics of the process and upon the half-period of the inflow oscillations. Its numerical value is given by

$$G = 3.14\,\frac{AR}{h}$$

The quantities "Ampl. of V," "Ampl. of Q," and "Ampl. of Q_s" are the magnitudes of the maximum variation of these variables on either side of their mean values, i.e., one half of their total variation.

The results of numerical substitution in the general formulas, for the assumed conditions (b_1) and (b_2), are included in Fig. 1, together with the curves of their solutions plotted against time. These curves show that the level and the outflow oscillate exactly in phase with one another, but that they are out of phase with the inflow.

The principal merit of this arrangement as a surge-absorbing system lies in its simplicity. Smoothing of the outflow versus the inflow is not impressive. The level can reach an eventual balance anywhere in the vessel, depending upon the average value of the inflow.

Independent Control of the Outflow

In this case a flow controller is installed directly on the outflow, as illustrated in Fig. 2, and is assumed to be completely successful in maintaining this flow at a constant value.

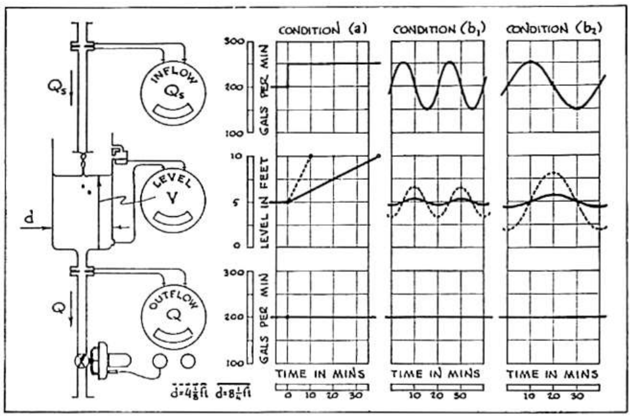
From the universally valid Equation [1]

$$(V)' = (Q_s - Q_m)/A \ldots\ldots\ldots\ldots\ldots [9]$$

Where the mean flow Q_m is the constant value at which the outflow happens to be controlled. Equation [9] may be written as the indefinite integral

$$V = \frac{1}{A}\int (Q_s - Q_m)dt$$

which is equivalent to the statement that the level "integrates" the excess of the inflow over the controlled outflow, and does so in inverse proportion to the capacity of the vessel.



Numerical Solutions		Independent Control of Outflow		
	$A = 5.88 d^2$	Condition (a)	Condition (b_1)	Condition (b_2)
Inflow Q_s gals/min		200 ---- ($t<0$) 250 ---- ($t>0$)	$200 + 50 \sin\left[180° \frac{t-0}{10}\right]$	$200 + 50 \sin\left[180° \frac{t-0}{20}\right]$
Level V feet	100 gals/foot -----	$5 + 0.5t$	$5 + 1.59 \sin\left[180° \frac{t-5}{10}\right]$	$5 + 3.18 \sin\left[180° \frac{t-10}{20}\right]$
	400 - ___	$5 + 0.125t$	$5 + 0.398 \sin\left[180° \frac{t-5}{10}\right]$	$5 + 0.796 \sin\left[180° \frac{t-10}{20}\right]$
Outflow Q gals/min	100 " -----	Constant at 200	Constant	Constant
	400 - ___	" " "	"	"
Time-Boundaries		($t<0$)	($-\infty < t < \infty$)	($-\infty < t < \infty$)

Fig. 2 Limiting Case; Independent Control of Outflow

For a sudden sustained increase in the inflow Q_s above Q_m, that is for condition (*a*), it is evident that the level assumes a constant rate of increase which depends upon the capacity A, as shown in Fig. 2.

* * *

Operationally

$$V = \frac{1}{Ap} \cdot (Q_s - Q_m) \dots\dots\dots\dots\dots [10]$$

In the case of continuous oscillation of the inflow Q_s, under conditions (b_1) and (b_2), the level response may be found by direct integration or by the formal $p = i\pi/h$ substitution already employed. Thus for sine-wave oscillations, we obtain the following response

$$\frac{\text{Ampl. of } V}{\text{Ampl. of } Q_s} = \frac{h}{\pi A}$$

$$\text{Lag of } V \text{ versus } Q_s = (h/\pi) \tan^{-1}(\infty) = h/2$$

$$Q \dots\dots\dots (\text{Constant})$$

* * *

The general form of the equations for the cyclic conditions (b_1) and (b_2) are

$$V = V_n + A_v \sin\left[180° \frac{t - T_v}{h}\right] \dots\dots\dots\dots [7]$$

$$Q = Q_m$$

in which

$$A_v = \text{Ampl. of } V = 0.318 \frac{h}{A} (\text{Ampl. of } Q_s)$$

$$= \text{level variation in feet}$$

$T_v = \dfrac{h}{2}$ = time in minutes by which the cycles of V lag behind cycles of Q_s

