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Necessary and sufficient conditions for robust reliable control in the presence of model uncertainties and system component failures☆

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ABSTRACT

This paper provides necessary and sufficient conditions for several forms of controlled system reliability. For comparison purposes, past results on the reliability analysis of controlled systems are reviewed and several of the past results are shown to be either conservative or have exponential complexity. For systems with real and complex uncertainties, conditions for *robust reliable stability and performance* are formulated in terms of the structured singular values of certain transfer functions. The conditions are necessary and sufficient for the controller to stabilize the closed-loop system while retaining a desirable level of the closed-loop performance in the presence of actuator/sensor faults or failures, as well as plant-model mismatches. The resulting conditions based on the structured singular value are applied to the decentralized control for a high-purity distillation column and singular value decomposition-based optimal control for a parallel reactor with combined precooling. Tight polynomial-time bounds for the conditions can be evaluated by using available off-the-shelf software.

1. Introduction

An inevitable consequence of industrial practice is that actuators and sensors can become faulty or fail, which motivates the development of methods to evaluate the reliability of the closed-loop system to such imperfect operations. A feedback-controlled system is said to be *reliable* if it is guaranteed to retain desired closed-loop system properties while tolerating faults or failures of actuators and/or sensors. Maximizing the reliability of a system concerns minimizing its potential performance degradation while retaining closed-loop stability when a fault or failure occurs in a control/measurement channel. In addition to the possibility of actuator/sensor faults or failures, plant-model mismatches are also inevitable, which motivates their incorporation into reliability and integrity analysis. This article is motivated by the need for nonconservative testing conditions to ensure closed-loop stability and to retain a satisfactory closed-loop performance in the presence of both plant-model mismatches and actuator/sensor faults or failures.

This paper primarily considers decentralized controlled systems and studies their *robust reliable stability and performance* in the presence of possible actuator/sensor faults or failures with consideration of the overall plant-model mismatches (i.e., model uncertainty) that are described in terms of bounded set-valued linear operators. The main purpose of this article is to present necessary and sufficient conditions for various types of *robust reliable stability and performance* of a set-valued plant model that is described by a linear fractional transformation (LFT) with structured uncertainties (Zhou et al., 1996). It is assumed that any failure of a local controller is detected and the controller is taken out of service whenever a failure occurs, so that any undesirable propagation of local failures to other parts of the system can be avoided. Although the main emphasis is on decentralized control systems, the proposed approach does not depend on the structure of the selected control schemes and can be applied to any type of linear controller and actuator–sensor selection.

Decentralized control depicted in Fig. 1a is ubiquitous in industrial applications, which is a special case of large-scale interconnected systems with interactions between subsystems and

☆ Part of the results of this article were presented in Braatz et al. (1994).

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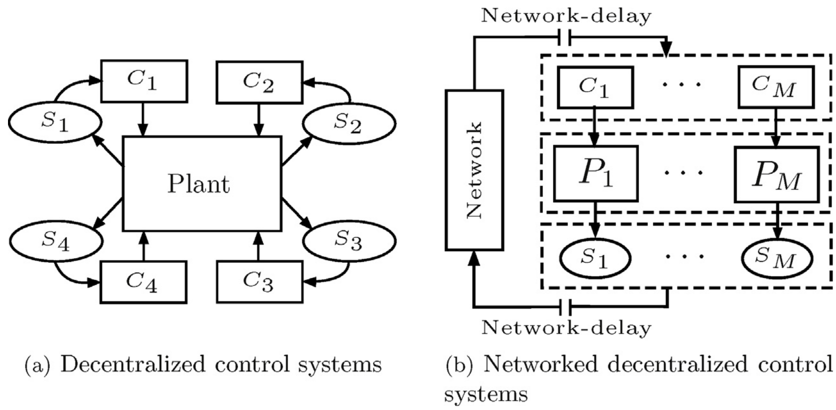


Fig. 1. Large-scale interconnected systems.

constraints on information flows. Extensive overviews on decentralized control are available (Bakule, 2008; Siljak, 1996). For decentralized controlled systems, actuator/sensor faults or failures can occur and the selection of a reliable actuator/sensor structure is an important consideration (Braatz et al., 1996; Khaki-Sedigh and Moaveni, 2009; Lee et al., 1995). A resurgent topic in systems and control theory related to *reliable decentralized control* is the study of the effect and propagation of communication link failures between several components of a *networked control system* (NCS) depicted in Fig. 1b on the stability and performance of the overall system (Imer et al., 2006; Tipsuwan and Chow, 2003; Walsh et al., 2002; Zhang et al., 2001). Although studied for decades, NCSs have received a large surge of interest in recent years. As time delays and communication losses are inevitable in an NCS, reliability analysis in the presence of faults and failures in communication networks is also important.

In Siljak (1978, 1980), multi-controller systems were introduced for reliable control and since then reliable stabilization problems under various failure and fault scenarios have been studied using decentralized configurations (Campo and Morari, 1994; Gündes, 1998; Morari, 1985; Morari and Zafiriou, 1989; Skogestad and Morari, 1992; Tan et al., 1992). In particular, the reliability of decentralized control with integral action was investigated in terms of steady-state gain matrices (Campo and Morari, 1994; Grosdidier et al., 1985; Morari, 1985) and existence conditions for a reliably stabilizing decentralized integral controller were derived in terms of the Niederlinski index (NI) and block relative gain (BRG) (Kariwala et al., 2005, 2006). Explicit conditions for reliable decentralized control of linear systems were derived for a two-channel decentralized feedback control configuration (Gündes, 1998), and coprime factorization methods and a design method for such controllers were proposed (Gündes and Kabuli, 2001).

In addition to the aforementioned frequency-domain approaches, some researchers have proposed design methods for reliable controllers in terms of state-space realizations of the plant and controller. Centralized reliable state feedback controllers have been investigated (Joshi, 1986; Mariton and Bertrand, 1986) and design methods for decentralized reliable observer-based output-feedback controllers were developed (Date and Chow, 1989; Veillette et al., 1992). Robust pole placement was used to design state feedback controllers for dynamical systems in the presence of actuator failures (Zhao and Jiang, 1998) while requiring redundant actuators to recover the normal level of operation. The design method of Zhao and Jiang (1998) was only applicable to state feedback control problems without any plant-model mismatch, so that the proposed design methods may perform poorly in the presence of model uncertainties. In Seo and Kim (1996), a simple high-gain state feedback control based on a Riccati-type equation was proposed with actuator redundancy for systems for some form of time-varying model uncertainties, but not fully structured and with no uncertainty allowed in the input channel matrices. The passivity theorem has been used to design a decentralized controller with some form of H_2 performance while maintaining stability when each control loop is detuned (Bao et al., 2002; Zhang et al., 2002).

The approaches described in this article are based on the structured singular value (μ) and a standard representation of uncertain systems known as the linear fractional transformation (LFT). Robust reliable control problems for large-scale systems with decentralized control are reformulated in terms of robustness analysis based on μ to model the effects of faults. The structures of interconnected sensors and actuators as well as the structure of the model uncertainties can be fully exploited to perform nonconservative or less conservative analysis. Some of the results in this article were presented in Braatz et al. (1994) and subsequently there were many research efforts such as the aforementioned works to develop robust reliable controllers. The main objective of this article is to provide an efficient framework for the analysis and synthesis of robust reliability. Faults and failures in process components are treated as parametric uncertainties that are compatible with μ. In response to the resurgence of research interest in robust reliable control for systems with integral action, this article extends and expands our past results (Braatz et al., 1994) to derive conditions for robust reliable stability of decentralized systems with integral action. Although the main focus of this article is on decentralized control problems, the methodology is not restricted to decentralized control and the results can be extended to general control structures in a straightforward manner.

