Recent Developments in Gain Scheduling Control

by

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Outline

- Introduction, definition and motivation.
- Classifications of gain-scheduling control techniques.
- Mathematical descriptions of linear time varying systems.
- Parameter dependent systems and control.
- Systems with linear fractional dependence in the parameters (LFT-systems).
- Linear parameter varying systems with induced quadratic performance.
- Summary.

What is gain scheduling control?

- Originally a control scheme to counteract nonlinear variations in the steady-state process gain.
- A gain scheduling controller is a parameterized set of linear controllers. In operation the parameter is measured (available) and the controller in action is scheduled (determined) according to the parameter.
- Example: Time varying PID control, where K_p (may also include T_i and T_d) is tabulated as a function of operating conditions.

ρ_2

ρ_1

Steps in the design of a gain scheduling controller.

1) Select a set of stationary operating points.
2) Design the controllers for each operating point.
3) Design the scheduling algorithm.
4) Implement the parameter-scheduled controller.

Motivating examples

1) Gain scheduling control of airplane. Scheduling variables:
   - Mach number (static air pressure and velocity).
   - Altitude (height above sea level).
2) Application of gain scheduling control in chemical process control.

Adaptive controller?

y linearly

Classification of gain-scheduling control techniques

Adam Lagerberg (1996):

- Linear Parameter Varying (LPV) approach.
- Gain-scheduling with Linear Fractional Transformations (LFT).
- Extended linearization and linearization families.
- The D-method.

In this presentation only the LFT and the LPV approaches will be considered.

Three types of “linear” system descriptions

- Linear Time Invariant (LTI) systems

$$\begin{bmatrix} \dot{x} \\ e \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ d \end{bmatrix}$$

- Linear Parameter Varying (LPV) systems

$$\begin{bmatrix} \dot{x} \\ e \end{bmatrix} = \begin{bmatrix} A(\rho) & B(\rho) \\ C(\rho) & D(\rho) \end{bmatrix} \begin{bmatrix} x \\ d \end{bmatrix}$$

- Linear Time Varying (LTV) systems

$$\begin{bmatrix} \dot{x} \\ e \end{bmatrix} = \begin{bmatrix} A(t) & B(t) \\ C(t) & D(t) \end{bmatrix} \begin{bmatrix} x \\ d \end{bmatrix}$$

Note, a LPV-system becomes a:

1) LTI-system for $\rho =$ const. and
2) LTV-system for $\rho = \rho(t)$ (along a time-varying trajectory).

Parameter-Dependent Systems

Parameter-Dependent linear plants:

$$\begin{bmatrix} \dot{x} \\ e \\ y \end{bmatrix} = \begin{bmatrix} A(\rho) & B_1(\rho) & B_2(\rho) \\ C_1(\rho) & D_{11}(\rho) & D_{12}(\rho) \\ C_2(\rho) & D_{21}(\rho) & D_{22}(\rho) \end{bmatrix} \begin{bmatrix} x \\ d \\ u \end{bmatrix}$$

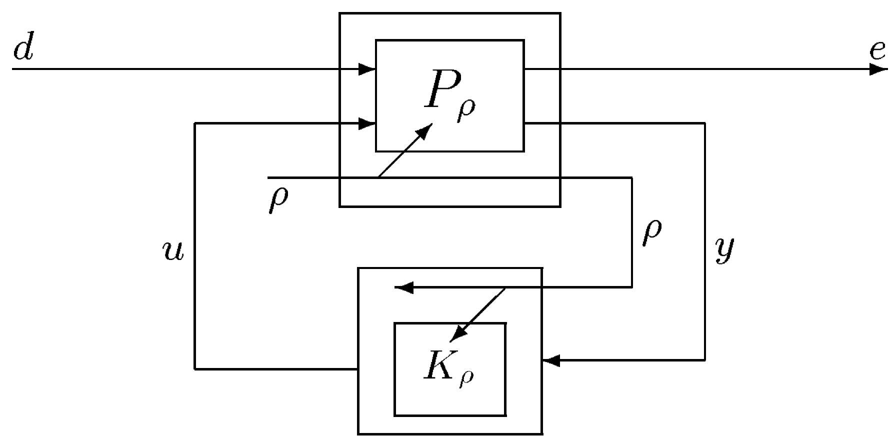
The parameter ρ:

- is time-varying, i.e. $\rho(t)$,
- takes values in a compact set $\mathcal{P}$, and
- there are known bounds on $\dot{\rho}$ (which may, or may not be exploited)

For control:

- The time variations are not known in advance.
- The parameter values $\rho(t)$ are measured in real-time with sensors.

Parameter-Dependent Control of Parameter-Dependent Systems



- Parameter dependent controller K_ρ which processes
- b and
  - ρ nonlinearly.

Optimizing closed-loop performance with respect to parameter variations.

Parameter-Dependent Systems

Consider two types of Parameter-Dependent systems

- Linear Parameter Varying (LPV), and
- Linear Fractional Transformation (LFT) models.

Distinctions:

- The allowable dependence that the state-space data has on the parameters.
- To which extent information about the parameter's variation are exploited in the analysis.
- The different techniques used to analyze the systems.

Obvious questions

Natural to look into:

- Stability.
- Stabilizability and Detectability.
- Parameterization of all stabilizing controllers.
- Choosing K to optimize performance.

Parameter-Dependent Systems: Stability Definitions

Two facts:

- Stability of LTV-system is well characterized.
- When evaluating along a allowable trajectory, the P-D system becomes an LTV-system.

LFT and LPV systems, two different approaches to stability tests:

- LFT-systems: Structured Small-Gain theorems.
- LPV-systems: Parameter dependent Lyapunov functions,

P-D Systems: Stabilizability and Detectability

- Stabilizability of $(A(\rho), B(\rho))$: If there exists a P-D state-feedback $u(t) = F(\rho)x(t)$, such that:

$$\dot{x} = A(\rho)x(t) + B(\rho)F(\rho)x(t)$$

  is stable for all possible parameters $\rho \in \mathcal{P}$

- Detectability of $(C(\rho), A(\rho))$: If there exists a P-D output to state-feedback $L(\rho)$, such that:

$$\dot{x} = A(\rho)x(t) + L(\rho)C(\rho)x(t)$$

  is stable for all possible parameters $\rho \in \mathcal{P}$

P-D Systems: All "stabilizing" controllers

Extension to Youla parameterization (Wei Min Lu) for

- the two types of P-D systems
- controlled by their corresponding P-D controllers
- with their corresponding notations on P-D stability.

Output feedback stabilization:

- Given an open-loop P-D system

$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \begin{bmatrix} A(\rho) & B(\rho) \\ C(\rho) & D(\rho) \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}$$

- There exists a finite-dimensional P-D controller that stabilizes the system if and only if

  1) The pair $(A(\rho), B(\rho))$ is stabilizable.