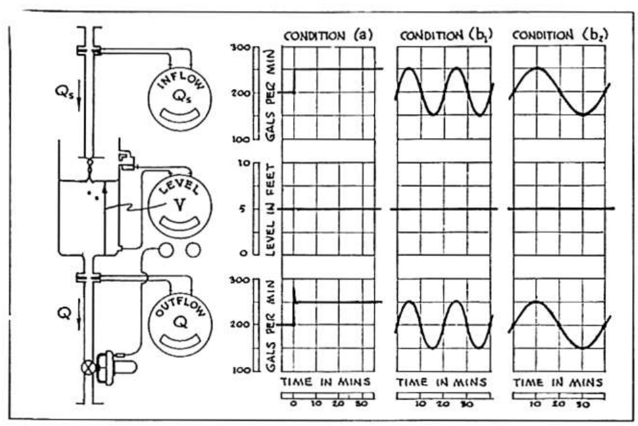
The results of numerical substitution in the general Equation [7], for the assumed conditions (b_1) and (b_2), are given in Fig. 2 together with curves of their time solutions. The behavior in this case under cyclic conditions (b_1) and (b_2) represents the limiting case of "perfect" averaging operation, based on oscillation of the inflow about a constant mean value. It is interesting to note, from the formulas for A_v and T_v, that the level variations are directly proportional to the period of the inflow variations and inversely proportional to the capacity or area of the vessel, and that the cycles of the level are exactly one-fourth period out of phase with the cycles of the inflow. This is evident also from the curves.

With this type of control, perfect smoothing of the outflow with respect to the inflow is made inevitable by the application of the flow controller on the outflow, but no recognition is taken of the level, which will gradually rise or fall, even to limits, depending upon the difference between the accumulated average of the inflow and the value at which the outflow is controlled.

In practical application of this method, periodic manual readjustment of the controlled outflow may in some cases be a satisfactory mode of operation, especially when the magnitude or the period of the oscillations encountered compares favorably with the size of the vessel. Such readjustment amounts to matching the controlled outflow to the average of the inflow taken over considerable periods of time. The aim of automatic averaging control is to make such readjustment continuous and automatic, to approach as nearly as possible to perfect smoothing of the outflow versus the inflow, consistent with keeping the level continuously within the vessel. Returning the level to a predetermined central value is also desirable as well, since this will permit optimum absorption both of sustained changes and of sudden surges, irrespective of the direction in which these occur.

Independent Control of the Level

Automatic control of a system involving only a single capacity unit can be carried out to any desired degree of effectiveness, even with types of control which in an operating sense may be



Numerical Solutions		Independent Control of Level		
	$A = 5.88 d^2$	Condition (a)	Condition (b_1)	Condition (b_2)
Inflow Q_s gals/min		200 ...... ($t<0$) 250 ...... ($t>0$)	$200 + 50 \sin\left[180° \frac{t-0}{10}\right]$	$200 + 50 \sin\left[180° \frac{t-0}{20}\right]$
Level V feet	100 gals/foot ----	Substantially constant at 5	Constant	Constant
	400 " ___	" " " "	"	"
Outflow Q gals/min	100 - -----	Similar to Q_s	Same as Q_s	Same as Q_s
	400 " ___	" " "	" " "	" " "
Time-Boundaries		($t>0$)	($-\infty < t < \infty$)	($-\infty < t < \infty$)

Fig. 3 Limiting Case; Independent Control of Level

called elementary. The problem is one exclusively of rapid measurement and manipulation. In Fig. 3 such a control system is assumed to be applied to maintain a constant level in the vessel. The level controller itself might have, for example, a proportional characteristic with an extremely narrow proportional or throttling band. In this sense the equations which are given later for proportional control may be considered to apply here, but with an extremely small value of proportional band b. Whatever means seem most proper actually to achieve a substantially constant level, we are for the moment concerned only with the effect on the outflow. As shown graphically in Fig. 3, this degree of level control is acquired at the cost of full variation of the outflow. The latter flow essentially duplicates the inflow, even to the point of being in phase with it.

From the point of view of automatic averaging control this example represents a limiting case, opposite to that of Fig. 2, and is brought in only as a logical step in the development.

Theoretically, the magnitude of the outflow variations is independent of the area of the vessel. Only the practical impossibility of reducing the proportional band precisely to zero, or some imperfection in the operation of the controls, could cause any reduction in the amplitude of the outflow cycles.

Cascaded Control

The term "cascaded control" appears appropriate to describe in general a system of control whereby the operating means of one controller automatically adjusts the control-point setting of one or more succeeding controllers, intermediate between the initial or master controller and the final controlling means or manipulated variable. In averaging level control, this would correspond to allowing the operating means of the level controller to "set the control point of" a special flow controller on the outflow.

Such inclusion of an auxiliary flow controller for the outflow has the advantage that it eliminates any direct dependence of the outflow upon the behavior of the level, or on external-pressure relationships such as changes in the drop across the outlet valve. It also eliminates similar dependence of the outflow upon whatever pressure may be impressed on the liquid surface, as shown symbolically in the last two figures of the paper. This method is a recognized procedure in control technique.

In the remaining examples it will be assumed, as in the earlier paper (6), that the cascaded method of control is employed. Thus, it is assumed that the "control point" of the flow controller on the outflow is set throughout its operating range by the operating means of the level controller, and that the relationship thus formed is uniform within that range.