1.1. Mathematical notation

The notation used in this paper is standard. $\|\cdot\|$ is the Euclidean norm for vectors or the corresponding induced matrix norm for matrices. $\mathbf{0}$ and $\mathbf{I}$ denote the null matrix whose components are all zeros and the identity matrix of compatible dimension, respectively. $\mathbb{C}_+$ denotes the open right-half plane, i.e., $\mathbb{C}_+ \triangleq \{s \in \mathbb{C} : s = \alpha + j\beta,\ \alpha > 0,\ \beta \in \mathbb{R} \cup \{\infty\}\}$. The set of eigenvalues is denoted by $\sigma(A) \triangleq \{\lambda \in \mathbb{C} : \det(\lambda \mathbf{I} - A) = 0\}$ and $\rho(A)$ refers to the spectral radius of A, i.e., $\rho(A) = \max_{\lambda \in \sigma(A)} |\lambda|$. The vector whose entries are all ones is represented by $1_m \triangleq [1, \ldots, 1]^T \in \mathbb{R}^m$. diag($A$, B) denotes a block-diagonal matrix whose diagonal entries are A and B. The argument s for a transfer function may be omitted for notational convenience, but will appear whenever required to avoid confusion. The standard LFT (or M–Δ configuration), uses the notation $\mathcal{F}_\ell(M, \Delta) \triangleq M_{11}(s) + M_{12}(s)\Delta(\mathbf{I} - M_{22}(s)\Delta)^{-1}M_{21}(s)$

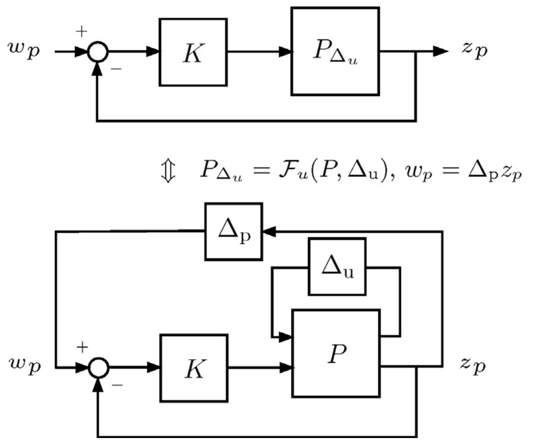


Fig. 2. Feedback-controlled closed-loop system with plant-model mismatch.

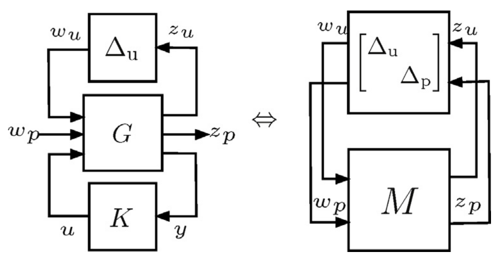


Fig. 3. Main-loop theorem.

and $\mathcal{F}_u(M, \Delta) \triangleq M_{22}(s) + M_{21}(s)\Delta(\mathbf{I} - M_{11}(s)\Delta)^{-1}M_{12}(s)$. For system performance, this paper focuses on $\mathcal{H}_\infty$ performance that is defined by $\sup_{\|w_p\|_2 \le 1} (\| z_p\|_2 / \| w_p\|_2)$ where w_p and z_p are the input and output from the $\mathcal{L}_2[0, \infty)$ space with the inner product defined by $\langle w_p, w_p \rangle \triangleq \int_0^\infty w_p^T(t) w_p(t) dt$.

2. Mathematical background on structure singular value theory and various types of reliability

2.1. Robust stability and performance

This section briefly reviews robust control theory, with most of materials adopted from Zhou et al. (1996). This article considers the system in Fig. 2, where K and P_{Δ_u} denote the linear time-invariant (LTI) controller and the real plant, respectively. The difference between the real plant P_{Δ_u} and its corresponding model P is represented by the set of uncertainties $\mathbf{\Delta}_u$ given in Definition 2.1 such that $P_{\Delta_u} \in \bigcup_{\Delta} \in \mathbf{\Delta}_u \mathcal{F}_u(P, \Delta)$.

Then, the system in Fig. 2 can be represented as the LFT (Zhou et al., 1996) in Fig. 3 with the matrix of transfer functions

$$G := \begin{bmatrix} P_{11} & \mathbf{0} & P_{12} \\ P_{21} & \mathbf{0} & P_{22} \\ -P_{21} & \mathbf{I} & -P_{22} \end{bmatrix}$$

and $M := \mathcal{F}_\ell(G, K)$. Now consider the nominal system $M(s)$ subject to norm-bounded perturbations, denoted by Δ_u, in Fig. 3. The structured singular value (μ) (Doyle, 1982; Zhou et al., 1996) framework provides a general approach for addressing multiple performance specifications in different locations. For a system with multiple sources of uncertainties, μ requires that all sources of uncertainties from their point of occurance be equivalently moved to a single reference location in the loop. Incorporating weights into $M(s)$, each element of the perturbation is assumed to be normalized to be of magnitude one, $\| \Delta_i \|_\infty \le 1$, where Δ_i has real values for representing parametric uncertainty and can have complex values for representing unmodeled dynamics. Without loss of generality, assume that Δ_u and each M_{ij} are square.

Definition 2.1. (Definition 11.1 in Zhou et al. (1996)) Let $M \in \mathbb{C}^{m \times m}$ be a square matrix with complex-valued entries and let $\mathbf{\Delta}_u$ be the set of matrices of block-diagonal perturbations given by

$$\mathbf{\Delta}_u \triangleq \{\mathrm{diag}(\delta_1^\mathrm{r}\mathbf{I}_{r_1}, \ldots, \delta_k^\mathrm{r}\mathbf{I}_{r_k}, \delta_1^\mathrm{c}\mathbf{I}_{r_{k+1}}, \ldots, \delta_k^\mathrm{c}\mathbf{I}_{r_\ell}, \Delta_{r_{\ell+1}}, \ldots, \Delta_{r_{m_c}}) : \\ \delta_i^\mathrm{r} \in \mathbb{R}, \delta_i^\mathrm{c} \in \mathbb{C}, \Delta_i \in \mathbb{C}^{m_i \times m_i}, \sum_{i=1}^{m_c} r_i = m\}. \quad (1)$$

If there does not exist $\Delta \in \mathbf{\Delta}_u$ such that $\det(\mathbf{I} - M\Delta) = 0$, then $\mu_{\mathbf{\Delta}_u}(M) = 0$; otherwise $\mu_{\mathbf{\Delta}_u}(M)$ is given by

$$\mu_{\mathbf{\Delta}_u}(M) \triangleq (\min_{\Delta \in \mathbf{\Delta}_u} \{\overline{\sigma}(\Delta) : \det(\mathbf{I} - M\Delta) = 0\})^{-1}.$$

Definition 2.2. (Definition 9.1, (Zhou et al., 1996)) Consider the closed-loop system in Fig. 2. Let Δ be a known conservative set of uncertainties. Then the system

- is nominally stable, if it is stable for $\Delta \equiv 0$;
- is robustly stable, if it is stable for all $\Delta \in \mathbf{\Delta}_u$;
- has nominal performance, if the performance specifications are satisfied for the case $\Delta \equiv 0$;
- has robust performance, if the performance specifications are satisfied for all $\Delta \in \mathbf{\Delta}_u$.

Below are μ tests for robust stability and robust performance for the systems in Fig. 3.

Lemma 2.1. *(Thm. 11.8 in Zhou et al. (1996)) The closed-loop system in Fig. 3 exhibits robust stability for all $\Delta_u \in \mathbf{\Delta}_u$ with $\| \Delta \|_\infty \le 1$, if and only if the closed-loop system is nominally stable, and $\mu_{\mathbf{\Delta}_u}(M_{11}(j\omega)) < 1$ for all $\omega \in \mathbb{R} \cup \{\infty\}$.*

Lemma 2.2. *(Thm. 11.9 in Zhou et al. (1996)) The closed-loop system in Fig. 3 exhibits robust performance and $\|(M, \Delta_u)\|_\infty \le 1$, for all $\Delta_u \in \mathbf{\Delta}_u$ with $\| \Delta \|_\infty \le 1$, if and only if the closed-loop system is nominally stable and $\mu_{\mathbf{\Delta}_c}(M(j\omega)) < 1$ for all $\omega \in \mathbb{R} \cup \{\infty\}$, where $\mathbf{\Delta}_c \triangleq \{\mathrm{diag}\{\Delta_u, \Delta_p\} : \Delta_u \in \mathbf{\Delta}_u, \Delta_p \in \mathbb{C}^{\ell \times \ell}, \| \Delta_p\|_\infty \le 1\}$, and ℓ is the dimension of the system output.*

The calculation of μ is NP-hard (Braatz et al., 1994) but many practical polynomial-time algorithms are available that compute upper and lower bounds on μ that have been tight for practical systems (Zhou et al., 1996).

2.2. Reliability of decentralized control

For notational convenience, the controller is assumed to be fully decentralized, i.e., the controller K is diagonal. Most of the results can be extended in an obvious manner to block-diagonal controllers and even to centralized controllers. Usually, the square plant P is assumed to be stable; the results do not carry over easily to plants that are open-loop unstable.

Several strong forms of reliability to failure of actuators or sensors are defined in the open literature for systems without plant-model mismatch. Below is a review of those forms of reliability and the extensions of the definitions to uncertain systems. To simplify the presentation, the primary focus is on a discussion of reliability to actuator faults or failures, although very similar definitions and the results can be trivially extended to the other process equipment.

