μ-Interaction measure for unstable systems

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Abstract: In this paper, we present a method for system stabilisation through independent designs of the decentralised controller. The proposed method extends the practical applicability of the conventional μ interaction measure (μ-IM) to unstable systems. The decentralised controller is designed based on a block diagonal approximation that is different from the block diagonal elements, but has the same number of unstable poles as the system. By expressing the μ-IM in terms of the transfer matrix between the disturbances and inputs, we show that the block diagonal approximation can be suboptimally selected by minimising the scaled $\mathcal{L}_\infty$ distance between the system and the approximation. We present a numerical method for choosing the block diagonal approximation and a simple method for designing the decentralised controller based on the approximation.

Keywords: decentralised control; large-scale systems; μ-interaction measure; structured singular value.

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1 Introduction

Despite the performance advantages of full multivariable controller, decentralised control is almost the exclusive choice for control of large-scale systems. For power systems, decentralised control is necessitated due to physical distances between different stations and the large cost of establishing a communication network. In process industries, the use of decentralised controllers is motivated by the ease of tuning and design. Decentralised control is also the preferred choice by nature, for example, the secretion of different enzymes and hormones in the human body is controlled by different sections of the brain.

Over the years, three different approaches have evolved for decentralised controller design:

1. *Simultaneous design using parametric search methods*: the decentralised controller is chosen to have a fixed structure (e.g. PID controller) with unknown parameters. The optimal value of these parameters is found by minimising the appropriate norm of the closed loop system. Though useful, this approach results in optimisation problems that are not usually convex and can be highly complicated even for simple systems (Bao et al., 1999).

2. *Sequential design*: the controllers are designed sequentially using a lexicographical ordering of the individual controllers. The lowest level controller is designed first and the loop is closed. The next controller is designed based on the partially closed loop system. The resulting performance strongly depends on the ordering of the loops and often a trial and error approach is required to obtain acceptable performance (Hovd and Skogestad, 1994; Mayne, 1973).

3. *Independent design*: the individual controllers are designed independently of each other based on a block diagonal approximation that is usually taken as the block diagonal elements of the system. Then, the decentralised controller design problem reduces to design a number of small dimensional full multivariable controllers. When the interactions are small, such a controller also stabilises the closed loop system with minimal loss of performance in comparison to the design basis (Hovd and Skogestad, 1993; Skogestad and Morari, 1989). This approach always results in suboptimal performance because the tuning of other controllers is neglected.

In this paper, we focus on the independent design approach. Although suboptimal, the controller design is much simpler as compared to other techniques.

Grosdidier and Morari (1986) proposed the use of μ-Interaction Measure (μ-IM) to assess the feasibility of system stabilisation through independent designs of individual loops. This approach yields sufficient conditions to ensure that the decentralised controller that stabilises the block diagonal part of the system also stabilises the system itself. The problem of decentralised controller synthesis through independent designs has also been studied by Limbeer (1982) and Ohta et al. (1986), who used the concepts of generalised block diagonal dominance and quasi-block diagonal dominance, respectively. The use of μ-IM is less conservative than these approaches because the controller structure is taken into account. A connection between these methods based on dominance and μ-IM is established by Kariwala et al. (2003a,b).

The conventional μ-IM requires that the system and its block diagonal part have the same Right Half Plane (RHP) poles. Grosdidier and Morari (1986) pointed out that this condition is not satisfied by most of the systems encountered in practice, limiting the applicability of μ-IM to open loop stable systems. Samyudia et al. (1995) have criticised the μ-IM for this limitation and have instead proposed a method based on ν-gap metric (Vinnicombe, 1999). In this paper, we present a modified μ-IM that easily handles unstable systems. The decentralised controller is designed based on a block diagonal approximation that is different from the block diagonal elements, but has the same number of unstable poles as the system.

Clearly, the number of block diagonal systems with the required number of unstable poles is infinite and the success of the modified μ-IM approach strongly depends on the choice of an appropriate approximation. We express the μ-IM in terms of the closed loop transfer matrix between disturbances and system input (or controller output). This alternate representation shows that the block diagonal approximation can be reasonably selected by minimising the scaled $\mathcal{L}_\infty$ distance between the system and the approximation. The problem of finding a structured approximation of a full multivariate system has earlier been considered by Li and Zhou (2002), but no numerical methods for solving the approximation problem are provided. In this paper, we present a numerical approach, where the approximation problem is first solved at a set of chosen frequencies followed by a parametric identification method.

Similar to the conventional μ-IM method, the stabilising decentralised controller can be synthesised using a loop shaping approach based on the block diagonal approximation. An advantage of alternate representation of μ-IM used here is that controller design can be much simplified using the results on input performance limitations (Glover, 1986; Kariwala et al., 2005). Although the focus of this paper is on finding stabilising decentralised controllers, we also show that the stabilising controller inherently minimises an upper bound on the input requirement for stabilisation.

The organisation of the remaining discussion in this paper is as follows: the available results of μ-IM are reviewed and their limitation is pointed out in § 2; the alternate representation of μ-IM is presented and upper bounds on closed loop input performance are derived in § 3; in § 4, we consider the problem of selecting the optimal block diagonal approximation; the simplified controller design method is presented in § 5; in § 6, a numerical example is presented to demonstrate the usefulness of proposed approach followed by conclusions in § 7.

Notation: before proceeding with the main discussion, we standardise the notation. We represent matrices by boldface uppercase letters and vectors by boldface lowercase letters. Given a matrix $\mathbf{A} \in \mathbb{C}^{m\times n}$, $\mathbf{A}^*$ is its conjugate transpose. The maximum singular value is represented as $\bar{\sigma}(\mathbf{A})$ and the Euclidian condition number as $\kappa(\mathbf{A})$. The symbol $\succeq$ denotes partial ordering, where $\mathbf{A} \succeq \mathbf{B}$ implies that $\mathbf{A} - \mathbf{B}$ is positive semi-definite. Let the set $\Delta \in \mathbb{C}^{n \times m}$ be defined as

$$\Delta = \{\mathrm{diag}(\Delta_i) : \Delta_i \in \mathbb{C}^{n_i \times m_i}, \bar{\sigma}(\Delta) \leq 1\}$$