Proportional Control of the Level, Cascaded

If the level instrument is assumed to be a proportional controller, as described in paper (6), we may write the controller equation as a relationship between the level V and the outflow Q, or as

$$Q - Q_m = (k/b)(V - V_n) \qquad [11]$$

where ($Q_{\min} < Q < Q_{\max}$), and in which it is assumed that the proportional band b is so located that V_n is in the middle of that band.

Combining Equation [1] for the "process" with Equation [11] for the controller and making the substitution

$$R_e = b/k \qquad [12]$$

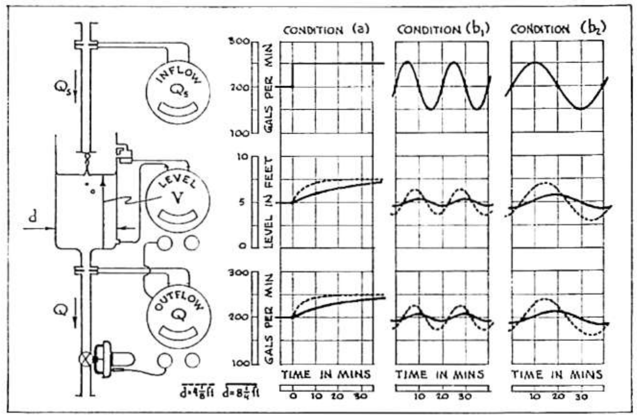
gives for the level and the outflow, respectively

$$AR_e(V - V_n)' + (V - V_n) = R_e(Q_s - Q_m) \qquad [13]$$

$$AR_e(Q - Q_m)' + (Q - Q_m) = (Q_s - Q_m) \qquad [14]$$

Equations [13] and [14] are similar to Equations [3] and [4] for the resistance-capacity unit. This fact is no coincidence as the systems are directly analogous. The ratio (b/k) for automatic control is analogous to the resistance R under self-regulation and may be thought of as an equivalent "resistance" R_e, so designated in the nomenclature in order to emphasize the analogy.

The response of the level and the outflow to the sudden sustained change in the inflow is obtained precisely as in the analogous case under self-regulation. The results are shown under condition (a) in Fig. 4.



NUMERICAL SOLUTIONS		CONTROL CONSTANTS: $b = 10$ ft., $k = 200$ gals/min		
	$A = 5.88d^2$	CONDITION (a)	CONDITION (b_1)	CONDITION (b_2)
INFLOW Q_s gals/min		200 ---- ($t<0$) 250 ---- ($t>0$)	$200 + 50 \sin\left[180° \frac{t-0}{10}\right]$	$200 + 50 \sin\left[180° \frac{t-0}{20}\right]$
LEVEL V feet	100 gals/foot ----	$7.5 - 2.5\, e^{-0.20t}$	$5 + 1.34 \sin\left[180° \frac{t-3.20}{10}\right]$	$5 + 1.96 \sin\left[180° \frac{t-4.23}{20}\right]$
	400 " ———	$7.5 - 2.5\, e^{-0.05t}$	$5 + 0.393 \sin\left[180° \frac{t-4.50}{10}\right]$	$5 + 0.759 \sin\left[180° \frac{t-8.04}{20}\right]$
OUTFLOW Q gals/min	100 " ----	$250 - 50\, e^{-0.20t}$	$200 + 26.85 \sin\left[180° \frac{t-3.20}{10}\right]$	$200 + 39.32 \sin\left[180° \frac{t-4.23}{20}\right]$
	400 " ———	$250 - 50\, e^{-0.05t}$	$200 + 7.86 \sin\left[180° \frac{t-4.50}{10}\right]$	$200 + 15.17 \sin\left[180° \frac{t-8.04}{20}\right]$
TIME-BOUNDARIES		($t > 0$)	($-\infty < t < \infty$)	($-\infty < t < \infty$)

Fig. 4 Proportional Level Controller

* * *

In operational form, Equations [13] and [14] become

$$V - V_n = \frac{R_e}{1 + AR_e p} \cdot (Q_s - Q_m) \qquad [15]$$

$$Q - Q_m = \frac{1}{1 + AR_e p} \cdot (Q_s - Q_m) \qquad [16]$$

Equations [15] and [16] are similar to Equations [5] and [6] for the resistance-capacity unit. The attenuation, or amplitude ratio, and the lag involved in the response of the level and outflow to continuous oscillation of the inflow are also given by analogous expressions and may be written

$$\frac{\text{Ampl. of } V}{\text{Ampl. of } Q_s} = \frac{R_e}{\sqrt{1 + G^2}}$$

where

$$(G = \pi A R_e / h)$$

$$\text{Lag of } V \text{ versus } Q_s = (h/\pi) \tan^{-1}(G)$$

$$\frac{\text{Ampl. of } Q}{\text{Ampl. of } Q_s} = \frac{1}{\sqrt{1 + G^2}}$$

$$\text{Lag of } Q \text{ versus } Q_s = \text{same as for } V$$

* * *

General equations for V and Q under cyclic conditions (b_1) and (b_2) may again be written

$$V = V_n + A_v \sin\left[180° \frac{t - T_v}{h}\right] \quad [7]$$

$$Q = Q_m + A_q \sin\left[180° \frac{t - T_q}{h}\right] \quad [8]$$

The numerical values of the constants in these equations may be determined from the following formulas

