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Convex initialization of the $\mathcal{H}_2$-optimal static output feedback problem

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***Abstract*— Recently we have established a link between invariants for quadratic optimization problems and linear-quadratic (LQ) optimal control [1]. The link is that for LQ control one invariant is $c_k = u_k - Kx_k$, which yields zero loss from optimality when controlled to a constant setpoint $c = c_s = 0$. In general there exists infinitely many such invariants to a quadratic programming (QP) problem. In [2] we show how the link can be used to generate output feedback control by using current and old measurements. In this paper we extend this approach by considering in more detail some interesting examples, and the use of additional (old) measurements. In particular, we show that if the number of measurements is less than the number of disturbances (initial states) plus independent inputs, we can not with this method find a policy $u_k = -K^y y_k$ that minimizes the original problem, because K^y is not optimally constant. However, this method may be used to find initial values for $\mathcal{H}_2$-optimal static output feedback synthesis.**

***Index Terms*— linear quadratic control, fixed-structure control**

I. INTRODUCTION

Consider a finite horizon LQ problem of the form

$$\min_{u_0,u_1,\ldots,u_{N-1}} J(u,x(0)) = E\{x_N^{\mathrm{T}} P x_N + \\ + \sum_{k=0}^{N-1} \left[x_k^{\mathrm{T}} Q x_k + u_k^{\mathrm{T}} R u_k\right]\} \tag{1}$$

$$\text{subject to } x_0 = x(0)$$
$$x_{k+1} = Ax_k + Bu_k, \quad k \geq 0$$
$$y_k = Cx_k + n_k^y,$$

where $x_k \in \mathbb{R}^{n_x}$ are the states, $u_k \in \mathbb{R}^{n_u}$ are the inputs and $y_k \in \mathbb{R}^{n_y}$ are the measurements. Further $P = P^{\mathrm{T}} > 0$, $Q \geq 0$, and $R > 0$ are matrices of appropriate dimensions, and $E\{\cdot\}$ is the expectation operator.

It is well-known that if $C = I$ and $n^y = 0$, such that $y_k = x_k$, the solution to (1) is state feedback $u_k = -Kx_k$, where the gain matrix K can be found by solving an iterative Riccati equation. For the case with white noise assumption on x_0 and y (n^y), the optimal solution is $u_k = -K\hat{x}_k$, where $\hat{x}_k$ is a state estimate from a Kalman filter [3], which in effect gives a dynamic compensator K^{lqg} (from y to u) of same order n_x as the plant.

In this paper we consider the static output feedback problem, $u_k = -K^y y_k$, where K^y is a static gain matrix. Note that the case with a fixed-order controller of order less than n_x may also be brought back to the static output feedback problem. A particular controller considered in this paper is the multi-input multi-output proportional-integral-derivative controller (MIMO-PID) where we have as many controlled outputs y^c as there are inputs u. The "allowed" measurements y_k in the formulation in (1) are the present value of the controlled output y_k^c (P), the integrated value $\sum_{i=0}^{k} y_i^c$ (I) and the derivative $\frac{\partial y_k^c}{\delta t}$ (D).

This optimal solution to this problem is unsolved [4] so one cannot expect to find an analytic or convex numerical solution. The contribution of this paper is therefore to propose a convex approach to find a good initial estimate for K^y, as a starting point for a numerical search.

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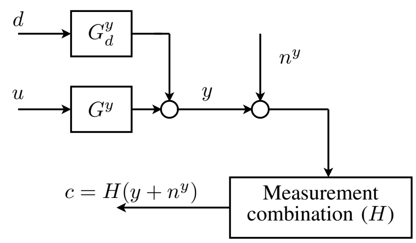

Fig. 1. Notation for self-optimizing control.

A. Notation

Notation adopted from self-optimizing control is summarized in figure 1. Typically, $u = (u_0, u_1, \ldots, u_{N-1})$, $d = x_0$ and $y = (x_0, u)$ or $y = (y_0, \frac{\partial y_0}{\partial t}, \ldots, u)$, but also other variables y will be considered.