  2) The pair $(C(\rho), A(\rho))$ is detectable.

- Usual observer structure.

LFT-systems

Given:

- Real-parameters δ_i which may be repeated, give rise to:

$$\left\{\mathcal{D}_{\mathcal{S}} \triangleq \mathrm{diag}\{\delta_1 I_{s_1}, \ldots, \delta_f I_{s_f}\} : \delta_i \in \mathbb{R}\right\} \subset \mathbb{R}^{s\times s}, \quad \mathcal{S} \triangleq (s_1, s_2, \ldots, s_f)$$

- System matrix M, partitioned according to:

$$\begin{bmatrix} \dot{x}(t) \\ e(t) \\ \alpha(t) \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} & M_{22} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \begin{bmatrix} x(t) \\ d(t) \\ \beta(t) \end{bmatrix}, \quad \beta(t) = \Delta(t)\alpha(t)$$

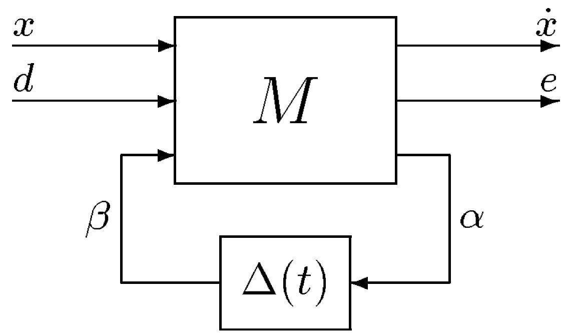
The LFT-system G_Δ is described by $G_\Delta = \mathcal{F}_l(M, \Delta(t))$, with state-space equations

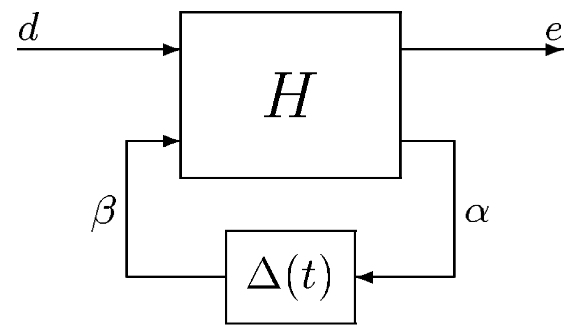
$$\begin{bmatrix} \dot{x}(t) \\ e(t) \end{bmatrix} = \underbrace{\left\{\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} + \begin{bmatrix} M_{13} \\ M_{23} \end{bmatrix} \Delta(t)(I - M_{33}\Delta(t))^{-1} \begin{bmatrix} M_{31} & M_{32} \end{bmatrix}\right\}}_{G_\Delta} \begin{bmatrix} x(t) \\ d(t) \end{bmatrix}$$

- where allowable piecewise continues $\Delta(t)$ trajectories satisfy:

$$\Delta(t) = \mathcal{D}_{\mathcal{S}}(\delta(t)), \quad |\delta_i(t)| \leq 1, \quad \text{however, no restrictions on } \dot{\delta}.$$

LFT-systems graphically





$$\begin{bmatrix} \dot{x}(t) \\ e(t) \end{bmatrix} = \mathcal{F}_l(M, \Delta(t)) \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \qquad e = \mathcal{F}_l(H, \Delta)d$$

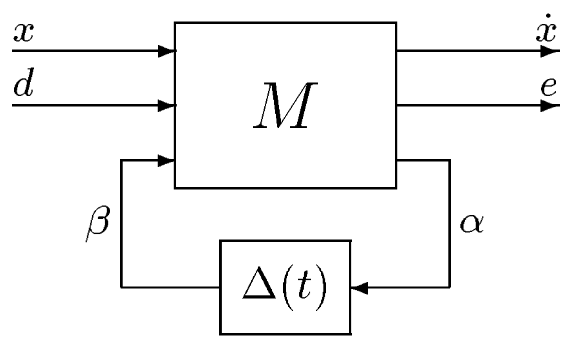
where $H(s)$ is the 2-input, 2-output transfer function such that

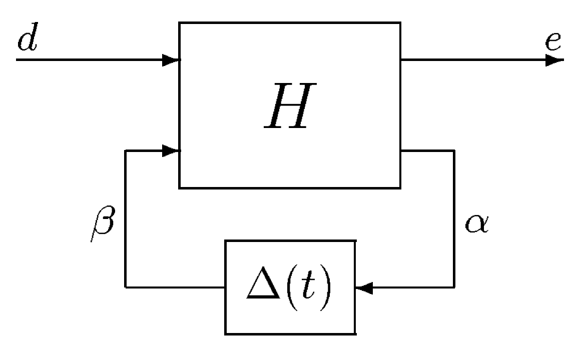
$$\begin{bmatrix} e \\ \alpha \end{bmatrix} = \begin{bmatrix} H_{11}(s) & H_{12}(s) \\ H_{21}(s) & H_{22}(s) \end{bmatrix} \begin{bmatrix} d \\ \beta \end{bmatrix}$$

$H(s)$ given in terms of M is

$$H(s) = \begin{bmatrix} M_{22} & M_{23} \\ M_{32} & M_{33} \end{bmatrix} + \begin{bmatrix} M_{21} \\ M_{31} \end{bmatrix} (sI - M_{11})^{-1} \begin{bmatrix} M_{12} & M_{13} \end{bmatrix}$$

Robust stability condition for LFT-systems with linear time varying Δ





The LFT-system $(M, \mathcal{S})$ is stable if

1) M_{11} is Hurwitz, and
2) there exists a constant matrix $Z = Z^T > 0$ of the form

$$Z = \text{diag}\{Z_1, Z_2, \dots, Z_f\}, \quad \text{with} \quad Z_i \in \mathbb{R}^{s_i \times s_i}, \quad \text{such that} \quad Z\Delta(t) = \Delta(t)Z$$

satisfying

$$\left\| Z^{\frac{1}{2}} H_{22} Z^{-\frac{1}{2}} \right\|_\infty < 1$$

If so, small-gain argument verify exponential stability of

$$\dot{x}(t) = \{M_{11} + M_{13}\Delta(t)(I - M_{33}\Delta(t))^{-1} M_{31}\} x(t)$$

Scaled bounded real lemma (Gahinet and Apkarian, 1994)

With $H_{22} = C(sI - A)^{-1}B + D$, then the following two statements are equivalent:

1) A is stable and there exists a constant diagonal matrix $Z = Z^T > 0$

$$Z = \mathrm{diag}\{Z_1, Z_2, \ldots, Z_f\}, \quad \text{with} \quad Z_i \in \mathbb{R}^{s_i}, \quad \text{such that} \quad Z\Delta(t) = \Delta(t)Z$$

   satisfying

$$\left\| Z^{\frac{1}{2}} H_{22} Z^{-\frac{1}{2}} \right\|_\infty < 1$$

2) There exist solutions $X = X^T > 0$ and $Z = Z^T > 0$ (with the structure give above) such that

$$\begin{bmatrix} A^T X + XA & XB & C^T \\ B^T X & -Z & D^T \\ C & D & -Z^{-1} \end{bmatrix} < 0$$

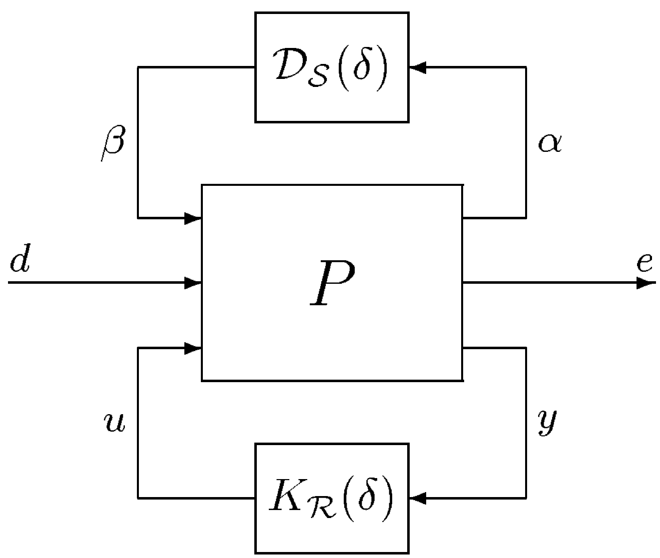
   is fulfilled.

LFT closed-loop system

Define time invariant rational transfer function matrix $P(s)$

$$\begin{bmatrix} \alpha \\ e \\ y \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \begin{bmatrix} \beta \\ d \\ u \end{bmatrix}$$

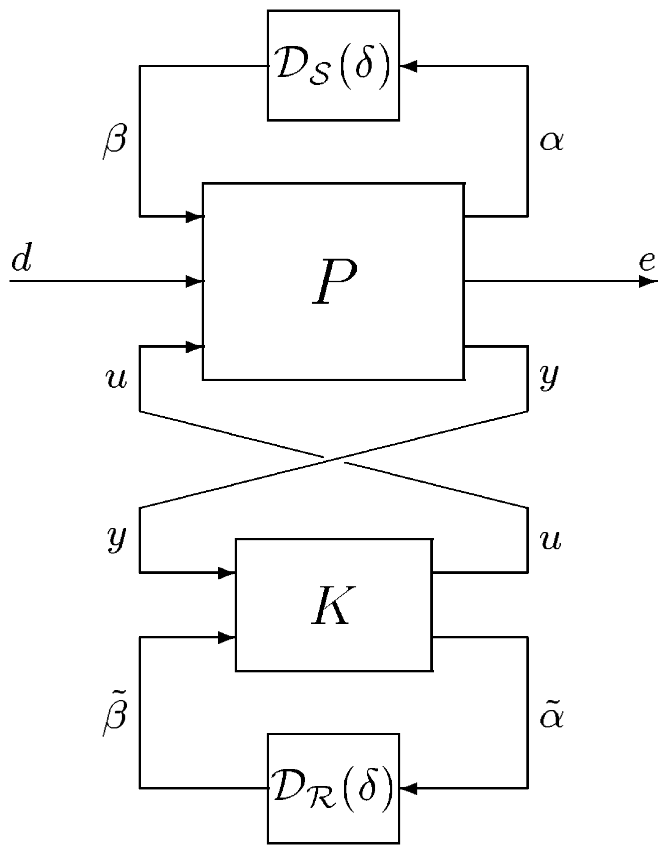
Interconnection of u and y, i.e. $u = K_{\mathcal{S}}(\delta)y$

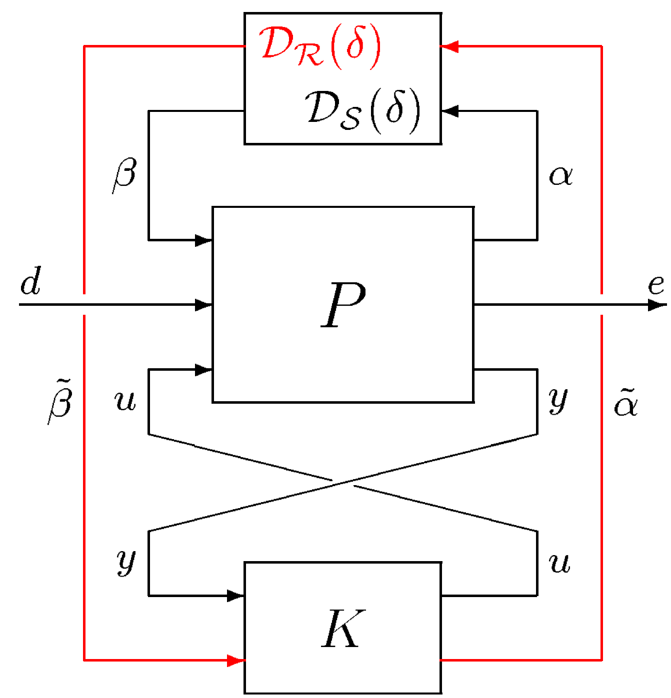


$$P(s) \overset{s}{=} \left[\begin{array}{c|ccc} A & B_\delta & B_1 & B_2 \\ \hline C_\delta & D_{\delta\delta} & D_{\delta 1} & D_{\delta 2} \\ C_1 & D_{1\delta} & D_{11} & D_{12} \\ C_2 & D_{2\delta} & D_{21} & D_{22} \end{array}\right]$$

The time varying parameters δ enter both in $\mathcal{D}_{\mathcal{S}}(\delta)$ and in $K_{\mathcal{R}}(\delta)$!