$$A_v = \text{Ampl. of } V = \frac{R_e \times (\text{Ampl. of } Q_s)}{\sqrt{1 + G^2}}$$

$$= \text{level variation in feet}$$

$$A_q = \text{Ampl. of } Q = \frac{(\text{Ampl. of } Q_s)}{\sqrt{1 + G^2}}$$

$$= \text{outflow variation in gpm}$$

$$T_v = T_q = \frac{h}{2}\frac{\tan^{-1}(G)}{90°}$$ = time in minutes by which cycles of V and of Q lag behind cycles of Q_s

in which

$$G = 3.14 \frac{AR_e}{h}$$

For the numerical examples considered

$$R_e = 0.005\, b \text{ and}$$

$$G = 0.0157 \frac{Ab}{h}$$

Results of numerical substitution in the general equations, for the assumed conditions (b_1) and (b_2), are included in Fig. 4, together with curves of the time solutions.

It is evident that the remarks already made on the performance of the simple resistance-capacity system, Fig. 1, apply almost equally well here. The use of the proportional type of level controller in this application merely imparts to the vessel a definite, mechanical, self-regulating property similar to that of the resistance-capacity system shown in Fig. 1, while the use of "cascaded control," as described, prevents alteration, by pressure changes in any form, of the already limited averaging characteristics of the system. In the case illustrated in Fig. 4, the proportional or throttling band is made equal to the full allowable range of the level. For proportional bands narrower than this value, the smoothing of the outflow is even less effective. Wider proportional bands, on the other hand, would not permit balance of the level within the allowable range, or within the confines of the vessel, for all values of inflow, even under steady conditions.

When the range of the instrument is so selected that it fits the allowable range of level variation, a proportional band having a width equal to this range, such as that chosen in Fig. 4, is generally referred to as a "100 per cent throttling range." From the viewpoint of averaging control this so-called 100 per cent throttling controller has a very limited ability toward smoothing of the outflow. Some of the limitations are shown by the following observations: (*a*) If the outlet resistance R of Fig. 1 had been located 5 ft below the bottom of the vessel, the value of R to give the same level in balance would have been equal to that of R_e in Fig. 4, and the results of self-regulation and of the 100 per cent throttling control would have been identical. (*b*) If, in such a system as is illustrated in Fig. 1, a constant static pressure of approximately 2 psi had been exerted on the liquid surface, the results of self-regulation and those of 100 per cent throttling control would have been identical. (*c*) The square-root characteristics of an ordinary valve, which could replace the resistance R in the system of Fig. 1, and which could be adjusted to give a level of 5 ft for a flow of 200 gpm, would provide the same averaging effect at the center of the level range as does the 100 per cent throttling control, although it would give less averaging at levels below the center and more above it.

Methods (*a*) and (*b*) of the preceding paragraph could be extended to increase the averaging effect throughout the allowable level range. This would be accomplished, however, at the cost of limitation of the range of inflow variations which would permit the level to remain within the allowable range. Adjustment of the outflow resistance in connection with any of methods (*a*) to (*c*) permits establishment of the value of the level for a given outflow and a given pressure drop across the resistance but does not permit adjustment of the range of level variation for a given variation of the inflow. The really practical advantages in using an automatic control instrument with adjustable proportional or throttling band lie in the fact that the level variation may be retained within a definite range for any specified variation in the outflow, and regardless of the pressure drops existing across the valve. The maximum capacity of the valve is the only factor limiting the range of outflow variation.

Proportional-Plus-Floating Control of the Level, Cascaded

For automatic averaging control, it is evident that there is a real advantage in the use of a level controller which controls to a single ultimate value rather than to within a band of values, i.e., in the use of a controller which has point-stability rather than band-stability alone. The severity of the corrective measures set up by such a controller may be moderated without simultaneously spreading out the band in which the level can ultimately balance, as is the case with the proportional form of instrument. The proportional-plus-floating type of controller, known to be a versatile form in other applications, fits this requirement and will be considered in an installation similar to that of the preceding section. The level controller, this time with a proportional-plus-floating characteristic, is again assumed to operate by setting the "control point" of a controller on the outflow.

As described in the authors' previous paper (6) and for the present installation, the proportional-plus-floating controller may be identified by the following equation

$$(Q)' = (k/b)[(V - V_n)' + r(V - V_n)] \quad [17]$$

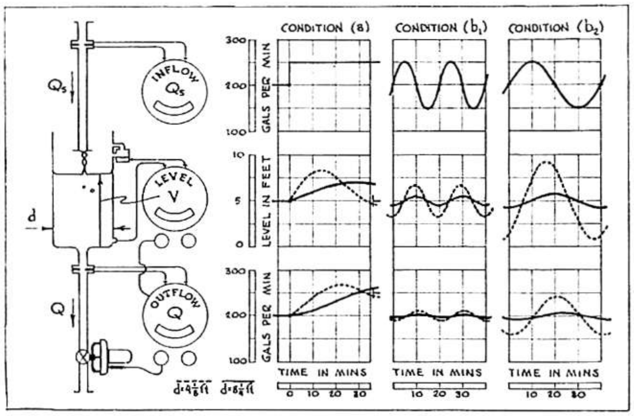
in which k and b have already occurred, and in which r is the so-called reset constant.