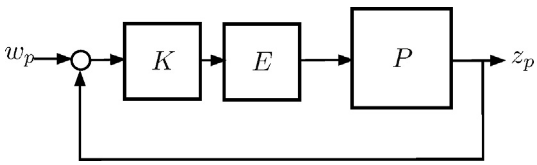


Fig. 4. Integrity under actuator faults/failures. For robust integrity, replace P by the set of uncertain plants P_{Δ_u}.

Integrity is defined as follows (Braatz et al., 1994; Morari, 1985; Morari and Zafiriou, 1989; Siljak, 1978, 1980).

Definition 2.3. The closed-loop system demonstrates *integrity* if $K_f(s) := EK(s)$ stabilizes $P(s)$ all for $E \in \mathcal{E}_{1/0} \triangleq \{\mathrm{diag}\{\epsilon_i\} : \epsilon_i \in \{0, 1\}, i = 1, \ldots, n\}$.

A closed-loop system that demonstrates integrity to actuator failures remains stable as actuators are arbitrarily brought in and out of service (Fig. 4). For a system to demonstrate integrity, the nominal plant model $P(s)$ must be stable. To have actuator failure tolerance when the controller is unstable, the failures must be recognized and the corresponding columns of the controller taken off-line. It is clear that the integrity of a system can be tested through 2^n stability (eigenvalue) determinations.

The following definition extends integrity to uncertain systems.

Definition 2.4. The closed-loop system demonstrates *robust integrity* if $K_f(s) := EK(s)$ stabilizes $P_{\Delta_u}(s)$ for all $E \in \mathcal{E}_{1/0}$ and all $\Delta_u \in \mathbf{\Delta}_u$ such that $\| \Delta_u \|_\infty \leq 1$.

An uncertain system demonstrates robust integrity to actuator failures if it remains stabilized for any plant given by the uncertainty description, as actuators are arbitrarily brought in and out of service. For a system to demonstrate robust integrity, the plant must be stable for all allowed perturbations. To have actuator failure tolerance when the controller is unstable, the failures must be recognized and the corresponding columns of the controller taken off-line, just as in the nominal case. Note that robust integrity implies integrity. It is clear that the robust integrity of a system can be tested through 2^n nominal stability (eigenvalue) and 2^n robust stability (μ) calculations.

A very strong notion of reliability was defined by Campo and Morari (1994) for decentralized integral controllers. The requirement is that the nominal closed-loop system remains stable under arbitrary independent detuning of the controller gains. For decentralized control systems, this is equivalent to arbitrary detuning of the actuator/sensor gains to zero. Having stability with detuning allows the operators to safely change the closed-loop speed of response depending on process operating conditions. Below is their definition of reliability extended to include control systems that do not necessarily have integral action in all, or any, channels.

Definition 2.5. The closed-loop system is *decentralized unconditionally stable (DUS)* if $K_f(s) := EK(s)$ stabilizes $P(s)$ for all $E \in \mathcal{E}_D \triangleq \{\mathrm{diag}\{\epsilon_i\} : \epsilon_i \in (0, 1)\}$.

The closed-loop system will not be DUS if either the plant $P(s)$ or controller $K(s)$ has poles in the open right-half plane (ORHP). To see this, consider the multivariable root locus (e.g., Skogestad and Postlethwaite, 2005) with equal detuning $\epsilon_i = \epsilon$ for all i. For small ϵ, the closed-loop poles approach the open-loop poles. Since the closed-loop poles are a continuous function of the controller gain, if any of the open-loop poles are in the ORHP then some of the closed-loop poles will be unstable for sufficiently small ϵ.

The following is the generalization of DUS to uncertain systems.

Definition 2.6. The closed-loop system is *robust decentralized unconditionally stable (RDUS)* if $K_f(s) := EK(s)$ stabilizes $P_{\Delta_u}(s)$ for all $E \in \mathcal{E}_D$ and all $\Delta_u \in \mathbf{\Delta}_u$ such that $\| \Delta_u \|_\infty \leq 1$.

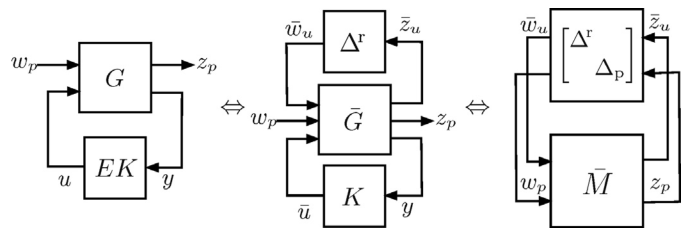


Fig. 5. Equivalent LFTs of fault tolerance.

By a similar argument as used for DUS, the closed-loop system will not be RDUS if any poles of the controller $K(s)$ or any plant given by the uncertainty description are in the ORHP. For open-loop unstable controllers or plants, some minimum amount of feedback is required for closed-loop stability.

Actually, the definition of DUS used by Campo and Morari (1994) requires that the closed-loop system be stable for all $\epsilon \in [0, 1]$. Here we refer to this notion as *closed decentralized unconditional stability (CDUS)*, with *closed robust decentralized unconditional stability (CRDUS)* defined similarly. These definitions of reliability require stability under total malfunctions of some actuators and allows perfect functioning of some actuators while other actuators are not working at all.

3. Analysis for reliability of decentralized control using μ

This section primarily focuses on the nominal and robust fault tolerance of systems that are affected by real parametric uncertainties and complex dynamic uncertainties. The detuned control gains of decentralized controllers are assumed to be real constants, unknown but bounded by open or closed intervals.

3.1. Modeling faults using μ

Braatz (1993) describes in some detail the modeling of faults with either uncertainty and/or performance descriptions. This modeling can be combined with requirements on the stability or performance during faulty operation to derive a μ condition that provides a test for system reliability. The following discussion illustrates how to model actuator gain variation for two cases: (i) without additional uncertainty (i.e., plant/model mismatch) and (ii) with additional uncertainty.

The nominal linear dynamic output feedback controller is defined to be $K(s) \in \mathbb{C}^{m \times m}$. Then the controller with gain variation can be described by $\tilde{K}(s) = EK(s)$, where $E \in \mathcal{E}[\epsilon_{\mathrm{low}}, \epsilon_{\mathrm{upper}}] \triangleq \{\mathrm{diag}\{\epsilon_i\} : \epsilon_i \in [\epsilon_{i,\mathrm{low}}, \epsilon_{i,\mathrm{upper}}]\}$. Any $E \in \mathcal{E}[\epsilon_{\mathrm{low}}, \epsilon_{\mathrm{upper}}]$ can be rewritten as

$$E \triangleq \bar{E} + W^r \Delta^r \tag{2}$$

where $\bar{E} = \mathrm{diag}(\bar{\epsilon}_i)$ with $\bar{\epsilon}_i \triangleq (\epsilon_{i,\mathrm{low}} + \epsilon_{i,\mathrm{upper}})/2$, $W^r = \mathrm{diag}\{\omega_i\}$ with $\omega_i \triangleq (\epsilon_{i,\mathrm{upper}} - \epsilon_{i,\mathrm{low}})/2$, and Δ^r is a diagonal real independent uncertainty, i.e., $\Delta^r = \mathrm{diag}\{\delta_i\}$ with $\delta_i \in [-1, 1]$, $i = 1, \ldots, m$.

Theorem 3.1. *Suppose that the model of a system in Fig. 2 is described with a transfer function matrix $P(s)$ without any additional uncertainty. The system remains stable under the gain variation defined with $E \in \mathcal{E}[\epsilon_{\mathrm{low}}, \epsilon_{\mathrm{upper}}]$ if and only if*

$$\mu^r_{\mathbf{\Delta}}(\bar{M}_{11}(j\omega)) < 1, \quad \forall \omega \in \mathbb{R} \cup \{\infty\}, \tag{3}$$

where $\bar{M}_{11}(s) = -K(s)(\mathbf{I} + P(s)\bar{E}K(s))^{-1} P(s) W^r$ *and* $\Delta^r \in \mathbf{\Delta}^r \triangleq \{\mathrm{diag}\{\delta_i\} : \delta_i \in [-1, 1], i = 1, \ldots, m\}$.

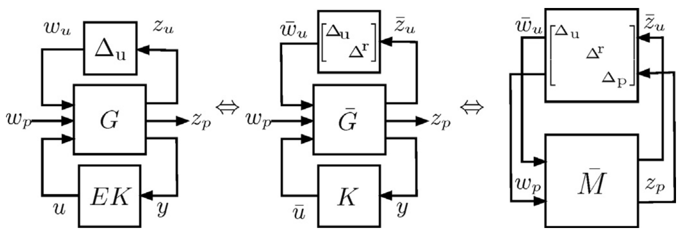


Fig. 6. Equivalent LFTs of robust fault tolerance.