LFT closed-loop system





LFT closed-loop system

Closed-loop transfer function is:

$$T(P, K, \mathcal{D}_{\mathcal{S}}(\delta), \mathcal{D}_{\mathcal{R}}(\delta)) = \mathcal{F}_l(\mathcal{F}_u(P, \mathcal{D}_{\mathcal{S}}(\delta)), \mathcal{F}_l(K, \mathcal{D}_{\mathcal{R}}(\delta))$$

Define $P_a(s)$

$$\begin{bmatrix} \tilde{\alpha} \\ \alpha \\ e \\ y \\ \tilde{\beta} \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & I_r \\ 0 & P_{11} & P_{12} & P_{13} & 0 \\ 0 & P_{21} & P_{22} & P_{23} & 0 \\ 0 & P_{31} & P_{32} & P_{33} & 0 \\ I_r & 0 & 0 & 0 & 0 \end{bmatrix}}^{P_a(s)} \begin{bmatrix} \tilde{\beta} \\ \beta \\ d \\ u \\ \tilde{\alpha} \end{bmatrix}$$

Closed-loop transfer function in terms of $P_a(s)$:

$$T(P_a, K, \mathcal{D}_{\mathcal{S}}(\delta), \mathcal{D}_{\mathcal{R}}(\delta)) = \mathcal{F}_u\left(\mathcal{F}_l(P_a(s), K(s)), \begin{bmatrix} \mathcal{D}_{\mathcal{R}}(\delta) & \\ & \mathcal{D}_{\mathcal{S}}(\delta) \end{bmatrix}\right)$$

- The LFT gain-scheduling control problem can be treated as a classical robust performance control problem with the nominal plant P_a, and with repeated real block uncertainty $\begin{bmatrix} \mathcal{D}_{\mathcal{R}}(\delta) & \\ & \mathcal{D}_{\mathcal{S}}(\delta) \end{bmatrix}$.

Induced L_2 performance in LFT-systems

- Given a time varying vector $e(t)$, then the L_2-norm of $e(t)$ is defined by

$$\|e(t)\|_2 \triangleq \left\{ \int_0^\infty \|e(\tau)\|^2 \, d\tau \right\}^{\frac{1}{2}} < \infty$$

- Induced L_2 performance metric

$$\max_{\text{allowable}\,\delta} \; \max_{d} \; \frac{\|G_\Delta d\|_2}{\|d\|_2}$$

  i.e. take the "worst-case" (over all parameter trajectories) induced L_2 gain from d to e.

- Induced L_2 performance bound γ $(L_2$-gain$)$

$$\int_0^T e^T(\tau) e(\tau) d\tau \leq \gamma^2 \int_0^T d^T(\tau) d(\tau) d\tau \quad \forall T \geq 0$$

LFT synthesis problem

Given:

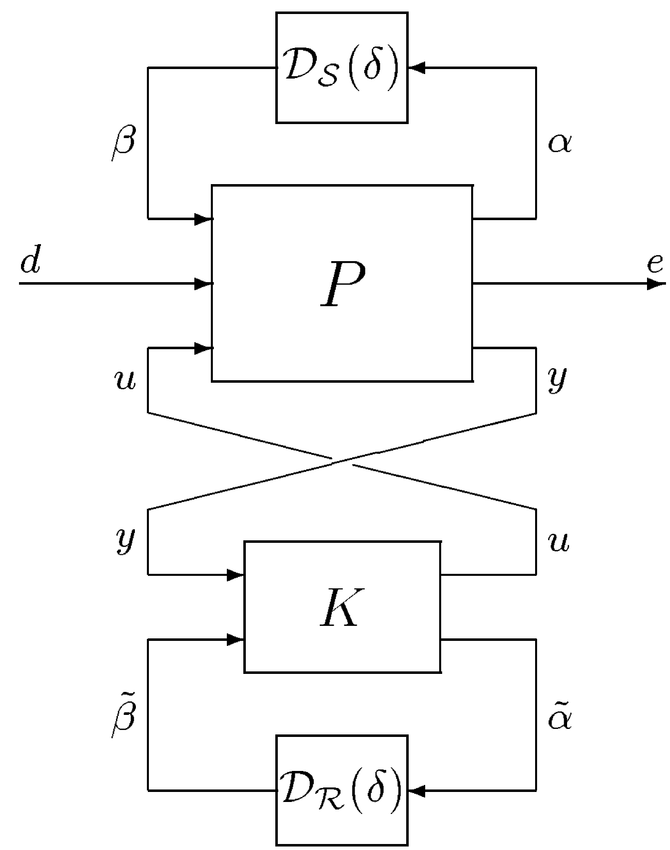
1) Finite dimensional linear P

$$P(s) \stackrel{s}{=} \left[\begin{array}{c|ccc} A & B_\delta & B_1 & B_2 \\ \hline C_\delta & D_{\delta\delta} & D_{\delta 1} & D_{\delta 2} \\ C_1 & D_{1\delta} & D_{11} & D_{12} \\ C_2 & D_{2\delta} & D_{21} & D_{22} \end{array}\right]$$

2) Vector of integers $\mathcal{S}$ describing the plant's parameter structure.

Find (if possible):

1) Finite dimensional linear K.
2) Vector of integers $\mathcal{R}$, describing the controller's parameter structure.



LFT synthesis setup

- $\Delta(t) \in \mathcal{D}_{\mathcal{S}}(\delta)$.
- Let Z_Δ denote the set of matrices such that for $Z \in Z_\Delta$ so that $Z\Delta = \Delta Z$
- Let $\mathcal{N}_X$ and $\mathcal{N}_Y$ be bases for the null spaces of

$$[\,B_2^T \quad D_{\delta 2}^T \quad D_{12}^T \quad 0\,] \quad \text{and} \quad [\,C_2 \quad D_{2\delta} \quad D_{21} \quad 0\,]$$

- Let $X_i \in \mathbb{R}^{s_i \times s_i}$, $Y_i \in \mathbb{R}^{s_i \times s_i}$ and define

$$\widehat{X} = \mathrm{diag}\{X_1, X_2, \ldots, X_f\} \quad \text{and} \quad \widehat{Y} = \mathrm{diag}\{Y_1, Y_2, \ldots, Y_f\}$$

  then $\widehat{X}, \widehat{Y} \in Z_\Delta$.
- Define

$$\widehat{B}_1 = [\,B_\delta \quad B_1\,], \quad \widehat{C}_1 = \begin{bmatrix} C_\delta \\ C_1 \end{bmatrix} \quad \text{and} \quad \widehat{D}_{11} = \begin{bmatrix} D_{\delta\delta} & D_{\delta 1} \\ D_{1\delta} & D_{11} \end{bmatrix}$$