It should be pointed out that the proportional or throttling band b, as defined in paper (6), need not exist in an entirely tangible form. The expressed value of this band may be considerably greater than the full available range of the level, in which case the controls act as though the full extent of such a band were really effective. This places no permanent restriction on the performance of the proportional-plus-floating controller since in operation this band is automatically moved in such a way that the level returns to the normal value for balanced conditions.

To determine the properties of the system under this form of control, we may combine Equation [17] for the controller with the "process" Equation [1]. By methods described in detail in the earlier paper (6), one obtains for the level V

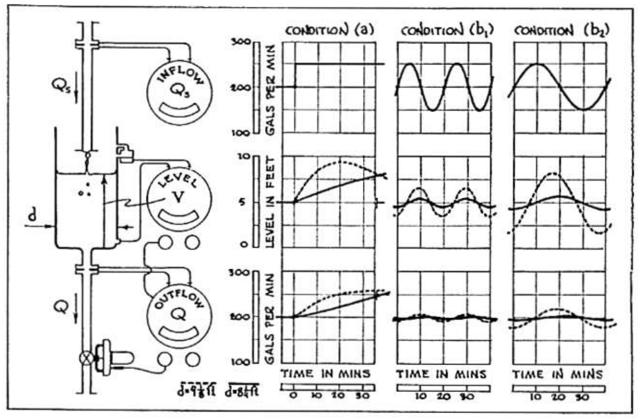
$$AR_e(V - V_n)'' + (V - V_n)' + r(V - V_n) = R_e(Q_s - Q_m)' \quad [18]$$

and for the outflow Q



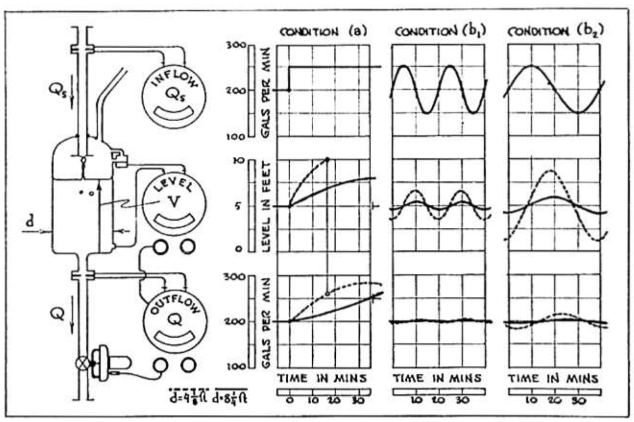
NUMERICAL SOLUTIONS		CONTROL CONSTANTS: $b = 30$ ft, $r = 0.15$ 1/min, $k = 200$ gals/min		
	$A = 5.88d^2$	CONDITION (a)	CONDITION (b_1)	CONDITION (b_2)
INFLOW Q_s gals/min		200 ---- ($t<0$) 250 ---- ($t>0$)	$200 + 50\sin\left[180°\frac{t-0}{10}\right]$	$200 + 50\sin\left[180°\frac{t-0}{20}\right]$
LEVEL V feet	100 $\frac{\text{gals}}{\text{foot}}$ ----	$5 + 5.30e^{-0.0333t}\cos\left[180°\frac{t-16.7}{33.3}\right]$	$5 + 1.72\sin\left[180°\frac{t-4.26}{10}\right]$	$5 + 4.35\sin\left[180°\frac{t-6.05}{20}\right]$
	400 " ___	$5 + 2.54e^{-0.00833t}\cos\left[180°\frac{t-31.9}{63.7}\right]$	$5 + 0.408\sin\left[180°\frac{t-4.83}{10}\right]$	$5 + 0.879\sin\left[180°\frac{t-9.85}{20}\right]$
OUTFLOW Q gals/min	100 " ----	$250 - 53.03e^{-0.0333t}\cos\left[180°\frac{t+3.6}{33.3}\right]$	$200 + 12.73\sin\left[180°\frac{t-5.68}{10}\right]$	$200 + 40.05\sin\left[180°\frac{t-10.91}{20}\right]$
	400 " ___	$250 - 50.71e^{-0.00833t}\cos\left[180°\frac{t+3.4}{63.7}\right]$	$200 + 3.01\sin\left[180°\frac{t-6.25}{10}\right]$	$200 + 8.11\sin\left[180°\frac{t-14.11}{20}\right]$
TIME-BOUNDARIES		($t>0$)	($-\infty < t < \infty$)	($-\infty < t < \infty$)