II. MAIN RESULTS

A. Results from self-optimizing control

1) Nullspace method: From [5] we have the following theorem:

Theorem 1: (Nullspace theorem = Linear invariants for quadratic optimization problem) Consider an unconstrained quadratic optimization problem in the variables u (input vector of length n_u) and d (disturbance vector of length n_d)

$$\min_u J(u,d) = \begin{bmatrix} u \\ d \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} J_{uu} & J_{ud} \\ J_{ud}^{\mathrm{T}} & J_{dd} \end{bmatrix} \begin{bmatrix} u \\ d \end{bmatrix}. \tag{2}$$

In addition, there are "measurement" variables $y = G^y u + G_d^y d$. If there exists $n_y \geq n_u + n_d$ independent measurements (where "independent" means that the matrix $\tilde{G}^y = \begin{bmatrix} G^y & G_d^y \end{bmatrix}$ has full rank), then the optimal solution to (2) has the property that there exists $n_c = n_u$ linear variable combinations (constraints) $c = Hy$ that are invariant to

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the disturbances d. The optimal measurement combination matrix H is found by selecting H such that

$$HF = 0, \tag{3}$$

where $F = \frac{\partial y^{\text{opt}}}{\partial d}$ is the optimal sensitivity matrix which can be obtained from

$$F = -(G^y J_{uu}^{-1} J_{ud} - G_d^y), \tag{4}$$

(That is, H is in the left nullspace of F.)

2) Generalization: Exact local method: A generalization of Theorem 1 is the following:

Theorem 2: (Exact local method = Loss by introducing linear constraint for noisy quadratic optimization problem [5]) Consider the unconstrained optimization problem in Theorem 1, (2), and a set of noisy measurements $y_m = y + n^y$, where $y = G^y u + G_d^y d$. Assume that $n_c = n_u$ constraints $c = Hy_m = c_s$ are added to the problem, which will result in a non-optimal solution with a loss $L = J(u,d) - J_{\text{opt}}(d)$. Consider disturbances d and noise n^y with magnitudes

$$d = W_d d'; \quad n^y = W_{n^y} n^y; \quad \left\| \begin{bmatrix} d' \\ n^{y'} \end{bmatrix} \right\|_2 \leq 1.$$

Then for a given H, the worst-case loss introduced by adding the constraint $c = Hy$ is $L_{wc} = \bar{\sigma}^2(M)/2$, where M is

$$\begin{aligned} M &\triangleq \begin{bmatrix} M_d & M_n \end{bmatrix} \\ M_d &= -J_{uu}^{1/2}(HG^y)^{-1}HFW_d \\ M_n &= -J_{uu}^{1/2}(HG^y)^{-1}HW_{n^y}, \end{aligned} \tag{5}$$

and $\bar{\sigma}(\cdot)$ is the maximum singular value. The optimal H that minimizes the loss can be found by solving the *convex* optimization problem

$$\begin{aligned} &\min_H \ \|H\tilde{F}\|_F \\ &\text{subject to } HG^y = J_{uu}^{1/2} \end{aligned} \tag{6}$$

Here $\tilde{F} = [FW_d \ \ W_{n^y}]$.

The reason for using the Frobenius norm is that minimization of this norm also minimizes $\bar{\sigma}(M)$ [6].

Remark 1: From [5] we have that any optimal H premultiplied by a non-singular matrix $n_c \times n_c$ D, i.e. $H_1 = DH$ is still optimal. One implication of this is that for a square plant, $n_c = n_u$, we can write $c = H_1 y = H_1^{y_m} y_m + Iu$. To see this, assume $y = (y_m, u)$, so $H = [H^{y_m} \ H^u]$, where H^u is a non-singular $n_u \times n_u$ matrix. Now, $H_1 = (H^u)^{-1}[H^{y_m} \ H^u] = [(H^u)^{-1}H^{y_m} \ I]$.

Remark 2: More generally, for the case when $\tilde{F}\tilde{F}^{\mathrm{T}}$ is singular, we can solve the convex problem (6) using for example CVX, a package for specifying and solving convex programs [7], with the following code:

```
cvx_begin
   variable H(N*nu,ny+nu*N);
   minimize norm(H*Ftilde,'fro')
   subject to
      H*Gy == sqrtm(Juu);
cvx_end
```

Remark 3: Noise will not be further discussed in this paper, but is covered in [8].

B. Some special cases

Some special cases will now be considered where explicit expressions can be found.

1) Full information: No new results are represented here, but we show the matrices G^y, G_d^y, J_{uu}, and J_{ud} for LQ-optimal control.

Assume that noise-free measurements of all the states are available. It is well known that the LQ problem (1) can be rewritten on the form in (2) (see for example [9]) by treating x_0 as the disturbance d, and letting $u = (u_0, u_1, \ldots, u_{N-1})$. Thus, from Theorem 1 we know that for the LQ problem there exists *infinitely* many invariants (but only one of these involves only present states).

Without loss of generality consider the case when the model in (1) is stable.