LFT synthesis solution

The LFT gain-scheduling control problem is solvable with closed-loop performance bound γ, if there exist matrices:

$$X_0 = X_0^T \in \mathbb{R}^{n\times n}, \quad Y_0 = Y_0^T \in \mathbb{R}^{n\times n}, \quad \widehat{X} = \widehat{X}^T \in Z_\Delta \quad \text{and} \quad \widehat{Y} = \widehat{Y}^T \in Z_\Delta$$

such that

$$\mathcal{N}_X \begin{bmatrix} AX_0 + X_0A^T & X_0\widehat{C}_1^T & \widehat{B}_1 \\ \widehat{C}_1X_0 & -\gamma\begin{bmatrix}\widehat{Y} & 0\\ 0 & I\end{bmatrix} & \widehat{D}_{11} \\ \widehat{B}_1^T & \widehat{D}_{11}^T & -\gamma\begin{bmatrix}\widehat{X} & 0\\ 0 & I\end{bmatrix} \end{bmatrix} \mathcal{N}_Y < 0 \tag{1}$$

$$\mathcal{N}_Y \begin{bmatrix} A^TY_0 + Y_0A & Y_0\widehat{B}_1 & \widehat{C}_1^T \\ \widehat{B}_1^TY_0 & -\gamma\begin{bmatrix}\widehat{X} & 0\\ 0 & I\end{bmatrix} & \widehat{D}_{11}^T \\ \widehat{C}_1 & \widehat{D}_{11} & -\gamma\begin{bmatrix}\widehat{Y} & 0\\ 0 & I\end{bmatrix} \end{bmatrix} \mathcal{N}_X < 0 \tag{2}$$

$$\begin{bmatrix} X_0 & I \\ I & Y_0 \end{bmatrix} \geq 0 \quad \text{and} \quad \begin{bmatrix} \widehat{X} & I \\ I & \widehat{Y} \end{bmatrix} \geq 0 \tag{3}$$

LFT synthesis solution summary

Sufficient condition for existence of LFT-controller such that the closed loop system has performance level $< \gamma$:

- if and only if there are matrices $X = X^T > 0$ and $Y = Y^T > 0$ with structure

  $$X = \mathrm{diag}\{X_0, X_1, \ldots, X_f\}, \quad Y = \mathrm{diag}\{Y_0, Y_1, \ldots, Y_f\},$$

  satisfying

  $$\text{AMI}_1(X, \gamma) < 0, \quad \text{AMI}_2(Y, \gamma) < 0 \quad \text{and} \quad \begin{bmatrix} X_i & I \\ I & Y_i \end{bmatrix} \geq 0$$

- γ-suboptimal controller of order k if

  $$\mathrm{rk}(I - X_0 Y_0) \leq k$$

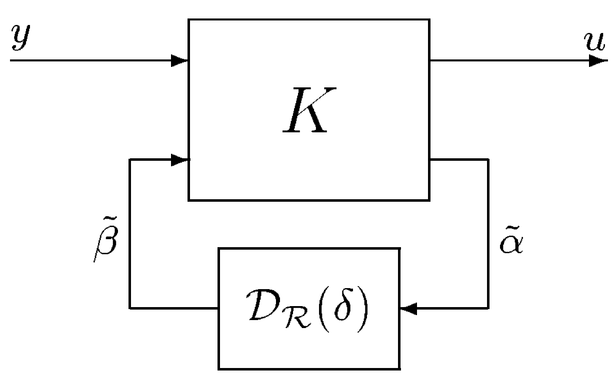
Some comments:

- The linear matrix inequalities are finite dimensional.
- Robust numerical methods for solving the affine linear matrix inequalities exist.

LFT-controller

Parameter dependent controller:

- $\mathcal{R} = \mathcal{S}$, i.e. the controller's parameter structure is the same as the plant's parameter structure.
- The LTI part of the LFT controller K, is reconstructed from X and Y, i.e. the solutions to the affine matrix inequalities.
- Implementation:



With δ given (measured):

$$K_{\mathcal{R}}(\delta) = \mathcal{F}_l(K, \mathcal{D}_{\mathcal{R}}(\delta))$$
$$u = K_{\mathcal{R}}(\delta) y$$

- The control action u is linear in y, nonlinear in δ.

LFT extensions and remarks

- Results on LFT synthesis derived independently by (Packard, 1997):
  - Andy Packard, Greg Becker and Fen Wu, and
  - Pierre Apkarian and Pascal Gahinet.

  But note, similar ideas have been proposed by Lu and Doyle (1992,1995).
- The Small-Gain LFT test may be conservative due to the realness of the parameters δ.
- Helmersson has generalized the ideas to exploit the realness of the parameters (upper bound on μ for repeated real parameters).
  - The synthesis conditions are still AMI's in block diagonal form.
  - Controller structure is still a LFT of a fixed LTI-system.

  An improved upper bound on μ for repeated real parameters is also given by Braatz and Morari (1997).
- The results may also be conservative since they do not take into account the rate variations in the parameters δ.

LPV-systems

Parameter set, $\mathcal{P} \subset \mathbb{R}^f$, LPV-system G_ρ on state-space form:

$$\begin{bmatrix} \dot{x} \\ e \\ y \end{bmatrix} = \begin{bmatrix} A(\rho) & B_1(\rho) & B_2(\rho) \\ C_1(\rho) & D_{11}(\rho) & D_{12}(\rho) \\ C_2(\rho) & D_{21}(\rho) & 0 \end{bmatrix} \begin{bmatrix} x \\ d \\ u \end{bmatrix}$$

where a allowable trajectory $\rho(\cdot)$ satisfy:

- $\rho(t) \in \mathcal{P}$, for all t and
- for each ρ_i and all t, $\underline{\nu}_i(\rho(t)) \leq \dot{\rho}_i(t) \leq \bar{\nu}_i(\rho(t))$, i.e. bounded rate of variation.

LPV-systems: control problem formulation

The gain-scheduling output feedback controller $K(\rho)$ is given by:

$$\begin{bmatrix} \dot{x}_K \\ u \end{bmatrix} = \begin{bmatrix} A_K(\rho,\dot{\rho}) & B_K(\rho,\dot{\rho}) \\ C_K(\rho,\dot{\rho}) & D_K(\rho,\dot{\rho}) \end{bmatrix} \begin{bmatrix} x_K(t) \\ y \end{bmatrix}$$

- Induced L_2 performance bound γ

$$\int_0^T e^T(\tau)e(\tau)d\tau \leq \gamma^2 \int_0^T d^T(\tau)d(\tau)d\tau \quad \forall T \geq 0$$

- Lyapunov function $V(x_{cl}, P(\rho)) = x_{cl} P(\rho) x_{cl}$

LPV-systems: Basic characterization (Apkarian and Adams,1998)

There exists a gain-scheduling output-feedback controller enforcing internal stability and a performance bound γ, whenever there exist P-D matrices $X = X^T > 0$ and $Y = Y^T > 0$ and a P-D quadruple $(\hat{A}_K, \hat{B}_K, \hat{C}_K, D_K)$, such that

$$\begin{bmatrix} \dot{X} + XA + \hat{B}_K C_2 + (\star) & \star & \star & \star \\ \hat{A}_K^T + A + B_2 D_K C_2 & -\dot{Y} + AY + B_2 \hat{C}_K + (\star) & \star & \star \\ (XB_1 + \hat{B}_K D_{21})^T & (B_1 + B_2 D_K D_{21})^T & -\gamma I & \star \\ C_1 + D_{12} D_K C_2 & C_1 Y + D_{12} \hat{C}_K & D_{11} + D_{12} D_K D_{21} & -\gamma I \end{bmatrix} < 0$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0$$