Proof. The sequence of equivalent representations in Fig. 5 is obtained with the system transfer function matrices

$$G := \begin{bmatrix} \mathbf{0} & P \\ \mathbf{I} & -P \end{bmatrix}, \bar{G} := \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{I} \\ PW^\mathrm{r} & \mathbf{0} & P\bar{E} \\ -PW^\mathrm{r} & \mathbf{I} & -P\bar{E} \end{bmatrix} \tag{4}$$

and

$$\overline{M} := \mathcal{F}_\ell(\bar{G}, K) = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ PW^\mathrm{r} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{I} \\ P\bar{E} \end{bmatrix} K(\mathbf{I} + P\bar{E}K)^{-1} \begin{bmatrix} -PW^\mathrm{r} & \mathbf{I} \end{bmatrix}. \tag{5}$$

The definition of the structured singular value (Doyle, 1982; Zhou et al., 1996) implies that the system is robustly stable under any gain variation $E \in \mathcal{E}[\epsilon_\mathrm{low}, \epsilon_\mathrm{upper}]$ if and only if $\mu^\mathrm{r}_{\mathbf{\Delta}}(\overline{M}_{11}(j\omega)) < 1$ for all $\omega \in \mathbb{R} \cup \{\infty\}$.□

Theorem 3.2. *Suppose that the model of a system in Fig. 2 is described with a transfer function matrix P(s) without any additional uncertainty. The system achieves unity (reliable) $\mathcal{H}_\infty$ performance under the gain variation defined with $E \in \mathcal{E}[\epsilon_\mathrm{low}, \epsilon_\mathrm{upper}]$ if and only if*

$$\mu_{\mathbf{\Delta}}(\overline{M}(j\omega)) < 1, \quad \forall \omega \in \mathbb{R} \cup \{\infty\}, \tag{6}$$

where $\Delta \in \mathbf{\Delta} \triangleq \{\mathrm{diag}(\Delta^\mathrm{r}, \Delta_\mathrm{p}) : \Delta^\mathrm{r} \in \mathbf{\Delta}^\mathrm{r}$ and $\Delta_\mathrm{p} \in \mathbb{C}^{m_2 \times \ell_2}\}$ *and the matrix transfer function*

$$\overline{M}(s) \triangleq \begin{bmatrix} -K(s)(\mathbf{I} + P(s)\bar{E}K(s))^{-1}P(s)W^\mathrm{r} & K(s)(\mathbf{I} + P(s)\bar{E}K(s))^{-1} \\ P(s)W^\mathrm{r} - P(s)\bar{E}K(s)(\mathbf{I} + P(s)\bar{E}K(s))^{-1}P(s)W^\mathrm{r} & P(s)\bar{E}K(s)(\mathbf{I} + P(s)\bar{E}K(s))^{-1} \end{bmatrix}. \tag{7}$$

Proof. Applying the main-loop theorem (Lemma 2.2) to the matrix transfer function $\overline{M}(s)$ given in (5) completes the proof. □

Testing the maintenance of closed-loop stability and/or performance with respect to both actuator gain variation and additional perturbations like plant-model mismatch involves more complicated expressions for M and G.

Theorem 3.3. *Suppose that the model of a system in Fig. 2 is described by the standard LFT with uncertainty Δ_u, i.e., $P_{\Delta_u} = \mathcal{F}_u(P, \Delta_u)$. The system remains stable under the gain variation defined with $E \in \mathcal{E}[\epsilon_\mathrm{low}, \epsilon_\mathrm{upper}]$ if and only if*

$$\mu_{\mathbf{\Delta}_\mathrm{a}}(\overline{M}_{11}(j\omega)) < 1, \quad \forall \omega \in \mathbb{R} \cup \{\infty\}, \tag{8}$$

where $\overline{M}_{11}$ is the submatrix transfer function corresponding to the uncertainty block $\Delta_\mathrm{a} \triangleq \mathrm{diag}\{\Delta_\mathrm{u}, \Delta^\mathrm{r}\}$ of the total transfer function matrix

$$\overline{M} \triangleq \begin{bmatrix} P_{11} - P_{12}\bar{E}K(\mathbf{I} + P_{22}\bar{E}K)^{-1}P_{21} & P_{12}W^\mathrm{r} - P_{12}\bar{E}K(\mathbf{I} + P_{22}\bar{E}K)^{-1}P_{22}W^\mathrm{r} & P_{12}\bar{E}K(\mathbf{I} + P_{22}\bar{E}K)^{-1} \\ -K(\mathbf{I} + P_{22}\bar{E}K)^{-1}P_{21} & -K(\mathbf{I} + P_{22}\bar{E}K)^{-1}P_{22}W^\mathrm{r} & K(\mathbf{I} + P_{22}\bar{E}K)^{-1} \\ P_{21} - P_{22}\bar{E}K(\mathbf{I} + P_{22}\bar{E}K)^{-1}P_{21} & P_{22}W^\mathrm{r} - P_{22}\bar{E}K(\mathbf{I} + P_{22}\bar{E}K)^{-1}P_{22}W^\mathrm{r} & P_{22}\bar{E}K(\mathbf{I} + P_{22}\bar{E}K)^{-1} \end{bmatrix}. \tag{9}$$

Proof. The sequence of equivalent representations in Fig. 6 is obtained with the system transfer function matrices

$$G := \begin{bmatrix} P_{11} & \mathbf{0} & P_{12} \\ P_{21} & \mathbf{0} & P_{22} \\ -P_{21} & \mathbf{I} & -P_{22} \end{bmatrix}, \quad \bar{G} := \begin{bmatrix} P_{11} & P_{12}W^\mathrm{r} & \mathbf{0} & P_{12}\bar{E} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \\ P_{21} & P_{22}W^\mathrm{r} & \mathbf{0} & P_{22}\bar{E} \\ -P_{21} & -P_{22}W^\mathrm{r} & \mathbf{I} & -P_{22}\bar{E} \end{bmatrix},$$

and $\overline{M}(s)$ is given in (9), which implies that the system is robustly stable for any uncertainty $\Delta_u \in \mathbf{\Delta}_\mathrm{u}$ and under any gain variation $E \in \mathcal{E}[\epsilon_\mathrm{low}, \epsilon_\mathrm{upper}]$ if and only if $\mu_{\mathbf{\Delta}_\mathrm{a}}(\overline{M}_{11}(j\omega)) < 1$ for all $\omega \in \mathbb{R} \cup \{\infty\}$.□

Theorem 3.4. *Suppose that the model of a system in Fig. 2 is described with a transfer function matrix P(s) without any additional uncertainty. The system achieves an $\mathcal{H}_\infty$ performance, $\sup_{\|w_p\|_2 \le 1} (\| z_p\|_2 / \| w_p\|_2) \le 1$, under the gain variation defined with $E \in \mathcal{E}[\epsilon_\mathrm{low}, \epsilon_\mathrm{upper}]$ if and only if*

$$\mu_{\mathbf{\Delta}}(\overline{M}(j\omega)) < 1, \quad \forall \omega \in \mathbb{R} \cup \{\infty\}, \tag{10}$$

where $\Delta \in \mathbf{\Delta} \triangleq \{\mathrm{diag}\{\Delta_\mathrm{u}, \Delta^\mathrm{r}, \Delta_\mathrm{p}\} : \Delta_\mathrm{u} \in \mathbf{\Delta}_\mathrm{u}, \quad \Delta^\mathrm{r} \in \mathbf{\Delta}^\mathrm{r}$, and $\Delta_\mathrm{p} \in \mathbb{C}^{m_2 \times \ell_2}, \| \Delta_\mathrm{p}\|_\infty \le 1\}$ *and $\overline{M}(s)$ is given as (9).*

Proof. Applying the main-loop theorem (Zhou et al., 1996) to the matrix transfer function $\overline{M}(s)$ given in (9) completes the proof.□

3.2. Conditions for reliability using μ

3.2.1. DUS and RDUS

The below necessary and sufficient conditions for DUS and RDUS can be tested approximately in polynomial time as a function of the plant dimension.

Corollary 3.1. *DUS Suppose that K(s) is decentralized. Define Δ^r to be a diagonal Δ-block with independent real uncertainties. Then the closed-loop system is DUS if and only if M(s) is internally stable and*

$$\mu^\mathrm{r}_{\mathbf{\Delta}}(M(j\omega)) \le 1, \quad \forall \omega \in \mathbb{R} \cup \{\infty\}, \tag{11}$$

where $M(s) = -(1/2)K(s)(\mathbf{I} + \frac{1}{2}P(s)K(s))^{-1}P(s)$.