The structured singular value of $\mathbf{A} \in \mathbb{C}^{m \times n}$ is given as (Doyle et al., 1982),

$$\mu_\Delta(\mathbf{A}) = \frac{1}{\min\{\bar{\sigma}(\tilde{\Delta}) : \tilde{\Delta} \in \Delta, \det(\mathbf{I} - \mathbf{A}\tilde{\Delta}) = 0\}}$$

unless no $\tilde{\Delta} \in \Delta$ makes $(\mathbf{I} - \mathbf{A}\tilde{\Delta})$ singular, in which case $\mu_\Delta(\mathbf{A}) = 0$. Let $\mathcal{D}_\mathrm{L}$, $\mathcal{D}_\mathrm{R}$ be set of matrices that commute with all elements of Δ or $\mathbf{D}_\mathrm{L}\tilde{\Delta} = \tilde{\Delta}\mathbf{D}_\mathrm{R}$ for all $\tilde{\Delta} \in \Delta, \mathbf{D}_\mathrm{L} \in \mathcal{D}_\mathrm{L}, \mathbf{D}_\mathrm{R} \in \mathcal{D}_\mathrm{R}$. Then,

$$\mu_\Delta(\mathbf{A}) \leq \inf_{\mathbf{D}_\mathrm{L} \in \mathcal{D}_\mathrm{L}, \mathbf{D}_\mathrm{R} \in \mathcal{D}_\mathrm{R}} \bar{\sigma}(\mathbf{D}_\mathrm{L}\mathbf{A}\mathbf{D}_\mathrm{R}^{-1}) \tag{1}$$

In this paper, we denote the upper bound given by (1) as $\bar{\mu}_\Delta(.)$.

The set of all rational stable systems is denoted as $\mathcal{RH}_\infty$. Let $\mathbf{G}(s) = \mathbf{G}_1(s) + \mathbf{G}_2(s)$ such that $\mathbf{G}_1(s) \in \mathcal{RH}_\infty^\perp$ and $\mathbf{G}_2(s) \in \mathcal{RH}_\infty$. Then $\mathbf{G}_1(s)$ is the unstable projection of $\mathbf{G}(s)$ represented as $\mathcal{U}(\mathbf{G}(s))$, where $\mathcal{U}(\mathbf{G}(s)) \in \mathcal{RH}_\infty^\perp$. The $\mathcal{H}_\infty$ or $\mathcal{L}_\infty$ norm of the transfer matrix $\mathbf{G}(s)$ is defined as (Zhou and Doyle, 1998)

$$\|\mathbf{G}(s)\|_\infty = \sup_{\mathrm{Re}(s)>0} \bar{\sigma}(\mathbf{G}(s)) = \sup_{\omega \in \mathbb{R}} \bar{\sigma}(\mathbf{G}(j\omega))$$

We represent the mininum Hankel singular value of $\mathbf{G}(s) \in \mathcal{RH}_\infty$ as $\underline{\sigma}_H(\mathbf{G}(s))$ (Glover, 1984; Zhou and Doyle, 1998).

2 μ-Interaction measure

In this section, we briefly review the available results on μ-IM (Grosdidier and Morari, 1986), point to their limitation and suggest a modification to overcome the same. Throughout this paper, we assume that the system does not contain any decentralised fixed modes (Wang and Davison, 1973). The absence of decentralised fixed modes is both necessary and sufficient for existence of a decentralised stabilising controller but only necessary, when individual loops of the decentralised controller are designed independently of each other.

With reference to Figure 1, let the system $\mathbf{G}(s)$ be partitioned as $\mathbf{G}(s) = \mathbf{G}_\mathrm{bd}(s) + \mathbf{G}_\mathrm{I}(s)$ such that

- $\mathbf{G}_\mathrm{bd}(s)$ contains the block-diagonal elements of $\mathbf{G}(s)$
- $\mathbf{G}_\mathrm{bd}(s)$ and $\mathbf{G}(s)$ have the same number of RHP poles.

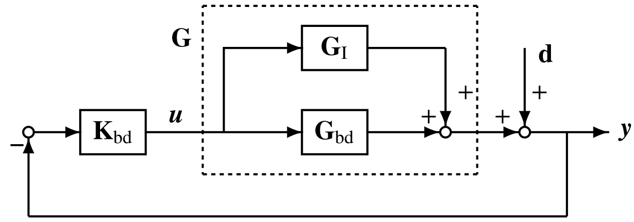
Define the transfer matrices $\mathbf{E}(s)$ and $\mathbf{T}_\mathrm{bd}(s)$ as,

$$\mathbf{T}_\mathrm{bd}(s) = \mathbf{G}_\mathrm{bd}\mathbf{K}_\mathrm{bd}(s)\,(\mathbf{I} + \mathbf{G}_\mathrm{bd}\mathbf{K}_\mathrm{bd}(s))^{-1} \tag{2}$$

$$\mathbf{E}(s) = (\mathbf{G}(s) - \mathbf{G}_\mathrm{bd}(s))\,\mathbf{G}_\mathrm{bd}(s)^{-1} \tag{3}$$

where $\mathbf{K}_\mathrm{bd}(s)$ is the block diagonal controller. $\mathbf{T}_\mathrm{bd}(s)$ can be interpreted as the complementary sensitivity function, if $\mathbf{G}_\mathrm{I}(s)$ were zero and $\mathbf{E}(s)$ as the multiplicative uncertainty in $\mathbf{G}_\mathrm{bd}(s)$. Let $\mathbf{K}_\mathrm{bd}(s)$ be designed such that $\mathbf{T}_\mathrm{bd}(s)$ is stable. The central question remains: Does $\mathbf{K}_\mathrm{bd}(s)$ also stabilise $\mathbf{G}(s)$? This issue has been addressed by Grosdidier and Morari (1986), who proposed the use of μ-IM for this purpose.

Figure 1 Partitioning of $\mathbf{G}(s)$ for μ-IM



Lemma 1: *Assume that $\boldsymbol{G}(s)$ and $\boldsymbol{G}_{bd}(s)$ have same number of RHP poles and $\boldsymbol{T}_{bd}(s)$ is stable. Then $\boldsymbol{T}(s) = \boldsymbol{G}\boldsymbol{K}_{bd}(s)\,(\boldsymbol{I} + \boldsymbol{G}\boldsymbol{K}_{bd}(s))^{-1}$ is stable if and only if (iff) the following conditions hold (Grosdidier and Morari, 1986)*

$$det(\boldsymbol{I} + \boldsymbol{E}\boldsymbol{T}_{bd}(s)) \neq 0 \tag{4}$$

$$N(0, det(\boldsymbol{I} + \boldsymbol{E}\boldsymbol{T}_{bd}(s))) = 0 \tag{5}$$

where $N(\alpha, .)$ denotes the winding number (Vinnicombe, 1999) or the number of clockwise encirclements of the point $(\alpha, 0)$ by the image of Nyquist D contour.