Let $y = (x_0, u_0, u_1, \ldots, u_{N-1}) = (x_0, u)$. Note that this includes also future inputs, but we will use the normal "trick" in MPC of implementing only the present (first) input change u_0. Since we have $n_y = n_d + n_u$ and no noise, we can use Theorem 1. The open loop model becomes:

$$\begin{aligned} y &= G^y u + G_d^y d \\ G^y &= \begin{bmatrix} 0_{n_x \times (n_u N)} \\ I_{n_u N} \end{bmatrix} \in \mathbb{R}^{(n_x + n_u N) \times (n_u N)} \\ G_d^y &= \begin{bmatrix} I_{n_u N} \\ 0_{(n_u N) \times n_x} \end{bmatrix} \in \mathbb{R}^{(n_x + n_u N) \times n_x} \end{aligned} \tag{7}$$

Here I_m is an $m \times m$ identity matrix and $0_{m \times n}$ is a $m \times n$ matrix of zeros.

The matrices J_{uu} and J_{ud} are the derivatives of the linear quadratic objective function.For the objective and process model in (1) we show in [10] that

$$\frac{J_{uu}}{2} = \begin{bmatrix} B^{\mathrm{T}}PB+R & B^{\mathrm{T}}A^{\mathrm{T}}PB & \ldots & B^{\mathrm{T}}(A^{N-1})^{\mathrm{T}}PB \\ B^{\mathrm{T}}PAB & B^{\mathrm{T}}PB+R & \ldots & B^{\mathrm{T}}(A^{N-2})^{\mathrm{T}}PB \\ \vdots & \vdots & \ddots & \vdots \\ B^{\mathrm{T}}PA^{N-1}B & B^{\mathrm{T}}PA^{N-2}B & \ldots & B^{\mathrm{T}}PB+R \end{bmatrix} \tag{8}$$

and

$$\frac{J_{ud}}{2} = \begin{bmatrix} B^{\mathrm{T}} & & & \\ & B^{\mathrm{T}} & & \\ & & \ddots & \\ & & & B^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} P \\ PA \\ \vdots \\ PA^{N-1} \end{bmatrix} A \tag{9}$$

The sensitivity matrix (optimal change in y when d is perturbed) becomes:

$$F = \frac{\partial y^{\text{opt}}}{\partial d^{\mathrm{T}}} = -(G^y J_{uu}^{-1} J_{ud} - G_d^y) = \begin{bmatrix} I_{n_x} \\ -J_{uu}^{-1} J_{ud} \end{bmatrix} \tag{10}$$

We can use Theorem 1 to get the combination matrix H, i.e. find an H such that $HF = 0$:

$$\begin{bmatrix} H_1 & H_2 \end{bmatrix} \begin{bmatrix} I_{n_x} \\ J_{uu}^{-1} J_{ud} \end{bmatrix} = H_1 - H_2(J_{uu}^{-1} J_{ud}) = 0 \tag{11}$$

To ensure a non-trivial solution we can choose $H_2 = I_{n_u N}$ and get the following optimal combination of x_0 and u:

$$c = Hy = J_{uu}^{-1} J_{ud} x_0 + u, \tag{12}$$

which can be interpreted as:

$$\begin{aligned}&\text{Invariant 1: } u_0 = K_0 x_0\\&\text{Invariant 2: } u_1 = K_1 x_0\\&\quad\vdots\\&\text{Invariant } N\text{: } u_{N-1} = K_{N-1} x_0\end{aligned} \tag{13}$$

From Theorem 1 implementation of (13) give zero loss from optimality, i.e. they correspond to the optimal input trajectory $u_0^*, u_1^*, \ldots, u_{N-1}^*$ from the solution of (1). Moreover, since the states capture all information, we must have that

$$u_1 = K_1 x_0 = \underbrace{K_1(A+BK_0)^{-1}}_{=K_0} x_1. \tag{14}$$

From this we deduce that the solution to (1) can be implemented as $u_k = K_0 x_k,\ k = 0, 1, \ldots$.

In [10] we prove that this gives the same result as conventional linear quadratic control, by conventional meaning for example equation (3) in Rawlings and Muske [9].

2) Output feedback: In this section we will show how Theorem 2 can be used for the special (but common) case when $y_k = Cx_k + 0 \cdot u_k,\ k = 0, 1, \ldots, N$ and we look for controllers on the form $u_k = -K^y y_k$. If C is full column rank, then we have full information (state feedback), but we here consider the general case where C has full row rank (independent measurements), but not full column rank.