Proof. Set $\bar{E} = W^r = (1/2)\mathbf{I}$ in (5). □

Corollary 3.2. *RDUS Suppose that K(s) is decentralized and the uncertain system is described by P(s) and Δ_u, i.e., $P_{\Delta_u} := \mathcal{F}_u(P, \Delta_\mathrm{u})$. Define Δ^r to be a diagonal Δ-block with independent real uncertainties. Then the closed-loop system is RDUS if and only if M(s) is internally stable and*

$$\mu_{\mathbf{\Delta}_\mathrm{a}}(M(j\omega)) \le 1, \quad \forall \omega \in \mathbb{R} \cup \{\infty\}, \tag{12}$$

where $\Delta_\mathrm{a} \in \mathbf{\Delta}_\mathrm{a} = \left\{\mathrm{diag}\{\Delta_\mathrm{u}, \Delta^\mathrm{r}\} : \Delta_\mathrm{u} \in \mathbf{\Delta}_\mathrm{u}, \ \Delta^\mathrm{r} \in \mathbf{\Delta}^\mathrm{r}\right\}$ *and the transfer function matrix*

$$M(s)=\begin{bmatrix} P_{11}(s)-\frac{1}{2}P_{12}(s)K(s)(\mathbf{I}+\frac{1}{2}P_{22}(s)K(s))^{-1}P_{21}(s) & \frac{1}{2}P_{12}(s)-\frac{1}{4}P_{12}(s)K(s)(\mathbf{I}+\frac{1}{2}P_{22}(s)K(s))^{-1}P_{22}(s) \\ -K(s)(\mathbf{I}+\frac{1}{2}P_{22}(s)K(s))^{-1}P_{21}(s) & -\frac{1}{2}K(s)(\mathbf{I}+\frac{1}{2}P_{22}(s)K(s))^{-1}P_{22}(s) \end{bmatrix}. \tag{13}$$

Proof. Set $\overline{E}=W^{\mathrm{r}}=\frac{1}{2}\mathbf{I}$ in (9). □

3.2.2. CDUS and RCDUS

When $K(s)$ is stable, a necessary and sufficient test for CDUS is given by Corollary 3.1, except with the condition $\mu<1$ replacing $\mu\leq 1$ in (11). When $K(s)$ includes integral action in all channels, μ in (11) will be equal to 1 at $\omega=0$, because setting the proportional gain to zero in a controller with integral action will remove the feedback around the integrators, which will then be a limit of instability. Thus, $\mu\leq 1$ in (11) is a tight necessary condition for CDUS. A simple example shows that $\mu\leq 1$ is not sufficient for CDUS:

Example 3.1. Consider the plant and controller:

$$P(s)=\frac{1}{s+1}\begin{bmatrix} s & -1 \\ 1 & 1 \end{bmatrix}, \quad K(s)=\frac{1}{s}\mathbf{I}.$$

The Routh criterion can be used to show that this system is DUS and $\mu\leq 1$. Loop # 1 is not stable (for any ϵ_1) when Loop # 2 is open (due to a pole-zero cancelation at $s=0$), and so the system does not possess integrity and is not CDUS.

A more involved example illustrates that a system can possess integrity and be DUS without being CDUS.

Example 3.2. Consider the plant and controller:

$$P(s)=\frac{1}{s+4}\begin{bmatrix} \dfrac{\gamma(s^2+s+10)}{s+\alpha} & 1 \\ 1 & 1 \end{bmatrix}, \quad K(s)=\frac{1}{s}\mathbf{I},$$

where $\gamma=(4-\sqrt{55}-256)/9$ and $\alpha=(62-8\sqrt{55})/9$. The Routh criterion can be used to show that this system is DUS and $\mu\leq 1$. It can also be shown that the first loop is not stable for $\epsilon_1=1/2$ and $\epsilon_2=0$ though it is stable for all other $\epsilon_i\in[0,1]$.

CDUS can be checked through a finite number of stability and μ tests, by using Corollary 3.1 to check the interior of the ϵ-hypercube, and testing the boundary (the points, edges, faces, etc.) through additional μ tests. The number of μ tests required grows rapidly with the number of actuators/sensors in the system. Though the above examples show that CDUS is not equivalent to DUS, the set of plants that are DUS but not CDUS is non-generic, i.e., any perturbation in such a plant will likely cause the plant to either become DUS or not be DUS. Since Corollary 3.1 provides an exact condition for DUS, finding computable exact conditions for CDUS is of diminished importance. A similar discussion applies for RDUS vs. CRDUS.

3.3. Sufficient conditions for robust reliability of decentralized integral control using μ

Now consider a special case of decentralized control in which there exists integral control action in each control loop. Its integrity is defined as follows.

Definition 3.1. (Definition 14.2-2 in Morari and Zafiriou (1989)) The system $L(s)=P(s)C(s)$ is *integral controllable (IC)* if there exists a $k>0$ such that (a) the closed-loop system in Fig. 7 is stable for $K=k\mathbf{I}$ and (b) the gains of the loops can be reduced to $K_\epsilon=\epsilon k\mathbf{I}$, $\epsilon\in(0,1]$ without affecting the closed-loop stability.

In decentralized integral control, the integral controllability of the closed-loop system can be related to the eigenvalues of the open-loop steady-state gain matrix.

Theorem 3.5. *(Theorems 14.3-2 in Morari and Zafiriou (1989) or (Morari, 1985)) Suppose that $K\in\mathbb{R}^{m\times m}$ is a diagonal constant gain matrix with positive entries, i.e., $K=\mathrm{diag}\{k_i\}$, $k_i>0$, $i=1,\ldots,m$ and $\Delta_{\mathrm{u}}=\mathbf{0}$ such that $P_\Delta(s)=P_{22}(s)$ that is the lower right block transfer function of $P(s)$. The closed-loop system is IC if the steady-state gain matrix $L(0)=P_{22}(0)C(0)$ is anti-Hurwitz, i.e., $\sigma(L(0))\subset\mathbb{C}_+$.*

A natural extension of integral controllability to uncertain systems can be defined as follows.

Definition 3.2. The system $L_\Delta(s)=P_\Delta(s)C(s)$ is *robust integral controllable (RIC)* if there exists a $k>0$ such that, for any $\Delta_{\mathrm{u}}\in\mathbf{\Delta}_{\mathrm{u}}$, (a) the closed-loop system shown in Fig. 7 is stable for $K=k\mathbf{I}$ and (b) the gains of the loops can be reduced to $K_\epsilon=\epsilon k\mathbf{I}$, $\epsilon\in(0,1]$ without affecting the closed-loop stability.

Similar to the integral controllability, the robust integral controllability of the closed-loop system can be related to the eigenvalues of the open-loop steady-state gain matrix of which robustness is required.

Corollary 3.3. *Suppose that $K\in\mathbb{R}^{m\times m}$ is a diagonal constant gain matrix with positive entries, i.e., $K=\mathrm{diag}\{k_i\}$, $k_i>0$, $i=1,\ldots,m$, and the uncertainty $\Delta_u\in\mathbf{\Delta}_u$. The closed-loop system in Fig. 7 is RIC if the steady-state gain matrix $L_\Delta(0)$ is anti-Hurwitz for all $\Delta_u\in\mathbf{\Delta}_u$, i.e., $\sigma(L_\Delta(0))\subset\mathbb{C}_+$ for all $\Delta_u\in\mathbf{\Delta}_u$.*

The proof of Corollary 3.3 follows from the application of Thm. 3.5 to each plant in the set of uncertain plants. The next result is a sufficient condition for RIC in terms of μ.