Lemma 1 was originally proven by Grosdidier and Morari (1986), except the requirement that (4) must hold. This is a minor technical requirement to ensure that the image of $\det(\mathbf{I} + \mathbf{T}_{bd}\mathbf{E}(s))$ does not pass through the origin of the complex plane. Lemma 1 forms the basis for a more important result, as presented next.

Theorem 1: *Let $\boldsymbol{G}(s)$ and $\boldsymbol{G}_{bd}(s)$ have same number of unstable poles. If $\boldsymbol{K}_{bd}(s)$ stabilises $\boldsymbol{G}_{bd}(s)$, then $\boldsymbol{K}_{bd}(s)$ also stabilises $\boldsymbol{G}(s)$, if*

$$\bar{\sigma}\,(\boldsymbol{T}_{bd}(j\omega)) < \mu_{\Delta}^{-1}\,(\boldsymbol{E}(j\omega)) \qquad \forall\omega \in \mathbb{R} \tag{6}$$

where Δ has the same block structure as $\boldsymbol{G}_{bd}(s)$ and, $\boldsymbol{T}_{bd}(s)$ and $\boldsymbol{E}(s)$ are defined by (2) *and* (3)*, respectively.*

Proof: The sufficiency of (6) for closed loop stability is proven by contradiction. Assume that (6) is satisfied, but $N(0, \det(\mathbf{I} + \mathbf{E}\mathbf{T}_{bd}(s))) > 0$. Thus the image of $\det(\mathbf{I} + \mathbf{E}\mathbf{T}_{bd}(s))$ intersects the negative real axis of complex plane at a certain frequency, which we denote as ω_o. Now, let us consider the variation of $\det(\mathbf{I} + \beta\mathbf{E}\mathbf{T}_{bd}(j\omega_o))$ with the scalar β. When $\beta = 0$, $\det(\mathbf{I} + \beta\mathbf{E}\mathbf{T}_{bd}(j\omega_o)) = \det(\mathbf{I}) > 0$. On the other hand, when $\beta = 1$, $\det(\mathbf{I}+\beta\mathbf{E}\mathbf{T}_{bd}(j\omega_o)) < 0$ due to intersection with negative real axis. Then due to continuity of the determinant function, there exists a β, $|\beta| < 1$ such that

$$\det(\mathbf{I} + \beta\mathbf{E}\mathbf{T}_{bd}(j\omega_o)) = 0$$

Similarly, let there exists a frequency ω_1 such that

$$\det(\mathbf{I} + \mathbf{E}\mathbf{T}_{bd}(j\omega_1)) = 0$$

Combining these two conditions, we notice that $\mathbf{T}(s)$ is unstable iff there exists a β, $|\beta| \leq 1$ such that $\det(\mathbf{I} + \beta\mathbf{E}\mathbf{T}_{bd}(j\omega)) = 0$ for some $\omega \in \mathbb{R}$. It follows from the definition of the structured singular value that the smallest $\beta\mathbf{T}_{bd}(j\omega)$ that destabilises $\mathbf{E}(j\omega)$ is given as $\bar{\sigma}^{-1}(\beta\mathbf{T}_{bd}(j\omega))$. When (6) holds, a β with $|\beta| < 1$ such that $\det(\mathbf{I} + \beta\mathbf{E}\mathbf{T}_{bd}(j\omega)) = 0$ for some $\omega \in \mathbb{R}$ does not exist and the closed loop system is stable. ■

Theorem 1 was proven by Grosdidier and Morari (1986) under the requirement that the unstable poles of $\mathbf{G}(s)$ and $\mathbf{G}_{bd}(s)$ be identical. It is clear from Lemma 1 and the proof of

Theorem 1 that the number of unstable poles of $\mathbf{G}(s)$ and $\mathbf{G}_{\mathrm{bd}}(s)$ being equal suffices. In either case, design of $\mathbf{K}_{\mathrm{bd}}(s)$ solely based on $\mathbf{G}_{\mathrm{bd}}(s)$ is equivalent to designing individual loops or control susbsystems independently. The Equation (6) is known as the μ-IM. This powerful result allows the designer to impose restrictions on individual controllers, but still design the decentralised controller solely based on $\mathbf{G}_{\mathrm{bd}}(s)$ such that closed loop stability is ensured.

As pointed by Grosdidier and Morari (1986) that in practice, $\mathbf{G}(s)$ and $\mathbf{G}_{\mathrm{bd}}(s)$ as defined above has same number of RHP poles only for open loop stable systems limiting the applicability of μ-IM. It is noted that this limitation only arises as $\mathbf{G}_{\mathrm{bd}}(s)$ is chosen as the block diagonal elements of $\mathbf{G}(s)$ and is easily overcome by relaxing this requirement. The decentralised controller can be designed based on $\mathbf{G}_{\mathrm{bd}}(s)$ that is different from the block diagonal elements but has the same number of RHP poles as $\mathbf{G}(s)$. This point is further illustrated using the following example.

Example 1: *Consider the following system*

$$\boldsymbol{G}(s) = \frac{1}{(s-1)(s-2)} \begin{bmatrix} (s+0.5) & 0.5 \\ (9s-3) & (s+1) \end{bmatrix} \tag{7}$$