- The controller (A_K, B_K, C_K, D_K) can be reconstructed from:

$$X, \quad Y \quad \text{and} \quad (\hat{A}_K, \hat{B}_K, \hat{C}_K, D_K)$$

LPV-systems: Projected solvability conditions (Apkarian and Adams,1998)

There exists a gain-scheduling output-feedback controller enforcing internal stability and a performance bound γ, whenever there exist P-D matrices $X = X^T > 0$ and $Y = Y^T > 0$, such that

$$\left[\begin{array}{c|c} \mathcal{N}_X & 0 \\ \hline 0 & I \end{array}\right]^T \left[\begin{array}{cc|c} \dot{X} + XA + A^T X & XB_1 & C_1^T \\ B_1^T X & -\gamma I & D_{11}^T \\ \hline C_1 & D_{11} & -\gamma I \end{array}\right] \left[\begin{array}{c|c} \mathcal{N}_X & 0 \\ \hline 0 & I \end{array}\right] < 0$$

$$\left[\begin{array}{c|c} \mathcal{N}_Y & 0 \\ \hline 0 & I \end{array}\right]^T \left[\begin{array}{cc|c} -\dot{Y} + YA^T + AY & YC_1^T & B_1 \\ C_1 Y & -\gamma I & D_{11} \\ \hline B_1^T & D_{11}^T & -\gamma I \end{array}\right] \left[\begin{array}{c|c} \mathcal{N}_X & 0 \\ \hline 0 & I \end{array}\right] < 0$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0$$

where $\mathcal{N}_X$ and $\mathcal{N}_Y$ are bases for the nullspaces of $[\, C_2 \quad D_{21} \,]$ and $[\, B_2^T \quad D_{12}^T \,]$.

LPV-systems: Projected solvability conditions (Apkarian and Adams,1998)

- Existence conditions become necessary and sufficient if quadratic stability is imposed through Lyapunov function:

$$V(x_{cl}, P(\rho)) = x_{cl}^T P(\rho) x_{cl}, \quad \text{with} \quad x_{cl} = \begin{bmatrix} x \\ x_K \end{bmatrix}$$

- The controller can be constructed from X and Y along the lines of (Gahinet, 1994).

LPV-systems: Summary and remarks

- Induced L_2 performance can be tested in terms of two affine linear matrix inequalities (AMI). However, these are dependent on the parameters ρ.
- This yields a infinite dimensional convex optimization problem.
- The suggested solution is to grid the parameter set $\mathcal{P}$.
- Pick a basis for the parameter dependent solutions $X(\rho)$ and $Y(\rho)$ to the AMI's.
- Solve the AMI's at the grid points.
- The number of inequalities grow exponentially with the number of parameters.
- The number of inequalities grow linearly with number of grid points.

LPV-systems: Summary and remarks

- If the state-space matrices $(A(\rho), B(\rho), C(\rho), D(\rho))$ depend in affine manner on the parameters ρ, i.e.

$$\begin{bmatrix} A(\rho) & B(\rho) \\ C(\rho) & D(\rho) \end{bmatrix} = \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix} + \sum_{i=1}^{f} \rho_i \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix}$$

  Then it is sufficient to test the corner points (Apkarian *et al.*, 1995). Then the infinite dimensional AMI's become finite dimensional.
- The controller becomes dependent on $\dot{\rho}$. In order to be practically valid, this dependence must be removed, for further details see (Apkarian and Adams, 1998).

Summary on “recent work” in gain-scheduling control.

- Renewed interest in gain-scheduling control and LPV-systems, due to new powerful techniques and computational schemes (interior point methods) which can be applied to LMI’s.
- Some nonlinear control problems can be solved.
- Focuses on analysis and theoretical development rather than ad. hoc. approaches.
- Applications using the LFT and LPV techniques start to emerge.
- Number of papers and the focus from academia is increasing.
- Several parallels between gain-scheduling control and model predictive control.

References

Apkarian, P. (1997). On the discretization of LMI-synthesized linear parameter-varying controllers, *Automatica* **33**(4): 655–661.

Apkarian, P. and Adams, R. J. (1998). Advanced gain-scheduling techniques for uncertain systems, *IEEE Transactions on Control Systems Technology* **6**(1): 21–32.

Apkarian, P. and Gahinet, P. (1995). A convex characterization of gain-scheduled $\mathcal{H}_\infty$ controllers, *IEEE Transactions on Automatic Control* **40**(5): 853–864.

Apkarian, P., Gahinet, P. and Becker, G. (1995). Self-scheduled $\mathcal{H}_\infty$ control of linear parameter-varying systems: a design example, *Automatica* **31**(9): 1251–1261.

Apkarian, P., Gahinet, P. and Biannic, J.-M. (1994). Self-scheduled $\mathcal{H}_\infty$ control of a missile via LMIs, *Proc. of the 33rd Conference on Decision and Control*, Lake Buena Vista, pp. 3312–3317.

$|| Astr\o m, K.\ J.(1983). Theory and applications of adaptive control - a survey, Automatica \mathbf{19}(5): 471--486.$

$|| Astr\o m, K.\ J.(1996). Tuning and adaption, Proc. from 13th IFAC World Congress, pp. 1--18.$

$|| Astr\o m, K.\ J. and\ Wittenmark, B.(1989). Adaptive Control, Addison -$
$Wesley Publishing Company, chapter Gain Scheduling, pp. 343--369.$

$|| Astr\o m, K.\ J. and\ Wittenmark, B.(1995). A survey of adaptive control applications, Proc. of the 34th Conference on Decis$
$-654.$

Balakrishnan, V., Huang, Y., Packard, A. and Doyle, J. (1994). Linear matrix inequalities in analysis with multipliers, *Proc. of the American Control Conference*, Baltimore, USA, pp. 1228–1232.

Balas, G. J., Fialho, I., Packard, A., Renfrow, J. and Mullaney, C. (1997). On the design of LPV controllers for the F-14 aircraft lateral-directional axis during powered approach, *Proc. of the American Control Conference*, New Mexico, pp. 123–127.

Banerjee, A., Arkun, Y., Pearson, R. and Ogunnaike, B. (1995). $\mathcal{H}_\infty$ control of nonlinear processes using multiple linear models, *Proc. of 3rd European Control Conference*, Rome, Italy, pp. 2671–2676.

Becker, G. (1996). Additional results on parameter-dependent controllers for LPV systems, *Proc. of IFAC 13th Triennial World Congress*, San Francisco, USA, pp. 351–356.

Becker, G. and Packard, A. (1994). Robust performance of linear parametrically varying systems using parametrically-dependent linear feedback, *Systems & Control Letters* **23**: 205–215.