FIG. 5 PROPORTIONAL-PLUS-FLOATING LEVEL CONTROLLER; CASE I



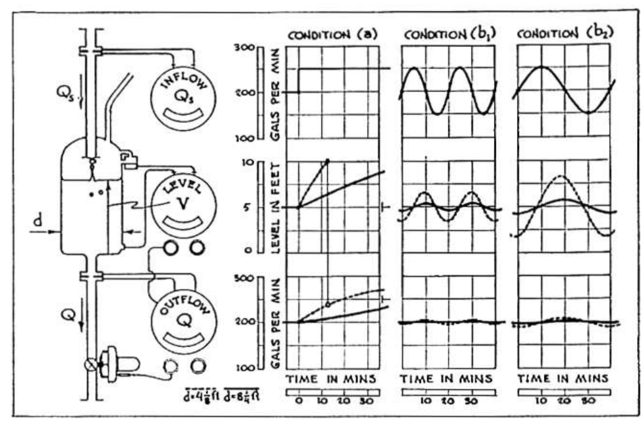
NUMERICAL SOLUTIONS		CONTROL CONSTANTS: $b = 30$ ft, $r = 0.05$ 1/min, $k = 200$ gals/min		
	$A = 5.88d^2$	CONDITION (a)	CONDITION (b_1)	CONDITION (b_2)
INFLOW Q_s gals/min		200 ---- ($t<0$) 250 ---- ($t>0$)	$200 + 50\sin\left[180°\frac{t-0}{10}\right]$	$200 + 50\sin\left[180°\frac{t-0}{20}\right]$
LEVEL V feet	100 $\frac{\text{gals}}{\text{foot}}$ ----	$5 + 10.61e^{-0.0333t}\cos\left[180°\frac{t-33.3}{66.7}\right]$	$5 + 1.61\sin\left[180°\frac{t-4.31}{10}\right]$	$5 + 3.30\sin\left[180°\frac{t-7.10}{20}\right]$
	400 " ___	$5 + 4.52e^{-0.00833t}\cos\left[180°\frac{t-56.8}{113.6}\right]$	$5 + 0.401\sin\left[180°\frac{t-4.83}{10}\right]$	$5 + 0.819\sin\left[180°\frac{t-9.30}{20}\right]$
OUTFLOW Q gals/min	100 " ----	$250 - 61.24e^{-0.0333t}\cos\left[180°\frac{t+13.0}{66.7}\right]$	$200 + 10.86\sin\left[180°\frac{t-4.81}{10}\right]$	$200 + 23.12\sin\left[180°\frac{t-9.60}{20}\right]$
	400 " ___	$250 - 52.22e^{-0.00833t}\cos\left[180°\frac{t-10.6}{113.6}\right]$	$200 + 2.70\sin\left[180°\frac{t-5.33}{10}\right]$	$200 + 5.73\sin\left[180°\frac{t-11.27}{20}\right]$
TIME-BOUNDARIES		($t>0$)	($-\infty < t < \infty$)	($-\infty < t < \infty$)

FIG. 6 PROPORTIONAL-PLUS-FLOATING LEVEL CONTROLLER; CASE II



NUMERICAL SOLUTIONS		CONTROL CONSTANTS: $b = 60$ ft, $r = 0.15$ 1/min, $k = 200$ gals/min		
	$A = 5.88d^2$	CONDITION (a)	CONDITION (b_1)	CONDITION (b_2)
INFLOW Q_s gals/min		200 ---- ($t<0$) 250 ---- ($t>0$)	$200 + 50\sin\left[180°\frac{t-0}{10}\right]$	$200 + 50\sin\left[180°\frac{t-0}{20}\right]$
LEVEL V feet	100 $\frac{\text{gals}}{\text{foot}}$ ----	$5 + 7.28e^{-0.0167t}\cos\left[180°\frac{t-22.9}{45.7}\right]$	$5 + 1.67\sin\left[180°\frac{t-4.65}{10}\right]$	$5 + 3.86\sin\left[180°\frac{t-8.34}{20}\right]$
	400 " ___	$5 + 3.64e^{-0.00417t}\cos\left[180°\frac{t-45.7}{91.4}\right]$	$5 + 0.414\sin\left[180°\frac{t-4.92}{10}\right]$	$5 + 0.837\sin\left[180°\frac{t-9.69}{20}\right]$
OUTFLOW Q gals/min	100 " ----	$250 - 61.84e^{-0.0167t}\cos\left[180°\frac{t+9.2}{45.7}\right]$	$200 + 6.15\sin\left[180°\frac{t-6.06}{10}\right]$	$200 + 17.78\sin\left[180°\frac{t-13.20}{20}\right]$
	400 " ___	$250 - 53.21e^{-0.00417t}\cos\left[180°\frac{t+10.1}{91.4}\right]$	$200 + 1.53\sin\left[180°\frac{t-6.33}{10}\right]$	$200 + 3.86\sin\left[180°\frac{t-14.50}{20}\right]$
TIME-BOUNDARIES		($t>0$)	($-\infty < t < \infty$)	($-\infty < t < \infty$)