Since all the minors of order 1 have $(s-1)(s-2)$ *as the denominator and*

$$\begin{aligned} det(\boldsymbol{G}(s)) &= \frac{(s+0.5)(s+1) - 0.5(9s-3)}{(s-1)^2(s-2)^2} \\ &= \frac{s^2 - 3s + 2}{(s-1)^2(s-2)^2} = \frac{1}{(s-1)(s-2)} \end{aligned}$$

the system (7) *has two unstable poles at 1 and 2 (MacFarlane and Karcanias, 1976; Skogestad and Postlethwaite, 2005). Let* $\boldsymbol{G}_{bd}(s)$ *be chosen as the diagonal elements of* $\boldsymbol{G}(s)$. *In this case,*

$$det(\boldsymbol{G}_{bd}(s)) = \frac{(s+0.5)(s+1)}{(s-1)^2(s-2)^2}$$

Due to absence of pole-zero cancellation, $\boldsymbol{G}_{bd}(s)$ *has poles at the same locations as* $\boldsymbol{G}(s)$, *but repeated twice and the assumption of* μ*—IM that* $\boldsymbol{G}(s)$ *and* $\boldsymbol{G}_{bd}(s)$ *have same number of unstable poles, is violated. Consider that* $\boldsymbol{G}_{bd}(s)$ *is chosen as,*

$$\boldsymbol{G}_{bd}(s) = \begin{bmatrix} \frac{1}{(s-\alpha_1)} f_1(s) & 0 \\ 0 & \frac{1}{(s-\alpha_2)} f_2(s) \end{bmatrix}$$

where $\alpha_1, \alpha_2 > 0$ *and* $f_1(s)$, $f_2(s)$ *are arbitrary stable transfer matrices. With this choice, the assumption that* $\boldsymbol{G}_{bd}(s)$ *and* $\boldsymbol{G}(s)$ *have the same number of unstable poles is easily satisfied. Note that for an arbitrary choice of* $\alpha_1, \alpha_2 > 0$, *the diagonal blocks of* $\boldsymbol{G}_I(s)$ *are not necessarily zero. A similar approach can be used for partitioning any arbitrary system.*

Remark 1: *The approach for choosing* $\boldsymbol{G}_{bd}$, *as illustrated above, still holds when some of the RHP poles of the system do not appear in any of its block diagonal elements. It is pointed out, however, that in this case, it may be very difficult to design a block diagonal controller* $\boldsymbol{K}_{bd}$ *to satisfy the* μ*-IM condition, as the corresponding diagonal blocks will have large element-wise uncertainties associated with them (up to* 100%, *if the diagonal block is* $\mathbf{0}$*).*

Though the generalisation used in choosing $\mathbf{G}_{\mathrm{bd}}(s)$ extends the practical applicability of μ-IM to unstable systems, the generalisation introduces an additional degree of freedom.

Clearly, whether the μ-IM condition (6) is satisfied depends on the choice of $\mathbf{G}_{\text{bd}}(s)$, which is dealt with in subsequent sections.

3 Alternate representation of μ-IM

For a given $\mathbf{G}_{\text{bd}}(s)$, a loop shaping approach can be used to find $\mathbf{K}_{\text{bd}}(s)$ for closed loop stability. In the present case, $\mathbf{G}_{\text{bd}}(s)$ can also be treated as a free parameter with the requirement of having the same number of unstable poles as $\mathbf{G}(s)$.

The task of jointly finding the pair $(\mathbf{G}_{\text{bd}}(s), \mathbf{K}_{\text{bd}}(s))$ such that the closed loop system is stable, is very difficult. We note in (6), both $\bar{\sigma}(\mathbf{T}_{\text{bd}}(j\omega))$ and $\mu_{\Delta}(\mathbf{E}(j\omega))$ depend on $\mathbf{G}_{\text{bd}}(j\omega)$, but $\mathbf{E}(j\omega)$ is independent of the controller. Then, a convenient (and not optimal) approach is to find $\mathbf{G}_{\text{bd}}(s)$ such that $\mu_{\Delta}(\mathbf{E}(j\omega))$ is minimised and then design the decentralised controlled based on it to satisfy the μ-IM condition; however, $\mathbf{E}(s)$ is not an affine function of $\mathbf{G}_{\text{bd}}(s)$. We next show that this difficulty can be overcome by representing μ-IM alternately in terms of transfer matrix between the disturbances and the inputs.

Proposition 1: *Let $\boldsymbol{G}(s)$ be partitioned as $\boldsymbol{G}(s) = \boldsymbol{G}_{bd}(s) + \boldsymbol{G}_I(s)$ such that $\boldsymbol{G}_{bd}(s)$ and $\boldsymbol{G}(s)$ have the same number of RHP poles. Define $\boldsymbol{S}_{bd}(s) = (\boldsymbol{I} + \boldsymbol{G}_{bd}\boldsymbol{K}_{bd}(s))^{-1}$. Then $\boldsymbol{K}_{bd}(s)$ stabilising $\boldsymbol{G}_{bd}(s)$ also stabilises $\boldsymbol{G}(s)$ if*

$$\bar{\sigma}(\boldsymbol{K}_{bd}\boldsymbol{S}_{bd}(j\omega)) < \mu_{\boldsymbol{\Delta}}^{-1}(\boldsymbol{G}_I(j\omega)) \qquad \forall\omega \in \mathbb{R} \tag{8}$$

where Δ has the same structure as $\boldsymbol{G}_{bd}(s)$.

Proof: Note that

$$\det(\mathbf{I} + \mathbf{E}\mathbf{T}_{\text{bd}}(s)) = \det(\mathbf{I} + \mathbf{G}_{\text{I}}(s)\mathbf{K}_{\text{bd}}\mathbf{S}_{\text{bd}}(s))$$

Now the sufficiency of (8) is shown by using Lemma 1 and following the proof of Theorem 1. ■

Since the RHS of (8) is affine in $\mathbf{G}_{\text{bd}}(s)$, the block diagonal approximation can be suboptimally selected by minimising $\mu_{\Delta}(\mathbf{G}_{\text{I}}(j\omega)$. This approach is suboptimal as the LHS of (8) also depends on $\mathbf{G}_{\text{bd}}(s)$. For a particular choice of $\mathbf{G}_{\text{bd}}(s)$ that optimally minimises $\mu_{\Delta}(\mathbf{G}_{\text{I}}(j\omega))$, there may not exist any controller satisfying (8) and *vice-versa*. This issue is further discussed later in this paper.