Becker, G., Packard, A., Philbrick, D. and Balas, G. (1993). Control of parametrically-dependent linear systems: A single quadratic lyapunov approach, *Proc. of the American Control Conference*, San Francisco, USA, pp. 2795–2799.

Becker, G. S. (1993). *Quadratic Stability and Performance of Linear Parameter Dependent Systems*, PhD thesis, University of California at Berkeley.

Bequette, B. W. (1997). Gain-scheduled process control: A review, *NATO ASI Nonlinear Model Based Process Control*, pp. 1–28.

Braatz, R. D. and Morari, M. (1997). On the stability of systems with mixed time-varying parameters, *International Journal of Robust and Nonlinear Control* **7**: 105–112.

Breedijk, T., Edgar, T. F. and Trachtenberg, I. (1994). Model-based control of rapid thermal processes, *Proc. of the American Control Conference*, Baltimore, USA, pp. 887–891.

Cardello, R. and San, K.-Y. (1987). Application of gain scheduling to the control of batch bioreactors, *Proc. of American Control Conference*, pp. 682–686.

Chilali, M. and Gahinet, P. (1996). $\mathcal{H}_\infty$ design with pole placement constraints : An LMI approach, *IEEE Transactions on Automatic Control* **41**(3): 358–367.

Doyle, J., Packard, A. and Zhou, K. (1991). Review of LFTs, LMIs, and μ, *Proc. of the 30th Conference on Decision and Control*, Brighton, England, pp. 1227–1232.

Driankov, D., Palm, R. and Rehfuess, U. (1996). A Takagi-Sugeno fuzzy gain-scheduler, *IEEE* pp. 1053–1059.

El-Zobaidi, H. M. H. and Jaimoukha, I. M. (1996). Robust normalised LPV gain scheduling, *Proc. of the 35th Conference on Decision and Control*, Kobe, Japan, pp. 3982–3983.

Engell, S. and Klatt, K. U. (1993). Nonlinear control of a non-minimum-phase CSTR, *Proc. of the American Control Conference*, San Francisco, USA, pp. 2941–2945.

Fei, S. and Huo, W. (1995). Robust $\mathcal{H}_\infty$ control for nonlinear systems with time-varying uncertainty, *Proc. of the American Control Conference*, Seattle, USA, pp. 2389–2390.

Feron, E., Apkarian, P. and Gahinet, P. (1996). Analysis and synthesis of robust control systems via parameter-dependent lyapunov functions, *IEEE Transactions on Automatic Control* **41**(7): 1041–1047.

Fossen, T. and Grøvlen,
$|| A.(1998).Nonlinear output feedback control of dynamically positioned ships using vectorial observer backstepping, IEEE$
$-128.$

Gahinet, P. and Apkarian, P. (1994). A linear matrix inequality approach to $\mathcal{H}_\infty$ control, *International Journal of Robust and Nonlinear Control* **1**: 421–448.

Gahinet, P., Apkarian, P. and Chilali, M. (1996). Affine parameter-dependent lyapunov functions and real parametric uncertainty, *IEEE Transactions on Automatic Control* **41**(3): 436–443.

Goodwin, G. C., Graebe, S. F. and Levine, W. S. (1993). Internal model control of linear systems with saturating actuators, *Proc. of European Control Conference*, pp. 1072–1077.

Graebe, S. F. and Ahlen, A. L. B. (1996). Dynamic transfer among alternative controllers and its relation to antiwindup controller design, *IEEE Transactions on Control Systems Technology* **4**(1): 92–99.

Häggblom, K. E. (1993). Experimental comparison of conventional and nonlinear model-based control of a mixing tank, *Ind. Eng. Chem. Res.* pp. 2653–2661.

Hägglund, T. and Tengvall, A. (1995). An automatic tuning procedure for unsymmetrical processes, *Proc. of the 3rd European Control Conference*, Rome, Italy, pp. 2450–2455.

Helmersson, A. (1995a). *Methods for Robust Gain Scheduling*, PhD thesis, Linkøping University, Sweden, Dep. of Electrical Engineering.

Helmersson, A. (1995b). μ synthesis and LFT gain scheduling with mixed uncertainties, *Proc. of 3rd European Control Conference*, Rome, Italy, pp. 153–158.

Helmersson, A. (1996). Application of real-μ gain scheduling, *Proc. of the 35th Conference on Decision and Control*, Kobe, Japan, pp. 1666–1671.

Hyde, R. A. and Glover, K. (1993). The application of scheduled $\mathcal{H}_\infty$ controllers to a VSTOL aircraft, *IEEE Transactions on Automatic Control* **38**(7): 1021–1039.

Johansen, T. A., Hunt, K. J., Gawthrop, P. J. and Fritz, H. (1998). Off-equilibrium linearisation and design of gain scheduled control with application to vehicle speed control, *unknown*.

Kamen, E. W. and Khargonekar, P. P. (1984). On the control of linear systems whose coefficients are functions of parameters, *IEEE Transactions on Automatic Control* **AC-29**(1): 25–33.

Kaminer, I., Pascoal, A. M., Khargonekar, P. P. and Coleman, E. E. (1995). A velocity algorithm for the implementation of gain-scheduled controllers, *Automatica* **31**(8): 1185–1191.

Khalil, H. K. (1993). Robustness issues in output feedback control of feedback linearizable systems, *Proc. of European Control Conference*, pp. 58–62.

Khargonekar, P. P. and Sontag, E. D. (1982). On the relation between stable matrix fraction factorizations and regulable realizations of linear systems over rings, *IEEE Transactions on Automatic Control* **AC-27**(3): 627–638.

Klatt, K.-U. and Engell, S. (1995). Gain scheduling trajectory control of neutralization processes in continuous stirred tank reactors, *Proc. of 3rd European Control Conference*, Rome, Italy, pp. 2665–2670.

Klatt, K.-U. and Engell, S. (1996). Nonlinear control of neutralization processes by gain-scheduling trajectory control, *Ind. Eng. Chem. Res.* pp. 3511–3518.

Lagerberg, A. (1996). Gain scheduling control and its application to a chemical reactor model, *Technical report*, School of Electrical and Computer Eng. Chalmers University of Tech, Sweden.

Lagerberg, A. and Breitholtz, C. (1997). A study of gain scheduling control applied to an exothermic CSTR, *Chem. Eng. Technol.* **20**: 435–444.

Lall, S. and Glover, K. (1995). Robust performance and adaptation using receding horizon $\mathcal{H}_\infty$ control of time varying systems, *Proc. of the American Control Conference*, Seattle, USA, pp. 2384–2388.

Lawrence, D. A. and Rugh, W. J. (1995). Gain scheduling dynamic linear controllers for a nonlinear plant, *Automatica* **31**(3): 381–390.