Let $y = (y_0, u)$ and as before $u = (u_0, u_1, \ldots, u_{N-1})$. The disturbance $d = x_0$. The open loop model is now

$$y = \begin{bmatrix} y_0 \\ u \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ I \end{bmatrix}}_{G^y} u + \underbrace{\begin{bmatrix} C \\ 0 \end{bmatrix}}_{G_d^y} d, \tag{15}$$

and the sensitivity matrix F is

$$F = -(G^y J_{uu}^{-1} J_{ud} - G_d^y) = \begin{bmatrix} C \\ -J_{uu}^{-1} J_{ud} \end{bmatrix} \tag{16}$$

Since we now have that $n_y = n_{\tilde{y}} + n_u < n_u + n_d$, where $n_{\tilde{y}}$ are the number of measurements from the plant, $n_{\tilde{y}} < n_d$, we cannot simply set $HF = 0$, but we need to solve (6).

Let us analyze this problem. For $G^{y\mathrm{T}} = [0\ I]$, $HG^y = J_{uu}^{1/2}$ is equivalent to $H_2 = J_{uu}^{1/2}$, where

$$H_{n_c \times (n_{\tilde{y}} + n_u)} = \begin{bmatrix} H_{1_{n_c \times n_{\tilde{y}}}} & H_{2_{n_c \times n_u}} \end{bmatrix} \tag{17}$$

With this partitioning we get that

$$H\tilde{F} = H[FW_d\ W_{n^y}] = [HFW_d\ HW_{n^y}], \tag{18}$$

and for $W_{n^y} = 0$, i.e. the noise-free case,

$$H\tilde{F} = [HFW_d\ 0]. \tag{19}$$

We want to minimize the Frobenius-norm of this matrix and we have that

$$\|[HFW_d\ 0]\|_F = \|HFW_d\|_F + \|0\|_F \tag{20}$$

Assume without loss of generality that $W_d = I$, and let $\tilde{J} = -J_{uu}^{-1} J_{ud}$. With $F^{\mathrm{T}} = [C^{\mathrm{T}}\ J^{\mathrm{T}}]$ we have that

$$HF = H_1 C + H_2 \tilde{J}\Big|_{H_2 = J_{uu}^{1/2}} = H_1 C - J_{uu}^{-1/2} J_{ud} \tag{21}$$

We want to minimize $\|H_1 C - J_{uu}^{-1/2} J_{ud}\|$, hence we look for a H_1 such that

$$H_1 C = J_{uu}^{-1/2} J_{ud}. \tag{22}$$

Using the pseudo-inverse, we find that

$$H_1 = J_{uu}^{-1/2} J_{ud} C^\dagger, \tag{23}$$

and we get that the optimal H is

$$H = [J_{uu}^{-1/2} J_{ud} C^\dagger\ \ J_{uu}^{1/2}]. \tag{24}$$

In the final implementation we can decouple the invariants in the inputs by

$$\tilde{H} = J_{uu}^{-1/2} H = [-J_{uu}^{-1} J_{ud} C^\dagger\ \ I]. \tag{25}$$

This means that the open-loop optimal "output feedback" is

$$u_k = -\underbrace{J_{uu}^{-1} J_{ud}}_{K^{\text{state feedback}}} C^\dagger y_k = -K^y y_k, \tag{26}$$

that is, for an optimal state feedback K, the optimal "output feedback" is $KC^\dagger$.

This means that for this case we have

$$\begin{aligned}&\text{"Invariant" 1: } u_0 = K_0 C^\dagger y_0\\&\text{"Invariant" 2: } u_1 = K_1 C^\dagger y_0\\&\quad\vdots\\&\text{"Invariant" } N\text{: } u_{N-1} = K_{N-1} C^\dagger y_0\end{aligned} \tag{27}$$

We have called these variable combinations "invariants" in quotation marks because they are not invariant to the solution of the original problem, but rather the variable combinations that minimize the (open-loop) loss. Indeed, the non-negative loss is

$$\begin{aligned}\|HF\| &= \|J_{uu}^{-1/2} J_{ud} C^\dagger C - J_{uu}^{-1/2} J_{ud}\|\\&= \|J_{uu}^{-1/2} J_{ud}(C^\dagger C - I)\|\\&\le \|J_{uu}^{-1/2} J_{ud}\| \|C^\dagger C - I\|\end{aligned} \tag{28}$$

For output feedback we have in the least squares sense

$$u_1 = \underbrace{K_1 C^\dagger C (A + BK_0 C^\dagger C)^{-1} C^\dagger}_{K_1} y_1. \tag{29}$$

Unfortunately, in general $K_1 \neq K_0$ and hence the open loop solution (27) cannot be implemented as a constant feedback $u_k = K_0 C^\dagger y_k$, as was the case for state feedback, see (14).