Remark 2: *Compared to the necessary and sufficient conditions provided by Lemma 1, the conditions provided by Theorem 1 and Proposition 1 are sufficient only. It is possible that there exists a controller $\boldsymbol{K}_{bd}(s)$ that violates* (6) *or* (8)*, but renders a stable closed loop system, which shows the conservatism of μ-IM. In this case, however, there exists some other controller $\bar{\boldsymbol{K}}_{bd}(s)$ that also violates* (8) *with $\bar{\sigma}(\bar{\boldsymbol{K}}_{bd}(j\omega)(\boldsymbol{I} + \boldsymbol{G}_{bd}\bar{\boldsymbol{K}}_{bd}(j\omega))^{-1}) = \bar{\sigma}(\boldsymbol{K}_{bd}\boldsymbol{S}_{bd}(j\omega))$ for some $\omega \in \mathbb{R}$ and $\bar{\boldsymbol{K}}_{bd}(s)(\boldsymbol{I} + \boldsymbol{G}_{bd}\bar{\boldsymbol{K}}_{bd}(s))^{-1}$ is unstable. Thus, the strength of μ-IM is that when* (6) *or* (8) *hold, any decentralised controller that stabilises $\boldsymbol{G}_{bd}(s)$ also stabilises $\boldsymbol{G}(s)$.*

Remark 3: *We note that in practice, only the upper and lower bounds on μ are computable. Hence, to assess the feasibility of independent designs, one needs to verify*

$$\bar{\sigma}(\boldsymbol{K}_{bd}\boldsymbol{S}_{bd}(j\omega)) < \bar{\mu}_{\boldsymbol{\Delta}}^{-1}(\boldsymbol{G}_I(j\omega)) \qquad \forall\omega \tag{9}$$

where $\bar{\mu}$ represents an upper bound on μ calculated by the D-scaling method with the left and right hand sides scaling matrices being $\boldsymbol{D}_L(\omega) \in \mathcal{D}_L$, $\boldsymbol{D}_R(\omega) \in \mathcal{D}_R$, respectively. Here, the sets $\mathcal{D}_L$ and $\mathcal{D}_R$ are given as

$$\mathcal{D}_L = \{diag(d_i \cdot \boldsymbol{I}_{m_i}), d_i \in \mathbb{R}\}$$
$$\mathcal{D}_R = \{diag(d_j \cdot \boldsymbol{I}_{m_j}), d_j \in \mathbb{R}\} \tag{10}$$

where the dimensions of individual blocks of $\boldsymbol{G}_{bd}(s)$ is $m_i \times m_j$.

Proposition 1 provides a sufficient condition to assess whether $\mathbf{K}_{\rm bd}(s)$ designed for $\mathbf{G}_{\rm bd}(s)$, can stabilise the closed loop system; however, it provides no information regarding the closed loop performance. Grosdidier and Morari (1986) pointed out, satisfying μ-IM condition guarantees closed loop stability, but the performance can be arbitrarily poor. We gain some insight into this issue by deriving an upper bound on the closed loop input performance, when the μ-IM condition (8) is satisfied.

Proposition 2: *Assume that $\boldsymbol{G}(s)$ and $\boldsymbol{G}_{bd}(s)$ have the same number of RHP poles and* (9) *holds. Then,*

$$\bar{\sigma}(\boldsymbol{K}_{bd}\boldsymbol{S}(j\omega)) \leq \frac{\kappa(\boldsymbol{D}_L(\omega))}{\bar{\sigma}^{-1}(\boldsymbol{K}_{bd}\boldsymbol{S}_{bd}(j\omega)) - \bar{\mu}_{\boldsymbol{\Delta}}(\boldsymbol{G}_I(j\omega))} \forall \omega \in \mathbb{R} \tag{11}$$

where $\boldsymbol{\Delta}$ has the same structure as $\boldsymbol{G}_{bd}(s)$ and $\boldsymbol{D}_L(\omega) \in \mathcal{D}_L$, $\boldsymbol{D}_R(\omega) \in \mathcal{D}_R$ are chosen to minimise $\bar{\sigma}\left(\boldsymbol{D}_L(\omega)\boldsymbol{G}_I(j\omega)\boldsymbol{D}_R^{-1}(\omega)\right)$.

Proof: Using $\mathbf{G}(s) = \mathbf{G}_{\rm bd}(s) + \mathbf{G}_{\rm I}(s)$,

$$\begin{aligned}
\mathbf{S}^{-1}\mathbf{K}_{\rm bd}^{-1}(s) &= (\mathbf{I} + \mathbf{G}\mathbf{K}_{\rm bd}(s))\mathbf{K}_{\rm bd}^{-1}(s) \\
&= \mathbf{K}_{\rm bd}^{-1}(s) + \mathbf{G}_{\rm bd}(s) + \mathbf{G}_{\rm I}(s) \\
&= (\mathbf{I} + \mathbf{G}_{\rm bd}\mathbf{K}_{\rm bd}(s))\mathbf{K}_{\rm bd}^{-1}(s) + \mathbf{G}_{\rm I}(s) \\
&= \mathbf{S}_{\rm bd}^{-1}\mathbf{K}_{\rm bd}^{-1}(s) + \mathbf{G}_{\rm I}(s)
\end{aligned} \tag{12}$$

Let $\mathbf{D}_{\rm L}(\omega) \in \mathcal{D}_{\rm L}$ and $\mathbf{D}_{\rm R}(\omega) \in \mathcal{D}_{\rm R}$, where $\mathcal{D}_{\rm L}$ and $\mathcal{D}_{\rm R}$ are defined by (10). Then, using (13) and singular value inequalities (Horn and Johnson, 1991; Skogestad and Postlethwaite, 2005),

$$\begin{aligned}
\underline{\sigma}\left(\mathbf{D}_{\rm L}(\omega)\mathbf{S}^{-1}\mathbf{K}_{\rm bd}^{-1}(j\omega)\mathbf{D}_{\rm R}^{-1}(\omega)\right) \geq\ & \underline{\sigma}\left(\mathbf{D}_{\rm L}(\omega)\mathbf{S}_{\rm bd}^{-1}\mathbf{K}_{\rm bd}^{-1}(j\omega)\mathbf{D}_{R}^{-1}(\omega)\right) \\
& - \bar{\sigma}\left(\mathbf{D}_{\rm L}(\omega)\mathbf{G}_{\rm I}(j\omega)\mathbf{D}_{\rm R}^{-1}(\omega)\right)
\end{aligned}$$