FIG. 7 PROPORTIONAL-PLUS-FLOATING LEVEL CONTROLLER; CASE III



NUMERICAL SOLUTIONS		CONTROL CONSTANTS: $b = 60$ ft, $r = 0.025$ 1/min, $k = 200$ gals/min		
	$A = 5.88d^2$	CONDITION (a)	CONDITION (b_1)	CONDITION (b_2)
INFLOW Q_s gals/min		200 ---- ($t<0$) 250 ---- ($t>0$)	$200 + 50\sin\left[180°\frac{t-0}{10}\right]$	$200 + 50\sin\left[180°\frac{t-0}{20}\right]$
LEVEL V feet	100 $\frac{\text{gals}}{\text{foot}}$ ----	$5 + 21.21e^{-0.0167t}\cos\left[180°\frac{t-66.9}{133.7}\right]$	$5 + 1.60\sin\left[180°\frac{t-4.66}{10}\right]$	$5 + 3.22\sin\left[180°\frac{t-8.62}{20}\right]$
	400 " ___	$5 + 9.05e^{-0.00417t}\cos\left[180°\frac{t-113.7}{227.3}\right]$	$5 + 0.399\sin\left[180°\frac{t-4.92}{10}\right]$	$5 + 0.801\sin\left[180°\frac{t-9.66}{20}\right]$
OUTFLOW Q gals/min	100 " ----	$250 - 117.3e^{-0.0167t}\cos\left[180°\frac{t+48.1}{133.7}\right]$	$200 + 5.34\sin\left[180°\frac{t-4.91}{10}\right]$	$200 + 10.86\sin\left[180°\frac{t-9.69}{20}\right]$
	400 " ___	$250 - 67.42e^{-0.00417t}\cos\left[180°\frac{t+53.2}{227.3}\right]$	$200 + 1.33\sin\left[180°\frac{t-5.17}{10}\right]$	$200 + 2.70\sin\left[180°\frac{t-10.67}{20}\right]$
TIME-BOUNDARIES		($t>0$)	($-\infty < t < \infty$)	($-\infty < t < \infty$)

FIG. 8 PROPORTIONAL-PLUS-FLOATING LEVEL CONTROLLER; CASE IV

$$AR_e(Q - Q_m)'' + (Q - Q_m)' + r(Q - Q_m) = r(Q_s - Q_m) + (Q_s - Q_m)' \qquad [19]$$

From the integration of these differential equations, we may determine the response of the level V and of the outflow Q when sudden changes occur in the inflow Q_s. The curves under condition (a) in Figs. 5, 6, 7, and 8 represent the response of the level and the outflow following the usual sudden disturbance, when various magnitudes of proportional band b and reset constant r are assumed for the proportional-plus-floating level controller. The numerical equations from which these curves were computed are included in the figures.

* * *

In operational form, Equations [18] and [19] become

$$V - V_n = \frac{R_e p}{r + p + AR_e p^2} \cdot (Q_s - Q_m) \qquad [20]$$

$$Q - Q_m = \frac{r + p}{r + p + AR_e p^2} \cdot (Q_s - Q_m) \qquad [21]$$

When a sudden change occurs in the inflow Q_s, the response of the level V and of the outflow Q may be found by classical methods from Equations [18] and [19] or by standard operational methods from Equations [20] and [21]. Under equilibrium conditions, after all transients have faded out and all derivatives have become zero, it is evident that $V = V_n$ and that $Q = Q_s$. Thus the level will ultimately balance out at the desired value for all of the values of flow.

As before, the response under permanently oscillatory conditions may be found by setting $p = i\pi/h$ in the operators of Equations [20] and [21]. For level and outflow, respectively, the operators yield the following complex expressions, where $G = \pi AR_e/h$ and $H = rh/\pi$

$$\frac{[1 - i(G - H)]R_e}{1 + (G - H)^2}$$

$$\frac{1 - H(G - H) - iG}{1 + (G - H)^2}$$

From these complex expressions, the amplitude ratios and the relative time lags may be found by methods already described. This information is completely descriptive of the behavior of level and outflow when the inflow is assumed to follow a given permanent harmonic oscillation about a constant mean value. Thus we find the following

$$\frac{\text{Ampl. of } V}{\text{Ampl. of } Q_s} = \frac{R_e}{\sqrt{1 + (G - H)^2}}$$

$$\text{Lag of } V \text{ versus } Q_s = (h/\pi) \tan^{-1} (G - H)$$

$$\frac{\text{Ampl. of } Q}{\text{Ampl. of } Q_s} = \sqrt{\frac{1 + H^2}{1 + (G - H)^2}}$$

$$\text{Lag of } Q \text{ versus } Q_s = (h/\pi) \tan^{-1} [G/(1 - GH + H^2)]$$

* * *

As in the previous cases, the general equations for V and Q under the cyclic conditions (b_1) and (b_2) may be written

$$V = V_n + A_v \sin \left[180° \frac{t - T_v}{h} \right] \qquad [7]$$

$$Q = Q_m + A_q \sin \left[180° \frac{t - T_q}{h} \right] \qquad [8]$$

The equations for the constants in these equations are again taken from the operational development, as outlined, and can be given as

$$A_v = \text{Ampl. of } V = \frac{R_e \times (\text{Ampl. of } Q_s)}{\sqrt{1 + (G - H)^2}}$$
$$= \text{level variation in feet}$$

$$A_q = \text{Ampl. of } Q = (\text{Ampl. of } Q_s) \times \sqrt{\frac{1 + H^2}{1 + (G - H)^2}}$$
$$= \text{outflow variation in gpm}$$

$$T_v = \frac{h}{2} \frac{\tan^{-1} (G - H)}{90°} = \text{time in minutes by which cycles of } V \text{ lag behind cycles of } Q_s$$

$$T_q = \frac{h}{2} \frac{\tan^{-1} [G/(1 - GH + H^2)]}{90°} = \text{time in minutes by which cycles of } Q \text{ lag behind cycles of } Q_s$$

in which $G = 3.14 \dfrac{AR_e}{h}$ and $H = 0.318rh$

Compared to those for the case of proportional control, these equations have become more complex, due to the inclusion of the reset constant r, but it is interesting to note the nature of the changes and the fact that the equations will reduce to those for proportional control on substituting $r = 0$. An important difference is that the cycles of the level and those of the outflow are no longer in phase with one another.