III. MAIN ALGORITHM

We now propose an algorithm for finding output feedback controllers. This is a two-step procedure where we first find initial values using Theorem 2. These initial values correspond to a controller that in the open-loop sense is closest to the optimal state feedback LQ controller. Thereafter we improve this controller by solving a closed-loop optimization problem where the controller parameters are the degrees of freedom.

In the previous section we showed that if $y = (Cx_0, u_0, \ldots, u_{N-1})$ Theorem 1 gives $u_0 = -K^y y_0 = K^{\text{state feedback}} C^\dagger y_0$. The algorithm presented here is more

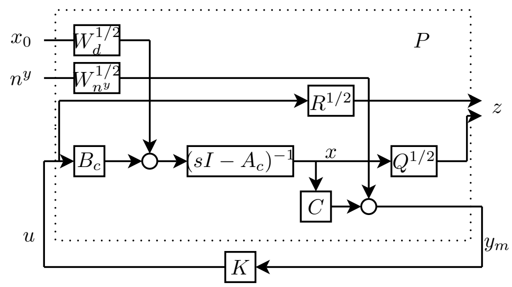

Fig. 2. Interconnection structure for closed-loop optimization of K.

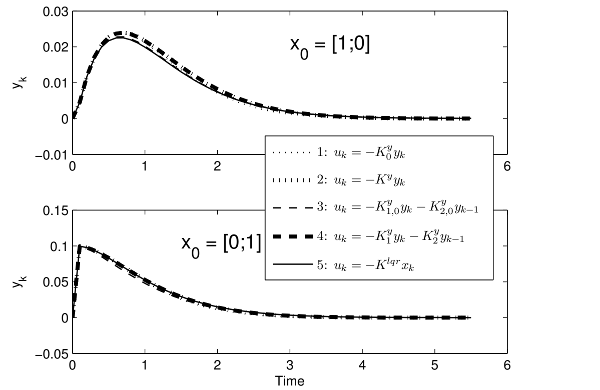

Fig. 3. Simulation results for disturbances in initial conditions, example 1.

general in the sense that it handles "measurements" such as $y = (y_0, y_1, \ldots, y_M, u_0, \ldots, u_{N-1})$. (In the latter case a casual controller is $u_M = -K^y[y_0^{\mathrm{T}} \ \ldots \ y_M^{\mathrm{T}}]^{\mathrm{T}}$.)

Algorithm 1 Low-order controller synthesis

1: Define a finite-horizon quadratic objective $J(u, x) = x_N^{\mathrm{T}} P x_N + \sum_{i=0}^{N-1} x_i^{\mathrm{T}} Q x_i u_i^{\mathrm{T}} R u_i + 2x_i^{\mathrm{T}} N u_i$.
2: Calculate J_{uu} and J_{ud} as in (8), (9).
3: Define candidate variables $y = G^y u + G_d^y x_0$, $u = (u_0, u_1, \ldots, u_{N-1})$.
4: Decide weights W_d and W_n^y (Default: $W_d = I, W_n^y = 0$)
5: Find $\tilde{H}$ by solving the convex optimization problem (6)
6: Optional: Improve control by closed-loop optimization. (Section III-A.)

A. Relationship between LQ-control and $\mathcal{H}_2$ optimal control

It is well-known that the LQG problem may be cast into the $\mathcal{H}_2$ framework and that a class of $\mathcal{H}_2$ optimal controllers may be implemented in an LQG-scheme with a Kalman estimator and a constant feedback gain from the estimated states [11]. In this paper we propose to improve the solution from the open-loop control by minimizing the $\mathcal{H}_2$ norm

$$\min_K \|F_l(P, K)\|_2. \tag{30}$$

In this context $\min_K$ means minimizing over the *parameters* in K. The lower-fractional transform $F_l(P, K) = P_{11} + P_{12}(I - P_{22}K)^{-1}P_{21}$ for a $P = \left[\begin{smallmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{smallmatrix}\right]$ [12]. The interconnection structure we use for P is shown in figure 2.

The last row of Algorithm 1 consists of solving (30) with initial values as $\tilde{H}$ on step 5 in algorithm 1, which is the solution to (6).

IV. EXAMPLES

In this section two examples will be considered. First we discuss P-control of a second order plant, then MIMO-PID control of a model of a distillation column.