Noting that the above expression holds for all $\mathbf{D}_{\rm L}(\omega) \in \mathcal{D}_{\rm L}$, $\mathbf{D}_{\rm R}(\omega) \in \mathcal{D}_{\rm R}$, we select these matrices to minimise $\bar{\sigma}\left(\mathbf{D}_{\rm L}(\omega)\mathbf{G}_{\rm I}(j\omega)\mathbf{D}_{\rm R}^{-1}(\omega)\right)$. Since $\mathbf{S}_{\rm bd}^{-1}\mathbf{K}_{\rm bd}^{-1}(j\omega) = \mathbf{D}_{\rm L}(\omega)\mathbf{S}_{\rm bd}^{-1}\mathbf{K}_{\rm bd}^{-1}(j\omega)\mathbf{D}_{\rm R}^{-1}(\omega)$,

$$\underline{\sigma}\left(\mathbf{D}_{\rm L}(\omega)\mathbf{S}^{-1}\mathbf{K}_{\rm bd}^{-1}(j\omega)\mathbf{D}_{\rm R}^{-1}(\omega)\right) \geq \underline{\sigma}\left(\mathbf{S}_{\rm bd}^{-1}\mathbf{K}_{\rm bd}^{-1}(j\omega)\right) - \bar{\mu}_{\Delta}(\mathbf{G}_{\rm I}(j\omega)) \tag{13}$$

For any $\omega \in \mathbb{R}$, using (13),

$$\bar{\sigma}(\mathbf{D}_{\rm L}(\omega))\bar{\sigma}(\mathbf{D}_{\rm R}^{-1}(\omega))\underline{\sigma}\left(\mathbf{S}^{-1}\mathbf{K}_{\rm bd}^{-1}(j\omega)\right) \geq \underline{\sigma}\left(\mathbf{S}_{\rm bd}^{-1}\mathbf{K}_{\rm bd}^{-1}(j\omega)\right) - \bar{\mu}_{\Delta}(\mathbf{G}_{\rm I}(j\omega)) \tag{14}$$

Since $\bar{\sigma}(\mathbf{D}_{\rm R}^{-1}(\omega)) = \bar{\sigma}(\mathbf{D}_{\rm L}^{-1}(\omega))$ by construction, $\bar{\sigma}(\mathbf{D}_{\rm L}(\omega))\bar{\sigma}(\mathbf{D}_{\rm R}^{-1}(\omega)) = \kappa(\mathbf{D}_{\rm L}(\omega))$. With this choice,

$$\kappa(\mathbf{D}_{\rm L}(\omega))\,\underline{\sigma}\left(\mathbf{S}^{-1}\mathbf{K}_{\rm bd}^{-1}(j\omega)\right) \geq \underline{\sigma}\left(\mathbf{S}_{\rm bd}^{-1}\mathbf{K}_{\rm bd}^{-1}(j\omega)\right) - \bar{\mu}_{\Delta}(\mathbf{G}_{\rm I}(j\omega))$$

Recognising that $\underline{\sigma}\left(\mathbf{S}^{-1}\mathbf{K}_{\rm bd}^{-1}(j\omega)\right) = \bar{\sigma}^{-1}\left(\mathbf{K}_{\rm bd}\mathbf{S}_{\rm bd}(j\omega)\right)$, the above expression can be rearranged to yield (11). ■

For stabilisation purposes, it is useful to minimise input usage as, the likelihood of input saturation is reduced and the disturbing effect of the stabilising control layer on the stabilised system is minimised (Havre and Skogestad, 2003). Based on (11), we note that when the decentralised controller stabilises the closed loop system, the input usage always remains finite. Further, we can express the sufficient condition for stabilisation in Proposition 1 as $\bar{\sigma}^{-1}\left(\mathbf{K}_{\rm bd}\mathbf{S}_{\rm bd}(j\omega)\right) > \bar{\mu}_{\Delta}(\mathbf{G}_{\rm I}(j\omega))\,\forall\omega$. Defining $\gamma(\omega) = \bar{\sigma}^{-1}\left(\mathbf{K}_{\rm bd}\mathbf{S}_{\rm bd}(j\omega)\right) - \bar{\mu}_{\Delta}(\mathbf{G}_{\rm I}(j\omega))$, we note that Proposition 1 is satisfied, if $\gamma(\omega) > 0\,\forall\omega$. Here, large $\gamma(\omega)$ can also be interpreted as the relative ease in stabilising the system using independent designs. The expression in (11) shows that large $\gamma(\omega)$ also ensures that input usage for stabilisation is small.

4 Block diagonal approximation

In this section, we consider the problem of finding an optimal block diagonal approximation $\mathbf{G}_{\rm bd}(s)$ for the given system $\mathbf{G}(s)$ such that $\mu_{\Delta}(\mathbf{G}(j\omega) - \mathbf{G}_{\rm bd}(j\omega))$ is minimised. Since only $\bar{\mu}_{\Delta}(.)$ is computable in practice, the block diagonal $\mathbf{G}_{\rm bd}(s)$ can be chosen by solving,

$$\min \bar{\sigma}\left(\mathbf{D}_{\rm L}(\omega)\big(\mathbf{G}(j\omega) - \mathbf{G}_{\rm bd}(j\omega)\big)\mathbf{D}_{\rm R}^{-1}(\omega)\right) \tag{15}$$

s.t.

$$\mathbf{D}_{\rm L}(\omega) \in \mathcal{D}_{\rm L}, \mathbf{D}_{\rm R}(\omega) \in \mathcal{D}_{\rm R}$$

where $\mathcal{D}_{\rm L}$ and $\mathcal{D}_{\rm R}$ are given by (10) and the number of unstable poles of $\mathbf{G}_{\rm bd}(s)$ and $\mathbf{G}(s)$ is same.

As mentioned earlier, the block diagonal elements of the system usually have more unstable poles than the system itself. Intuitively, a suboptimal solution to the optimisation problem (15) can be obtained by simply reducing the order of the block diagonal elements of $\mathbf{G}(s)$. We next show that the diagonal blocks optimally approximate a complex matrix partitioned into 2 blocks. This result indicates that order reduction of diagonal elements is likely to yield a nearly optimal solution for systems decomposed into 2 blocks.

Lemma 2: *For complex matrices $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$*

$$\mu_{\boldsymbol{\Delta}}\left(\begin{bmatrix} \mathbf{0} & \boldsymbol{A}_1 \\ \boldsymbol{A}_2 & \mathbf{0} \end{bmatrix}\right) = \sqrt{\bar{\sigma}(\boldsymbol{A}_1)\bar{\sigma}(\boldsymbol{A}_2)}$$

where $\boldsymbol{\Delta} = diag(\boldsymbol{\Delta}_1, \boldsymbol{\Delta}_2)$ and the full complex matrices $\boldsymbol{\Delta}_1, \boldsymbol{\Delta}_2$ have the same dimensions as $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$, respectively (Skogestad and Morari, 1988).