Theorem 3.6. *Suppose that $K\in\mathbb{R}^{m\times m}$ is a diagonal constant gain matrix with positive entries, i.e., $K=\mathrm{diag}\{k_i\}$, $k_i>0$, $i=1,\ldots,m$, and the uncertainty $\Delta_u\in\mathbf{\Delta}_u$. The closed-loop system in Fig. 7 is RIC if*

$$\mu_{\mathbf{\Delta}_u^0}(M(j\omega))<\left(\sup_{\Delta\in\mathbf{\Delta}_u^0}\overline{\sigma}(\Delta)\right)^{-1}, \quad \forall\omega\in\mathbb{R}\cup\{\infty\}, \tag{14}$$

where $\mathbf{\Delta}_u^0\triangleq\{\mathbf{\Delta}_u(0):\mathbf{\Delta}_u\in\mathbf{\Delta}_u\}$ and

$$M(s)\triangleq\mathcal{F}_u\left(\begin{bmatrix} -P_{11}(0)C(0) & -P_{12}(0) \\ P_{21}(0)C(0) & P_{22}(0) \end{bmatrix}, \frac{1}{s}\mathbf{I}\right).$$

Proof. The steady-state gain matrix $L_\Delta(0)$ is anti-Hurwitz if and only if the linear system $\dot{x}=-L_\Delta(0)x$ is globally asymptotically

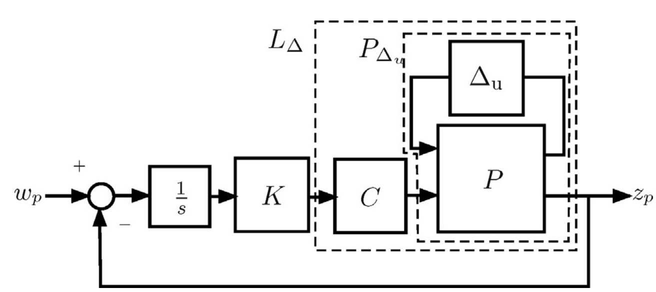


Fig. 7. Closed-loop uncertain system with integrator and diagonal compensator.

stable (g.a.s.) (or equivalently, globally exponentially stable (g.e.s.)). Furthermore, $L_\Delta(0)$ can be rewritten as

$$\mathcal{F}_\ell\left(\begin{bmatrix} P_{11}(0)C(0) & P_{12}(0) \\ P_{21}(0)C(0) & P_{22}(0)\end{bmatrix}, \Delta_\mathrm{u}(0)\right).$$

Now, for each $\Delta_u(0) \in \mathbf{\Delta}_u^0$, $\dot{x} = -L_\Delta(0)x$ is g.a.s. if and only if $\det(\mathbf{I} + M(s)\Delta_u(0)) \neq 0$ for all $s \in \mathbb{C}_+$. From the subharmonic property of μ and the homotopy condition on the uncertainty set $\mathbf{\Delta}_\mathrm{u}^0$ (i.e., $\Delta_u(0) \in \mathbf{\Delta}_\mathrm{u}^0$ implies that $\tau\Delta_u(0) \in \mathbf{\Delta}_\mathrm{u}^0$ for any $\tau \in [0, 1]$), the determinant condition can be reduced to the frequency-domain condition on μ in (14).□

Conditions for robust integral controllability have been derived in some past studies. In particular, conditions with respect to the relative gain array of the nominal linear plant are presented in some past research work, e.g., (Firouzbahrami and Nobakhti, 2011; Haggblom, 2008; Kariwala et al., 2006; Yu and Luyben, 1987), in which only additive uncertainty is addressed. In contrast, this article considers general linear fractional uncertainties, which include multiplicative and additive uncertain systems, and their combinations, as special cases.

3.4. Remarks on decentralized detunability

Detuning a controller refers to changing some parameter in the controller or in the control synthesis procedure so that the control action becomes less aggressive. For example, in linear quadratic (LQ) optimal control, detuning refers to increasing the magnitude of the weight of control action in the quadratic cost function—exactly opposite of *cheap control* in which control weights are very small (Seron et al., 1999). In decentralized internal model control (IMC), detuning refers to increasing the IMC filter time constants (or equivalently, decreasing the bandwidth of the IMC filter) in each single-loop controller (Hovd, 1992; Hovd et al., 1993). The special case of detuning the single-loop controller gains in a decentralized controller was discussed earlier in the sections on DUS and RDUS.

Hovd (1992) introduced a very general definition for robust decentralized detunability.

Definition 3.3. For a given design method, a closed-loop system is robust decentralized detunable (RDD) if each single-loop controller can be detunable independently by an arbitrary amount without losing robust stability in the closed-loop system.

Whenever a controller is detuned by varying a parameter in the controller, RDD can be tested via a μ test where the variation in parameters is covered by real uncertainty (the real uncertainty must be independent for arbitrary detuning). Both the robust performance and the RDD loopshaping bounds are plotted and the most restrictive of the bounds are used in the design. The resulting controller meets robust performance and gives a system that is RDD. This loopshaping design procedure is illustrated in Braatz (1993), where interested readers can go for details and examples.

4. Discussion

4.1. Review of previous research with illustrative examples

4.1.1. Integrity

Most research on reliability analysis considers only system integrity without considering plant-model mismatch (Delich, 1992; Fujita and Shimemurab, 1988; Morari, 1985; Morari and Zafiriou, 1989). Controller-independent conditions that can establish necessary and sufficient conditions for the existence or non-existence of a controller such that the system possesses integrity have been derived (Gündes and Kabuli, 2001; Kariwala et al., 2005), but these conditions are also only applicable to perfectly known linear time-invariant systems.

Fujita and Shimemurab (1988) state that a necessary and sufficient condition for integrity with stable controllers is that all the principal minors of $\mathbf{I} + PK$ are minimum phase. This condition is theoretically interesting, because this test does not require the calculation of matrix inverses. However, since the number of principal minors of matrix grows exponentially with its dimension, the calculation required by this test grows exponentially as a function of the plant dimension. Fujita and Shimemurab (1988) also provide a sufficient condition for integrity when the controller is stable, in terms of the generalized diagonal dominance of $\mathbf{I} + P(j\omega)K(j\omega)$. Applying the Perron–Frobenius Theorem (Horn and Johnson, 1985) gives the following lemma (for details, see Delich (1992)).

Lemma 4.1. *Assume $P(s)$ and $K(s)$ are stable, the diagonal elements of $\mathbf{I} + P(s)K(s)$ are minimum phase, and $P(s)$ is irreducible. Then the closed-loop system demonstrates integrity if*

$$\rho(|H(j\omega)(\overline{H}(j\omega))^{-1}|) < 2, \quad \forall\omega \in \mathbb{R} \cup \{\infty\}, \tag{15}$$

where $H = \mathbf{I} + PK$, $\overline{H}$ refers to the matrix with all off-diagonal elements of H replaced by zeros, and $|A|$ denotes the matrix with each element of A replaced by its magnitude.

The above assumption that P is irreducible can be removed with some added complexity in the theorem statement (Delich, 1992). The spectral radius is readily computable with polynomial growth ($\sim n^3$) as a function of the plant dimension. However, the lemma might be conservative as shown in the following example.

Example 4.1. Consider the closed-loop system with the plant and controller:

$$P(s) = \frac{1}{75s+1}\begin{bmatrix} -0.878 & 0.014 \\ -1.082 & -0.014 \end{bmatrix};$$

$$K(s) = \frac{75s+1}{\lambda s+1}\begin{bmatrix} -\dfrac{1}{0.878} & 0 \\ 0 & -\dfrac{1}{0.014}\end{bmatrix}; \quad \lambda = 4.$$

The system demonstrates integrity but the condition in (15) is not satisfied for this system ($\rho \approx 2.1 < 2$), which indicates that the test (15) can be conservative, even for 2×2 systems.

4.1.2. Robust integrity

Laughlin et al. (1993) provide computationally simple tests for robust integrity that are useful for cross-directional processes (see VanAntwerp et al. (2007) for a review of cross-directional process control problems). Their results do not extend to general plants and so are not further discussed here.

4.1.3. Decentralized unconditional stability

Morari (1985) considers stability with simultaneous detuning of all loops, which leads to a number of computationally simple necessary conditions for DUS that are surveyed in the monograph by Morari and Zafiriou (1989). However, all these conditions can be conservative for testing DUS, as illustrated by examples in that monograph.

4.1.4. CDUS

Nwokah and Perez (1991) considered conditions for which a system with controller $K(s) = (1/s)\mathbf{I}$ is CDUS, including the claim that a necessary condition for $K(s) = (1/s)\mathbf{I}$ to provide CDUS is that the steady-state matrix $P(0)$ is *all gain positive stable*. A matrix P is all gain positive stable if P, P^{-1}, and all their corresponding principal submatrices are D-stable. A matrix P is D-stable if $\sigma(PD) \subset \mathbb{C}_+$ for all positive diagonal matrices D. Example 4.2 shows that the condition in Nwokah and Perez (1991) is not necessary.