Example 1: (P-control of second order plant) Consider the plant $g(s) = \frac{2}{s^2+3s+2}$. The plant is sampled with $T_s = 0.1$ to get

$$\begin{aligned} x_{k+1} &= \begin{bmatrix} 0.7326 & -0.1722 \\ 0.0861 & 0.9909 \end{bmatrix} x_k + \begin{bmatrix} 0.1722 \\ 0.0091 \end{bmatrix} u_k \\ y_k &= \begin{bmatrix} 0 & 1 \end{bmatrix} x_k \end{aligned} \tag{31}$$

The objective is to derive two LQ-optimal controllers for this process, one P-controller on the form $u_k = -K^y y_k$ and a PD-controller $u_k = -(K_1^y y_k + K_2^y y_{k-1})$.

In the synthesis of the controllers we use algorithm 1. The open loop objective to be minimized is $\tilde{J}(u, x) = \sum_{i=0}^{\infty} x_i^{\mathrm{T}} Q x_i + u_i^{\mathrm{T}} R u_i$ with $Q = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $R = 1$. The infinite horizon objective can be approximated by the following objective:

$$J(u, x) = x_N^{\mathrm{T}} P x_N + \sum_{i=0}^{N-1} x_i^{\mathrm{T}} Q x_i + u_i^{\mathrm{T}} R u_i, \tag{32}$$

with $P = \left[\begin{smallmatrix} 0.8333 & 2.4917 \\ 2.4917 & 9.6667 \end{smallmatrix}\right]$ and $N = 10$. (P is a solution to the discrete Lyapunov equation $P = A^{\mathrm{T}} P A + Q$.) The objective is now on the form of step 1 in the algorithm.

P-control: For the P-controller, the variables to combine are $y^1 = (y_0, u_0, \ldots, u_{N-1})$. The matrices J_{uu} and J_{ud} are the same as those reported in equations (8) and (9). Since we do not consider noise $W_d = I$ and $W_{n^y} = 0$. We can now find $\tilde{H}$ either by solving the convex problem in (6), or we can simply use the explicit formula in (23), i.e. $H_1 = J_{uu}^{-1} J_{ud} C^{\dagger}$. As shown in section II-B.2 (see equation 27) we now get $N = 10$ "invariants" to the solution to the original optimization problem in (32). The first one of these invariants is reported as controller 1 in table I. We observe that $\|H\tilde{F}\| > 0$, which is expected from Theorem 1, as $n_y < n_u + n_d$ in this case ($n_y = 1 + 10, n_u = 10, n_d = 2$).

Using this P-controller ($u_k = -0.404 y_k$) as the initial estimate, the $\mathcal{H}_2$-optimal closed-loop controller K in Figure 2 is obtained numerically. Note from row 2 in Table I that the $\mathcal{H}_2$-norm is only reduced slightly (from 0.2993 to 0.2981), although K changes from -0.404 to -0.313.

PD-control: For the synthesis of a PD controller we again use algorithm 1. The variables to combine are now $y^2 = (y_0, y_1, u_0, \ldots, u_{N-1})$. The objective function and the matrices J_{uu} and J_{ud} remain the same. The open-loop model

TABLE I
CONTROLLERS FOR EXAMPLE 1.

Controller	$\|H\tilde{F}\|_F$	$\|F_l(P,K)\|_2$	
1: $u_k = -0.404 y_k$	0.390	0.2993	First invariant using Theorem 2 for $y^1 = (y_0, u_0, \ldots, u_{N-1})$
2: $u_k = -0.313 y_k$	–	0.2981	Closed-loop optimal P-controller
3: $u_k = -(1.49 y_k - 1.11 y_{k-1})$	0	0.3176	Second invariant using Theorem 2 for $y^2 = (y_0, y_1, u_0, \ldots, u_{N-1})$
4: $u_k = -(0.416 y_k - 0.109 y_{k-1})$	–	0.2979	Closed-loop optimal controller PD-controller
5: $u_k = -[0.131\ 0.396] x_k$	0	0.2972	LQR

$y = G^y u + G_d^y x_0$ is now:

$$y^2 = \begin{bmatrix} y_0 \\ y_1 \\ u \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ CB & 0 \\ I & 0 \\ 0 & I \end{bmatrix} u + \begin{bmatrix} C \\ CA \\ 0 \end{bmatrix} x_0. \qquad (33)$$

For this particular variable combination (23) cannot be used, as the variables occur on different instances in time. We therefore solve the optimization problem (6) using cvx, as shown in remark 2. The solution is again on the form of (27), for which the second invariant is reported as controller 3 in table I. The solution (all the invariants) gives $\|H\tilde{F}\| = 0$, which is expected from Theorem 1, as $n_y = n_u + n_d$ and no noise is present. Further numerical optimization reduces the $\mathcal{H}_2$ norm from 0.3176 to 0.2979.