Proposition 3: *Consider a complex matrix $\boldsymbol{A} \in \mathbb{C}^{p\times q}$, which is partitioned as,*

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix}$$

Then, $\boldsymbol{A}_{bd} = diag(\boldsymbol{A}_{11}, \boldsymbol{A}_{22})$ minimises $\mu_{\boldsymbol{\Delta}}(\boldsymbol{A} - \boldsymbol{A}_{bd})$, where $\boldsymbol{A}_{bd}$ and $\boldsymbol{\Delta}$ have the same structure as $diag(\boldsymbol{A}_{11}, \boldsymbol{A}_{22})$ and

$$\min_{\boldsymbol{A}_{bd}} \mu_{\boldsymbol{\Delta}}(\boldsymbol{A} - \boldsymbol{A}_{bd}) = \sqrt{\bar{\sigma}(\boldsymbol{A}_{12})\bar{\sigma}(\boldsymbol{A}_{21})} \tag{16}$$

Proof: Using Lemma 2, it follows that $\mu_\Delta(\mathbf{A} - \mathrm{diag}(\mathbf{A}_{11}, \mathbf{A}_{22})) = \sqrt{\bar{\sigma}(\mathbf{A}_{12})\bar{\sigma}(\mathbf{A}_{21})}$. Then, it suffices to show that for all $\mathbf{A}_{\mathrm{bd}}$ having the required structure, the minimum achievable value of $\mu_\Delta(\mathbf{A} - \mathbf{A}_{\mathrm{bd}})$ is given by (16).

Let $\mathbf{A}_{\mathrm{bd}} = \mathrm{diag}(\mathbf{A}_{11} + \mathbf{B}_1, \mathbf{A}_{22} + \mathbf{B}_2)$. Since Δ has two complex blocks (Zhou and Doyle, 1998),

$$\begin{aligned}\mu_\Delta(\mathbf{A} - \mathbf{A}_{\mathrm{bd}}) &= \inf_{\mathbf{D}_\mathrm{L} \in \mathcal{D}_\mathrm{L}, \mathbf{D}_\mathrm{R} \in \mathcal{D}_\mathrm{R}} \bar{\sigma}(\mathbf{D}_\mathrm{L}(\mathbf{A} - \mathbf{A}_{\mathrm{bd}})\mathbf{D}_\mathrm{R}^{-1}) \\ &= \inf_{d_1, d_2 \in \mathbb{R}} \bar{\sigma}\left(\begin{bmatrix} \mathbf{B}_1 & \frac{d_1}{d_2}\mathbf{A}_{12} \\ \frac{d_2}{d_1}\mathbf{A}_{21} & \mathbf{B}_2 \end{bmatrix}\right)\end{aligned}$$

Let $\mathbf{U}$ be a unitary matrix that permutes the off-diagonal blocks of $\mathbf{D}_\mathrm{L}(\mathbf{A} - \mathbf{A}_{\mathrm{bd}})\mathbf{D}_\mathrm{R}^{-1}$ to diagonal blocks and *vice versa*. Without loss of generality, we can choose $d_1 = 1$ (Zhou and Doyle, 1998). Since the largest singular value of a matrix is larger than or equal to largest singular value of the submatrices of the matrix (Horn and Johnson, 1991),

$$\begin{aligned}&\bar{\sigma}(\mathbf{D}_\mathrm{L}(\mathbf{A} - \mathbf{A}_{\mathrm{bd}})\mathbf{D}_\mathrm{R}^{-1}) \\ &\quad = \bar{\sigma}(\mathbf{D}_\mathrm{L}(\mathbf{A} - \mathbf{A}_{\mathrm{bd}})\mathbf{D}_\mathrm{R}^{-1}\mathbf{U}) \\ &\quad \geq \max(\bar{\sigma}(d_2^{-1}\mathbf{A}_{12}), \bar{\sigma}(d_2\mathbf{A}_{21})) \quad \forall d_2 \in \mathbb{R} \\ &\quad \geq \max\left(\left|d_2^{-1}\right|\bar{\sigma}(\mathbf{A}_{12}), |d_2|\,\bar{\sigma}(\mathbf{A}_{21})\right) \quad \forall d_2 \in \mathbb{R} \\ &\quad \geq \sqrt{\bar{\sigma}(\mathbf{A}_{12})\bar{\sigma}(\mathbf{A}_{21})}\end{aligned}$$

The result follows by noting that the above expression is independent of the scaling matrices. ■

Note that Proposition 3 says nothing about the uniqueness of the optimal solution. For $(\mathbf{A} - \mathbf{A}_{\mathrm{bd}})$ partitioned and permuted as done in the proof of Proposition 3 (Zhou and Doyle, 1998, p.218),

$$\mu_\Delta(\mathbf{A} - \mathbf{A}_{\mathrm{bd}}) \leq \max(\bar{\sigma}(\mathbf{A}_{12}), \bar{\sigma}(\mathbf{A}_{21})) + \sqrt{\bar{\sigma}(\mathbf{B}_1)\bar{\sigma}(\mathbf{B}_2)}$$

If $\mathbf{B}_1 = \mathbf{0}$ and $\bar{\sigma}(\mathbf{A}_{12}) = \bar{\sigma}(\mathbf{A}_{21})$, the upper bound on $\mu_\Delta(\mathbf{A} - \mathbf{A}_{\mathrm{bd}})$ is the same as the lower bound. This shows that there exists an infinite number of $\mathbf{B}_2$ and thus block diagonal matrices which achieve the lower bound.

Unfortunately, Proposition 3 does not hold for matrices partitioned into more than 2 blocks; (see Kariwala, 2004) for numerical experiments. We next present an algorithm that provides a locally optimal solution for the optimisation problem (15).

Algorithm 1: For a given system $\mathbf{G}(s)$ with n unstable poles, a locally optimal solution to the block diagonal approximation problem is obtained by the following steps:

1 Solve the optimisation problem (15) at a set of chosen frequencies to yield $\mathbf{G}_{\mathrm{bd},j\omega}$.

2 Solve a parametric optimisation problem to find $\tilde{\mathbf{G}}_{\mathrm{bd}}(s)$ that has at least n unstable poles and minimises the worst case error between $\tilde{\mathbf{G}}_{\mathrm{bd}}(j\omega)$ and $\mathbf{G}_{\mathrm{bd},j\omega}$.