Example 4.2. Consider the plant (Campo and Morari, 1994):

$$P(s) = \begin{bmatrix} 1 & 0 & 2 \\ \frac{1}{s+1} & 1 & \frac{-4s}{s+1} \\ 0 & 4 & 1 \end{bmatrix}.$$

It can be shown that the Routh–Hurwitz stability criteria that the closed-loop system for the above plant is stable for $K(s) = (1/s)\mathbf{I}$ and remains stable with arbitrary detuning of the SISO loop gains. But, $\sigma(P(0)) = \{\pm i\sqrt{3}, 3\}$, so $P(0)$ is not D-stable, and $P(0)$ is not all gain positive stable. We note here without details that this plant also shows that all of the theorems in Nwokah and Perez (1991) regarding decentralized integral controllability are also not necessary.

4.1.5. RDUS and RCDUS

To our knowledge, it seems that RDUS and RCDUS have not been considered in the open literature, except for a thesis (Braatz, 1993) and the proceedings paper (Braatz et al., 1994) that contains some of the results of this manuscript. Note that Section 3.2 showed that conditions for RDUS and RCDUS can be represented as evaluating the structured singular value of the associated transfer function. Although its exact computation is NP-hard (Braatz et al., 1994), upper and lower bounds on μ are computable in polynomial-time (Balas et al., 1998).

4.2. Illustrative examples: fault-tolerant decentralized control

4.2.1. High-purity distillation column

We now illustrate the investigation of robust stability and performance of a decentralized controller for the high-purity distillation column under fault/failure scenarios. A high-purity distillation column is given in Skogestad and Morari (1989) and discussed in more detail in Skogestad et al. (1988). The nominal model is

$$P_{\mathrm{n}}(s) = \frac{1}{75s+1}\begin{bmatrix} -0.878 & 0.014 \\ -1.082 & -0.014 \end{bmatrix},$$

which uses distillate and boilup as manipulated inputs to control top and bottom composition using measurements of the top and bottom compositions. The plant has a large condition number, so input uncertainty strongly affects robust performance (Skogestad et al., 1988). The uncertainty and performance weights are

$$w_{\mathrm{I}}(s) = 0.1\frac{5s+1}{0.25s+1}, \quad w_{\mathrm{P}}(s) = 0.025\frac{7s+1}{7s}.$$

The input uncertainty includes actuator uncertainty and neglected right-half plane zeros of the plant. The performance bound implies zero steady-state error and a closed-loop time constant of 7 min. The uncertainty block Δ_{I} is a diagonal 2×2 matrix (independent actuators) and the performance block Δ_{P} is a full 2×2 matrix.

In Braatz (1993), loopshaping bounds are used to design the decentralized controller

$$K(s) = \frac{75s+1}{4s}\begin{bmatrix} -\frac{1}{0.878} & 0 \\ 0 & -\frac{1}{0.014} \end{bmatrix}.$$

We will now analyze the closed-loop system with the designed controller to show that it satisfies integrity, robust integrity, DUS, and RDUS.

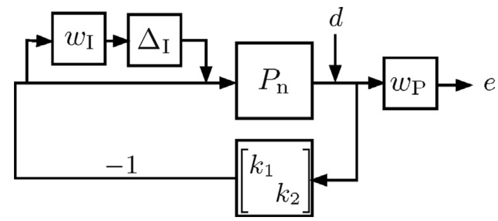


Fig. 8. The plant with input uncertainty Δ_{I} of magnitude $w_{\mathrm{I}}(s)$ and the performance specification $w_{\mathrm{P}}(s)$.

4.2.1.1. *Integrity.*

The four transfer functions

$$(\epsilon_1, \epsilon_2) = (0, 0) \Rightarrow P_{\mathrm{n}},$$
$$(\epsilon_1, \epsilon_2) = (1, 1) \Rightarrow -w_{\mathrm{I}}K(\mathbf{I}+P_{\mathrm{n}}K)^{-1}P_{\mathrm{n}},$$
$$(\epsilon_1, \epsilon_2) = (1, 0) \Rightarrow -w_{\mathrm{I}}K_1(\mathbf{I}+P_{\mathrm{n},11}K_1)^{-1}P_{\mathrm{n},11},$$
$$(\epsilon_1, \epsilon_2) = (0, 1) \Rightarrow -w_{\mathrm{I}}K_2(\mathbf{I}+P_{\mathrm{n},22}K_2)^{-1}P_{\mathrm{n},22},$$

are stable, so the closed-loop system has integrity.

4.2.1.2. *Robust integrity.*

Robust integrity for a 2×2 system can be evaluated by checking the robust stability for four conditions. Nominal stability was tested above (for testing integrity), so only the μ conditions are tested here. The system has robust stability when all loops are turned off provided that $P_{\mathrm{n}}(\mathbf{I}+w_{\mathrm{I}}\Delta_{\mathrm{I}})$ is stable. Since P_{n}, w_{I}, and Δ_{I} are stable, $P_{\mathrm{n}}(\mathbf{I}+w_{\mathrm{I}}\Delta_{\mathrm{I}})$ is stable. Robust stability for the overall system is satisfied since $\mu_{\boldsymbol{\Delta}_{\mathrm{I}}}(-w_{\mathrm{I}}K(\mathbf{I}+P_{\mathrm{n}}K)^{-1}P_{\mathrm{n}}) = 0.3 < 1$. Robust stability for the cases when exactly one loop has failed is satisfied since

$$(\epsilon_1, \epsilon_2) = (1, 0) \Rightarrow \quad \mu_{\boldsymbol{\Delta}_{\mathrm{I},11}}(-w_{\mathrm{I}}K_1(\mathbf{I}+P_{\mathrm{n},11}K_1)^{-1}P_{\mathrm{n},11}) = 0.12 < 1;$$
$$(\epsilon_1, \epsilon_2) = (0, 1) \Rightarrow \quad \mu_{\boldsymbol{\Delta}_{\mathrm{I}},22}(-w_{\mathrm{I}}K_2(\mathbf{I}+P_{\mathrm{n},22}K_1)^{-1}P_{\mathrm{n},22}) = 0.12 < 1.$$

Since all four μ conditions are satisfied, the system demonstrates robust integrity.

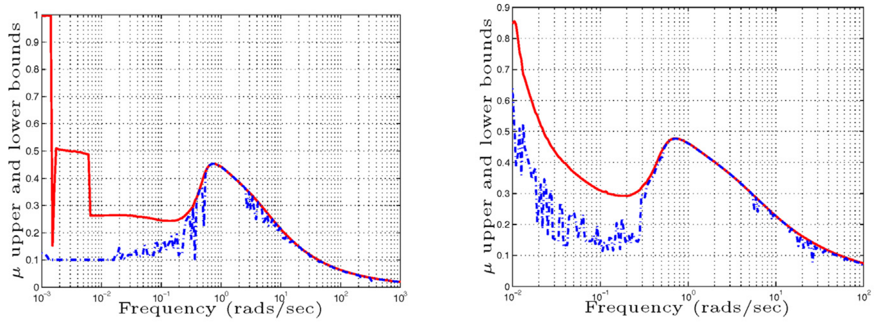
4.2.1.3. *DUS and RDUS.*

First let's test RDUS. The transfer function matrices P, G, $\overline{G}$, and Δ_{a} needed to apply Theorems 3.3 and 3.4 are derived directly from the block diagram in Fig. 8:

$$P = \begin{bmatrix} \mathbf{0} & -w_{\mathrm{I}}\mathbf{I} \\ P_{\mathrm{n}} & -P_{\mathrm{n}} \end{bmatrix}, \quad G = \begin{bmatrix} \mathbf{0} & \mathbf{0} & -w_{\mathrm{I}}\mathbf{I} \\ w_{\mathrm{P}}P_{\mathrm{n}} & \mathbf{0} & -w_{\mathrm{P}}P_{\mathrm{n}} \\ -P_{\mathrm{n}} & \mathbf{I} & P_{\mathrm{n}} \end{bmatrix},$$
$$\overline{G} = \begin{bmatrix} \mathbf{0} & -w_{\mathrm{I}}W^{\mathrm{r}} & \mathbf{0} & -w_{\mathrm{I}}\overline{E} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \\ w_{\mathrm{P}}P_{\mathrm{n}} & -w_{\mathrm{P}}P_{\mathrm{n}}W^{\mathrm{r}} & \mathbf{0} & -w_{\mathrm{P}}P_{\mathrm{n}}\overline{E} \\ -P_{\mathrm{n}} & P_{\mathrm{n}}W^{\mathrm{r}} & \mathbf{I} & P_{\mathrm{n}}\overline{E} \end{bmatrix}, \tag{16}$$
$$\Delta_{\mathrm{a}} = \mathrm{diag}\{\Delta_{\mathrm{I}}, \Delta^{\mathrm{r}}\}.$$

Fig. 9a is the μ plot to test condition (8) in Theorem 3.3 for evaluating RDUS. As expected, the value of μ approaches 1 at zero frequency due to the integrators as either of the ϵ_i approach zero. We see that $\mu \ll 1$ for all frequencies away from $\omega = 0$. Since $\mu \le 1$, the system demonstrates RDUS. Since DUS is implied by RDUS, DUS does not need to be numerically tested for this example.