Lin, J.-Y. and Yu, C.-C. (1993). Automatic tuning and gain scheduling for pH control, *Chemical Engineering Science* **48**(18): 3159–3171.

Lind, R., Balas, G. J. and Packard, A. (1995). Optimal scaled $\mathcal{H}_\infty$ full information synthesis with real parametric uncertainty, *Proc. of the American Control Conference*, Seattle, USA, pp. 3463–3467.

Ling, C. and Edgar, T. F. (1992). A new fuzzy gain scheduling algorithm for process control, *Proc. of the*

American Control Conference, pp. 2284–2290.

Lu, W.-M. and Doyle, J. C. (1994). $\mathcal{H}_\infty$ control of nonlinear systems: A convex characterization, *Proc. of the American Control Conference*, Baltimore, USA, pp. 2098–2102.

Lu, W.-M. and Doyle, J. C. (1995). $\mathcal{H}_\infty$ control of nonlinear systems: A convex characterization, *IEEE Transactions on Automatic Control* **40**(9): 1668–1675.

Megretski, A. (1996). L_2 bibo output feedback stabilization with saturated control, *Proc. of IFAC 13th Triennial World Congress*, San Francisco, USA, pp. 435–440.

Muske, K. R., Logue, D. A., Keaton, M. M. and Hayward, R. A. (1991). Gain scheduled model predictive control of a crude oil distillation unit, *unpublished* pp. 1–12.

Nichols, R. A., Reichert, R. T. and Rugh, W. J. (1993). Gain scheduling for H-infinity controllers: A flight control example, *IEEE Transactions on Control Systems Technology* **1**(2): 69–79.

Packard, A. (1994). Gain scheduling via linear fractional transformations, *Systems & Control Letters* **22**: 79–92.

Packard, A. and Kantner, M. (1996). Gain scheduling the LPV way, *Proc. of the 35th Conference on Decision and Control*, Kobe, Japan, pp. 3938–3941.

Puruchuri, V. P. and Rhinehart, R. R. (1994). Experimental demonstration of nonlinear model-based-control of a heat exchanger, *Proc. of the American Control Conference*, Baltimore, USA, pp. 3533–3537.

Rugh, W. J. (1991). Analytical framework for gain scheduling, *IEEE Control System Magazine* pp. 79–84.

Russell, E. L. and Braatz, R. D. (1996). Analysis of large scale systems with multiple time delay uncertainties and process constraints, *Presented at the 1996 AIChE meeting, Chicago, USA*, pp. 1–16.

Schei, T. S. (1993). Automatic tuning of simple decouplers in multivariable control systems, *Proc. of IFAC 12th Triennial World Congress*, Sydney, Australia, pp. 65–71.

Scherer, C., Gahinet, P. and Mahmoud, C. (1997). Multiobjective output-feedback control via LMI optimization, *IEEE Transactions on Automatic Control* **42**(7): 896–911.

Seron, M. M., Graebe, S. F. and Goodwin, G. C. (1994). All stabilizing controllers, feedback linearization and anti-windup: A unified review, *Proc. of the American Control Conference*, Baltimore, USA, pp. 1685–1689.

Shahruz, S. M. and Behtash, S. (1992). Design of controllers for linear parameter-varying systems by the gain scheduling technique, *Journal of Mathematical Analysis and Applications* **168**: 195–217.

Shamma, J. S. and Athans, M. (1990). Analysis of gain scheduled control for nonlinear plants, *IEEE Transactions on Automatic Control* **35**(8): 898–907.

Shamma, J. S. and Athans, M. (1991). Guaranteed properties of gain scheduled control for linear parameter-varying plants, *Automatica* **27**(3): 559–564.

Shamma, J. S. and Athans, M. (1992). Gain scheduling: Potential hazards and possible remedies, *IEEE Control Systems Magazine* pp. 101–107.

Shamma, J. S. and Cloutier, J. R. (1992). A linear parameter varying approach to gain scheduled missile autopilot design, *Proc. of American Control Conference*, pp. 1317–1321.

Shamma, J. S. and Cloutier, J. R. (1993). Gain-scheduled missile autopilot design using linear parameter varying transformations, *Journal of Guidance, Control, and Dynamics* **16**(2): 256–263.

Shue, S. P., Sawan, M. E. and Rokhsaz, K. (1997). Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ method suitable for gain scheduled aircraft control, *Journal of Guidance, Control and Dynamics* **20**(4): 699–706.

Stuber, J. D., Trachtenberg, I., Edgar, T. F., Elliott, J. K. and Breedijk, T. (1994). Model-based control of rapid thermal processes, *Proc. of the 33rd Conference on Decision and Control*, Lake Buena Vista, pp. 79–85.

Sugie, T. and Kawanishi, M. (1995). μ analysis/synthesis based on exact expression of physical parameter variations, *Proc. of 3rd European Control Conference*, Rome, Italy, pp. 159–164.

Tetley, B. and Ulff, A. (1989). Gain scheduling for process control, *C & I* pp. 83–85.

Walker, D. J. (1997). Gain-scheduled flight control via two-degree-of-freedom $\mathcal{H}_\infty$ optimization, *Proc Instn Mech Engrs* **211**: 263–268.

Wang, J. and Rugh, W. J. (1987). Feedback linearization families for nonlinear systems, *IEEE Transactions on Automatic Control* **AC-32**(10): 935–940.

Whatley, M. J. and Pott, D. C. (1984). Adaptive gain improves reactor control, *Hydrocarbon Processing* pp. 75–78.

Wu, F. (1995). *Control of Linear Parameter Varying Systems*, PhD thesis, University of California at Berkeley.

Wu, F. and Packard, A. (1995a). Optimal LQG performance of linear uncertain systems using state-feedback, *Proc. of the American Control Conference*, Seattle, USA, pp. 4435–4439.

Wu, F. and Packard, A. (1995b). LQG control design for LPV systems, *Proc. of the American Control Conference*, Seattle, USA, pp. 4440–4444.

Wu, F., Packard, A. and Balas, G. (1995). LPV control design for pitch-axis missile autopilots, *Proc. of the 34th Conference on Decision & Control*, New Orleans, USA, pp. 188–193.

Wu, F., Yang, X. H., Packard, A. and Becker, G. (1995). Induced L_2-norm control for LPV system with bounded parameter variation rates, *Proc. of the American Control Conference*, Seattle, USA, pp. 2379–2383.

Wu, F., Yang, X. H., Packard, A. and Becker, G. (1996). Induced L_2-norm control for LPV systems with bounded parameter variation rates, *Submitted to Int. Journal of Nonlinear and Robust Control* .

Yang, X. H., Wu, F. and Packard, A. (1995). Adaptive control of full information problem, *Proc. of the American Control Conference*, Seattle, USA, pp. 3371–3372.