In Figs. 5 through 8 are shown numerical and graphical examples of the application of the general formulas obtained. Four different cases are taken, covering four different sets of adjustments incorporated in the proportional-plus-floating level controller. Otherwise the conditions assumed are the same as were those for the previously considered system. The values of (effective) proportional band b and of reset constant r assigned in the various cases are given in tabular form as follows:

	Proportional band (b)	Reset constant (r)
Case I (Fig. 5)	30 Ft (300 per cent)	0.15 Inverse min
Case II (Fig. 6)	30 Ft (300 per cent)	0.05 Inverse min
Case III (Fig. 7)	60 Ft (600 per cent)	0.15 Inverse min
Case IV (Fig. 8)	60 Ft (600 per cent)	0.025 Inverse min

With proportional bands wider than 100 per cent, it is necessary to consider the effect of sustained changes in the inflow. Figs. 5 through 8, under condition (a), show the response of the level and the outflow following a sudden sustained change in the inflow. After such a change, the duty of the installation is to bring the outflow as smoothly as possible into equality with the new inflow, and also to return the level to the normal value. The more time allowed for these operations, the better the duties of smoothing may be performed. Shown in the figures are the initial portions of the level and outflow transients following the instantaneous disturbance of condition (a).

In case III, Fig. 7, and in case IV, Fig. 8, the size of the sudden change in the inflow is such that (for the low-capacity vessel) the level reaches its high limit in about $16^1/_2$ and $12^1/_2$ min, respectively. The practical design of proportional-plus-floating control instruments, capable of utilizing such excessive magnitudes of proportional band, must include mechanical means for decreasing the effective "throttling range" in the immediate region of the high and low limits. Details of such mechanism and of its operation are discussed in the paper (1) by J. B. McMahon.

It is readily evident that the general equations for the cyclic conditions can be put to practical use in determining many of the important relationships in actual averaging-control installations. The substitution of known factors permits concrete determination of other factors or relationships between them, as in connection with (a) vessel areas to give desired smoothing of the outflow for various types of instruments; (b) periods and magnitudes of oscillation which could be tolerated in existing installations; (c) economic considerations of instrument investment against increased equipment costs, etc.

The particular process and conditions selected for consideration in this paper illustrate many of the common circumstances met with in commercial installations. Much general information can be gained by exploring the hidden "intelligence" of these equations. Space will only permit us a brief discussion of one series of observations which appears to shed light on the nature of desirable instrument adjustments.

Under case I, Fig. 5, the response for the smaller vessel and the longer period of oscillation shows that both the level and the outflow variations have been increased over those for the 100 per cent throttling control, Fig. 4, although the proportional band has been trebled. Furthermore, the level variation has been increased beyond that of Fig. 2, where the outflow was perfectly constant. This circumstance is a result of the fact that the reset constant is too great for the existing conditions of vessel area and period of oscillation.

In case II, Fig. 6, the reset constant is made one third of its value in case I, Fig. 5, but the proportional band is kept at the same value. For the same vessel area and period of oscillation, a marked improvement is discernible in the variation of the outflow. Some reduction is also made in the level variation, although this variation is still in excess of that for the ideal case of Fig. 2.

In case III, Fig. 7, the proportional band is increased to 6 times that used in the 100 per cent throttling control, Fig. 4, or to twice that used in cases I and II, Figs. 5 and 6, but the same reset constant is applied as in case I. A still further reduction in outflow variation is obtained, but the variation of the level is greater than that of case II, Fig. 6. This means that the reset constant could still be reduced.

In case IV, Fig. 8, the same proportional band is used as in case III, Fig. 7, but the reset constant is reduced to one sixth that of case III, Fig. 7, or to one half that of case II, Fig. 6. Another marked improvement is evident in the smoothing of the outflow variations, as well as a further reduction in those of the level. It is interesting to observe, in this case, for both of the vessel areas and for both periods of inflow oscillation, how closely the magnitude and lag of the level variations have approached those seen under the ideal case of constant outflow, Fig. 2.

Even this brief introductory treatment and these few observations seem to have established certain of the characteristic properties of the proportional band and reset adjustments in connection with automatic averaging control. The authors feel that a considerable amount of investigation remains to be done in this direction and that such work could be of tremendous practical value to those in industry who are faced with averaging-control problems. We have only endeavored to point out a possible approach. Even from the quantitative material presented here, tables could be compiled or charts prepared which would facilitate the engineering of installations.

Acknowledgment

Acknowledgment is wholeheartedly made to all those who have contributed, through their experiences and discussion, to the developments here presented. Appreciation is especially due to R. A. Rockwell of The Foxboro Company for his valuable assistance and advice concerning a rational development of the substance of the text. The authors are grateful also to many members of the Society, and in particular to P. W. Keppler for his encouragement in the preparation of the paper.

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