It can be verified that the variable combination is indeed optimal after one step with the following calculations:

$$\begin{aligned} u_0 = -Kx_0, \quad u_1 = -K_1^y y_1 - K_2^y y_0 \\ \Rightarrow u_1 = -\underbrace{\left(K_1^y C + K_2^y C(A-BK)^{-1}\right)}_{=K} x_1, \end{aligned} \qquad (34)$$

where K is the LQR controller. For implementation some sub-optimality must be expected since we are not starting the control with LQR, rather we use the PD controller at all time instances.

Simulations: From the closed-loop norms reported in table I the controllers are expected to perform similarly in closed loop. This is confirmed in the closed loop simulations of disturbances in initial states, see figure 3.

Example 2: (Linear dynamic model of distillation column.) In this example we consider MIMO-PI and -PID control of "column A" in [13]. The model is used as an example for offset-free control in [14]. The model is based on the following assumptions:

- binary separation,
- 41 stages, including reboiler and total condenser,
- each stage is at equilibrium, with constant relative volatility $\alpha = 1.5$,
- linearized liquid flow dynamics,
- negligible vapor holdup,
- constant pressure.

The feed enters on stage 21. $u = \left[\begin{smallmatrix} L \\ V \end{smallmatrix}\right]$ and $y = \left[\begin{smallmatrix} x_D \\ x_B \end{smallmatrix}\right]$.

We here consider the LV-configuration, where D and B are used to control the levels. With level controllers implemented (P-control with $K_c = 10$) the rest of the column is stable.

Balanced reduction is used to reduce the number of states to 16. Then integrated outputs are added to the model, resulting in a model with 18 states. If we let the outputs of the model be P, I, and D, we get a model with the following structure:

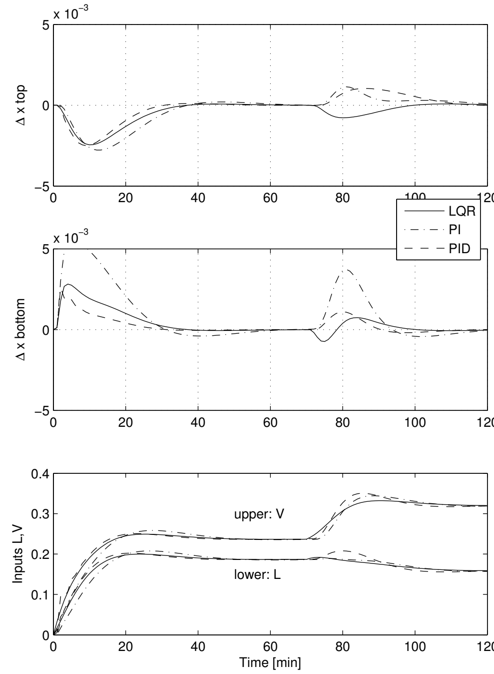

Fig. 4. Simulation results for example 2. At $t = 0$ a step-change of 0.1 in F occurs, and at $t = 70$ z_F is changed from 0.5 to 0.6.

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{\sigma} \end{bmatrix} &= \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} \begin{bmatrix} x \\ \sigma \end{bmatrix} + \begin{bmatrix} b \\ d \end{bmatrix} u \\ \begin{bmatrix} y^P \\ y^I \\ y^D \end{bmatrix} &= \begin{bmatrix} c & 0 \\ 0 & I \\ ca & 0 \end{bmatrix} \begin{bmatrix} x \\ \sigma \end{bmatrix} + \begin{bmatrix} d \\ 0 \\ cb \end{bmatrix} u \end{aligned}. \qquad (35)$$

This model is sampled with $T_s = 1$ to get a discrete time model. Again we set up an infinite time objective function, with $Q = C^{\mathrm{T}} \left[\begin{smallmatrix} 0 & \\ & I \\ & & 0 \end{smallmatrix}\right] C$, and $R = 0.1 \cdot I$, and for intermediate calculations we approximate this by a finite horizon objective with $N = 150$ and $P = Q$.

We now look for controllers on the form

$$u_k = -\left(K^P y_k^P + K^I y_k^I + K_D y_k^D\right) \qquad (36)$$

and we assume measurements of the compositions with a sample time of 1 minute is available.