Robust performance under arbitrary detuned control gains (0–100% of the nominal value) can also be studied using condition (10) in Theorem 3.4. Consider the analysis of robust performance under a reduced performance defined by $w_{\mathrm{P}}(s) = 0.025(7s+1)/(7s)$, which is 1/10 of the nominal performance specified by Skogestad et al. (1988) when there is no faults in the system. Fig. 9b



(a) μ plot for evaluation of RDUS

(b) μ plot for evaluation of robust performance

Fig. 9. μ plots for evaluating reliability to uncertainties and reduction of actuator/sensor/controller gains for the high-purity distillation column. The relatively smooth red curve is the upper bound for μ and the rough blue curve is its lower bound. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of the article.)

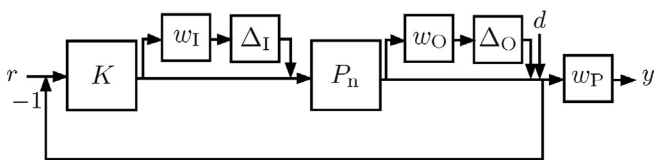


Fig. 10. The plant with input and output uncertainties Δ_I and Δ_O of magnitude $w_I(s)$ and $w_O(s)$, and the performance specification $w_P(s)$.

shows that $\mu_{\boldsymbol{\Delta}}(j\omega) \le 1$ for all frequency, so the reduced level of robust performance is achieved for the specified range of detuned control gains.

4.2.2. Parallel reactors with combined precooling

Now consider the robust stability and performance of an SVD optimal controller for a parallel reactor with combined precooling. In Hovd et al. (1997), a simplified model of four parallel reactors with combined precooling is

$$G(s) = \frac{1}{100s+1}\begin{bmatrix} 1 & 0.7 & 0.7 & 0.7 \\ 0.7 & 1 & 0.7 & 0.7 \\ 0.7 & 0.7 & 1 & 0.7 \\ 0.7 & 0.7 & 0.7 & 1 \end{bmatrix}.$$

Consider the input and output uncertainty in the system shown in Fig. 10. The input uncertainty Δ_I and output uncertainty Δ_O are assumed to have independent diagonal and uncertainty weights given by $w_I := 0.125(5s+1)/(0.5s+1)\mathbf{I}$ and $w_O := 0.125(2.5s+1)/(0.25s+1)\mathbf{I}$, respectively. To reject disturbances at the system output, the weighted performance specification is $\| w_P S_p \|_\infty < 1$, where S_p is the transfer function mapping d to y, with the performance weight $w_P(s) := 0.125(125s+1)/(125s)\mathbf{I}$. An SVD optimal controller was designed using DK-iteration and reported in Hovd et al. (1997). This example considers the reliability of this controller design to 80% independent detuning of the controller gains. The performance weight models partially degraded performance, compared to the case when there is no fault or failure of controllers in Hovd et al. (1997).

4.2.2.1. Robust reliability.

The transfer function matrices $P, G, \overline{G}$, and Δ_a needed to apply Theorems 3.3 and 3.4 are derived directly from the block diagram in Fig. 10:

$$P = \begin{bmatrix} \mathbf{0} & \mathbf{0} & -w_I\mathbf{I} \\ w_O P_n & \mathbf{0} & -w_O P_n \\ P_n & \mathbf{I} & -P_n \end{bmatrix}, \quad G = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & -w_I\mathbf{I} \\ w_O P_n & \mathbf{0} & \mathbf{0} & -w_O P_n \\ w_P P_n & w_P\mathbf{I} & \mathbf{0} & -w_P P_n \\ -P_n & -\mathbf{I} & \mathbf{I} & P_n \end{bmatrix},$$

$$\overline{G} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & -w_I W^r & \mathbf{0} & -w_I\overline{E} \\ w_O P_n & \mathbf{0} & w_O P_n W^r & \mathbf{0} & -w_O P_n \overline{E} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \\ w_P P_n & w_P \mathbf{I} & w_P P_n W^r & \mathbf{0} & -w_P P_n \overline{E} \\ -P_n & -\mathbf{I} & -P_n W^r & \mathbf{I} & P_n \overline{E} \end{bmatrix}, \tag{17}$$

$$\Delta_a = \mathrm{diag}\{\Delta_I, \Delta_O, \Delta^r\}.$$

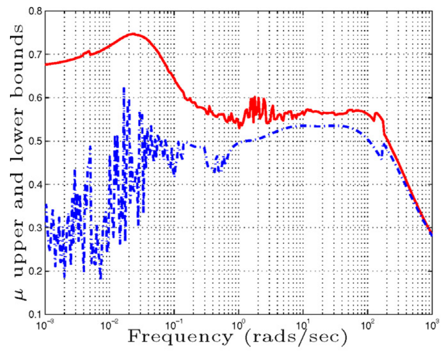
To assess whether the closed-loop uncertain system remains stable with up to 80% independent detuning of the actuator/sensor/controller gains, set $\epsilon_i \in [0.2, 1]$ for all i and $W^r = 0.4\mathbf{I}$ and $\overline{E} = 0.6\mathbf{I}$. The μ plot in Fig. 11a to test condition (8) in Theorem 3.3 shows that $\mu(j\omega) < 1$ for all frequencies, which implies that the system is robust to this degree of control detuning and to model uncertainties. This μ plot also implies that the nominal system is reliable to 80% independent detuning of the actuator/sensor/controller gains.

Robust performance under detuned control gains can be studied using condition (10) in Theorem 3.4. Fig. 11b shows that $\mu_{\boldsymbol{\Delta}}(j\omega) \le 1$ for all frequencies, so robust performance is achieved with up to 80% independent detuning of the controller gains. Various degrees of degraded closed-loop performance could be defined for different degrees of detuning, by plotting a different μ plot for each performance weight and range of detuning.

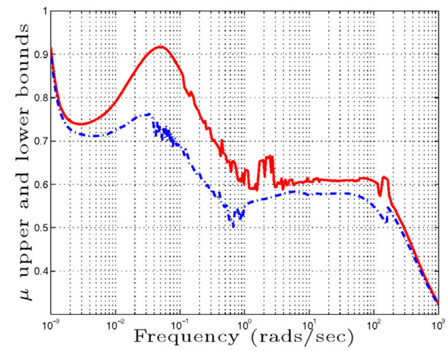
4.3. Related topics

4.3.1. Fault detection and diagnosis

For systems affected by time-varying parametric uncertainties and time-varying detuned gain of decentralized controllers, it might be natural to discuss the design of linear parametrically varying (LPV) controllers or gain-scheduled controllers when the time-varying parameters are not known *a priori*, but are online measurable. In that control framework, faults in the actuators and/or sensors can be detected and LPV control laws give a natural way to remedy those faults.



(a) μ plot for robust stability



(b) μ plot for robust performance

Fig. 11. μ plots for evaluating reliability to uncertainties and up to 80% independent detuning of controller gains for the parallel reactors with combined precooling. The red curve is the upper bound for μ and the blue curve is its lower bound. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of the article.)

4.3.2. Reliable networked control systems

In a networked control system, most communication links introduce variable and unpredictable time delays in the information flow, which are called *network-induced delays* (Zhang et al., 2001). This application problem has motivated the analysis of the effects of time delays among interconnecting elements of a decentralized or distributed network control system on the closed-loop system stability and performance. The problem formulation and conditions for robust reliability analysis of decentralized control systems can be extended to the robust stability and performance analysis of networked control systems under intermittent communication losses between distributed sensors and actuators.

5. Conclusions

Robust reliability of closed-loop systems is an important issue in control systems engineering and for large-scale interconnected systems. This article considers the analysis of the reliability of controlled systems with and without model uncertainties. Necessary and sufficient conditions for robust fault-tolerant stability and performance under constant but unknown gain variation are derived for uncertain systems that are affected by real parametric and complex dynamic uncertainties. The proposed conditions are represented in terms of the structured singular value and are nonconservative in the sense that locations and structures of potential faults and failures can be fully exploited, and structured plant-model mismatches are considered, in the derivation of necessary and sufficient conditions for system reliability. Upper and lower bounds on μ can be computed in polynomial-time by using off-the-shelf software (Balas et al., 1998) and provide computationally tractable tools for verifying reliability of the controllers. Numerical case studies for high-purity distillation column and parallel reactors with combined precooling are presented for illustration of the application of the proposed reliability conditions.

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