Table II shows "first-move" PI (= first move invariant realized as feedback), closed loop PI and PID controllers,

TABLE II

CONTROLLERS FOR EXAMPLE 2.

Description	Control equation	$\|H\tilde{F}\|_F$	$\|F_l(P,K)\|$
"First-move" PI	$u_k = -\left(\begin{bmatrix} 5.316 & 0.25664 \\ -3.1953 & -3.3371 \end{bmatrix} y_k^P + \begin{bmatrix} 2.6897 & -0.5975 \\ -0.13498 & -2.5939 \end{bmatrix} y_k^I\right)$	4.28	3.99
Closed-loop optimal PI	$u_k = -\left(\begin{bmatrix} 16.0156 & -5.17125 \\ 0.541199 & -9.57775 \end{bmatrix} y_k^P + \begin{bmatrix} 2.7148 & -0.715 \\ 0.33949 & -2.7672 \end{bmatrix} y_k^I\right)$	–	3.65
"First-move" PID	$u_k = -\left(\begin{bmatrix} 9.9305 & -0.96741 \\ -5.2025 & -3.5369 \end{bmatrix} y_k^P + \begin{bmatrix} 2.6891 & -0.62581 \\ -0.13969 & -2.6454 \end{bmatrix} y_k^I \dots\right.$ $\left.\dots + \begin{bmatrix} 1.0724 & -0.22799 \\ -0.53974 & -0.43514 \end{bmatrix} y_k^D\right)$	3.44	3.78
Closed-loop optimal PID	$u_k = -\left(\begin{bmatrix} 17.5043 & -8.22394 \\ 3.48592 & -17.7333 \end{bmatrix} y_k^P + \begin{bmatrix} 2.743 & -1.3167 \\ 0.15148 & -4.3547 \end{bmatrix} y_k^I \dots\right.$ $\left.\dots + \begin{bmatrix} -1.78427 & -6.07614 \\ -9.79446 & -13.3285 \end{bmatrix} y_k^D\right)$	–	3.63
LQR	$u_k = -\begin{bmatrix} -0.0022 & 0.0002 & -0.0004 & -0.0007 & 0.0016 & -0.0097 \\ 0.0008 & 0.0015 & -0.0016 & -0.0037 & 0.0079 & -0.0074 \end{bmatrix} x_k(1:6)\dots$ $\dots - \begin{bmatrix} -0.0036 & 0.0048 & 0.0116 & -0.0011 & -0.0213 & 0.0305 \\ -0.0066 & 0.0262 & 0.0610 & 0.0044 & 0.0093 & -0.0148 \end{bmatrix} x_k(7:12)\dots$ $\dots - \begin{bmatrix} 0.0149 & 0.0521 & 0.1349 & 0.1034 & 2.6897 & -0.5975 \\ 0.0233 & -0.0372 & -0.1607 & 0.0895 & -0.1350 & -2.5939 \end{bmatrix} x_k(13:18)$	–	3.61

TABLE III

ITERATION COUNT USING FMINUNC (MATLAB© R2008A) WITH DIFFERENT INITIALIZATIONS

	Algorithm 1	$K^0 = 0$	SIMC-tuned PI controllers
PI	44	–	71
PID	91	–	123

and the LQR controller for reference. In addition to the initialziation proposed in this paper we tried to initialize the numerical search with $K^0 = 0$ and two SIMC-tuned [15] PI controllers with $\tau_c = 10$ minutes, leading to $K^0_{\text{SIMC}} = \left[\begin{smallmatrix} 14.63 & 0 & 0.37 & 0 & 0 & 0 \\ 0 & -10.91 & 0 & -0.27 & 0 & 0 \end{smallmatrix}\right]$. As reported in table III did $K^0 = 0$ not converge, whereas intializing with two SIMC-tuned PI controllers converged in both cases (both for PI and PID design), though with some more iterations than the method proposed in this paper.

Figure 4 shows simulation results where we at $t = 0$ introduce a step in the feed rate and at $t = 70$ a step in the feed composition. 'PI' and 'PID' refers to the closed-loop optimal controllers. As one observes is the MIMO-PID controller quite close in performance to the LQR controller.

V. CONCLUSIONS

In this paper we have discussed synthesis of $\mathcal{H}_2$-optimal static output feedback, and in particular the MIMO-PID. We have shown that initial conditions for closed loop optimization can be found by solving a convex program, and that the resulting closed loop optimization problem converges for some interesting cases.

VI. ACKNOWLEDGMENTS

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