3 If $\tilde{\mathbf{G}}_{\mathrm{bd}}(s)$ has more than n unstable poles, the order of $\tilde{\mathbf{G}}_{\mathrm{bd}}(s)$ is reduced to n through optimal Hankel norm approximation to get $\mathbf{G}_{\mathrm{bd}}(s)$.

The role of these steps becomes clear by noting,

$$\mu_\Delta(\mathbf{G}(j\omega) - \mathbf{G}_{\mathrm{bd}}(j\omega)) \leq \mu_\Delta(\mathbf{G}(j\omega) - \mathbf{G}_{\mathrm{bd},j\omega}) + \bar{\sigma}(\mathbf{G}_{\mathrm{bd},j\omega} - \tilde{\mathbf{G}}_{\mathrm{bd}}(j\omega)) + \bar{\sigma}(\tilde{\mathbf{G}}_{\mathrm{bd}}(j\omega) - \mathbf{G}_{\mathrm{bd}}(j\omega)) \tag{17}$$

It follows from (17) that every step in the proposed method minimises the contribution of one of terms on RHS of (17) to the total approximation error. Thus, Algorithm 1 inherently minimises an upper bound on the objective function of the optimisation problem in (15). In the following sections, the individual steps of the proposed method are discussed. For the sake of brevity, the discussion is brief at places and additional details can be found in Kariwala (2004).

4.1 Frequency wise approximation

The first step of the proposed method for finding the optimal block diagonal approximation consists of minimising (15) at a set of chosen frequencies. The (possibly non-uniformly spaced) set of frequencies can be selected based on $\bar{\sigma}(\mathbf{G}(j\omega))$, that is, a larger number of frequencies can be chosen around the peaks of $\bar{\sigma}(\mathbf{G}(j\omega))$. In the remaining discussion, the frequency argument of the scaling matrices is dropped for notational convenience. Using similar arguments as used in calculating $\bar{\mu}(.)$ by solving a Linear Matrix Inequality (LMI) (Boyd et al., 1994),

$$\bar{\sigma}(\mathbf{D}_\mathrm{L}(\mathbf{G}(j\omega) - \mathbf{G}_{\mathrm{bd},j\omega})\mathbf{D}_\mathrm{R}^{-1}) \leq \gamma \tag{18}$$

$$\Leftrightarrow \mathbf{D}_\mathrm{R}^{-*}(\mathbf{G}(j\omega) - \mathbf{G}_{\mathrm{bd},j\omega})^*\mathbf{D}_\mathrm{L}^*\mathbf{D}_\mathrm{L}(\mathbf{G}(j\omega) - \mathbf{G}_{\mathrm{bd},j\omega})\mathbf{D}_\mathrm{R}^{-1} \preceq \gamma^2\mathbf{I} \tag{19}$$

$$\Leftrightarrow (\mathbf{G}(j\omega) - \mathbf{G}_{\mathrm{bd},j\omega})^*\mathbf{P}_L(\mathbf{G}(j\omega) - \mathbf{G}_{\mathrm{bd},j\omega}) \preceq \gamma^2\mathbf{P}_\mathrm{R} \tag{20}$$

where $\mathbf{P}_\mathrm{L} = \mathbf{D}_\mathrm{L}^*\mathbf{D}_\mathrm{L} \in \mathcal{D}_\mathrm{L}$, $\mathbf{P}_\mathrm{R} = \mathbf{D}_\mathrm{R}^*\mathbf{D}_\mathrm{R} \in \mathcal{D}_\mathrm{R}$ and $\mathbf{P}_\mathrm{L}, \mathbf{P}_\mathrm{R} \succ 0$. Note that unlike the calculation of $\bar{\mu}(.)$ (Boyd et al., 1994), (20) is a Bilinear Matrix Inequality (BMI) and thus not affine in the decision variables $\mathbf{G}_{\mathrm{bd},j\omega}$, $\mathbf{P}_\mathrm{L}$ and $\mathbf{P}_\mathrm{R}$; however, a locally optimal solution can be found using an iterative approach.

Using the Schur complement lemma (Boyd et al., 1994), (19) can be equivalently expressed as,

$$\begin{bmatrix} -\gamma\mathbf{I} & \mathbf{D}_\mathrm{R}^{-*}(\mathbf{G}(j\omega) - \mathbf{G}_{\mathrm{bd},j\omega})^*\mathbf{D}_\mathrm{L}^* \\ \mathbf{D}_\mathrm{L}(\mathbf{G}(j\omega) - \mathbf{G}_{\mathrm{bd},j\omega})\mathbf{D}_\mathrm{R}^{-1} & -\gamma\mathbf{I} \end{bmatrix} \preceq 0 \tag{21}$$

Note that for fixed $\mathbf{D}_\mathrm{L}, \mathbf{D}_\mathrm{R}$, (21) is an LMI in $\mathbf{G}_{\mathrm{bd},j\omega}$. Now, a locally optimal solution for the frequency wise approximation problem can be found by using an iterative approach, where (21) is solved for $\mathbf{G}_{\mathrm{bd},j\omega}$ by fixing $\mathbf{D}_\mathrm{L}$ and $\mathbf{D}_\mathrm{R}$, and (20) is solved for $\mathbf{P}_\mathrm{L}, \mathbf{P}_\mathrm{R}$ using a bisection search method by fixing $\mathbf{G}_{\mathrm{bd},j\omega}$. This iterative method can be initialised by setting $\mathbf{D}_\mathrm{L} = \mathbf{D}_\mathrm{R} = \mathbf{I}$. Note that unlike a general BMI problem, the sequence of solutions obtained using this iterative approach is guaranteed to converge (Kariwala, 2004).

Remark 4: *Since the approximation problem has multiple local minima and the converged solution depends on the initial value, the iterative procedure can converge to a minima that is worse than using the diagonal blocks. This difficulty is overcome by using diag($\boldsymbol{G}_{ii}(j\omega)$) as an initial guess, which is equivalent to replacing $\boldsymbol{G}(j\omega)$ by $\boldsymbol{G}(j\omega)$ − diag($\boldsymbol{G}_{ii}(j\omega)$).*

Then, the locally optimal solution is given as $\boldsymbol{G}^{sub}_{bd,j\omega} + diag(\boldsymbol{G}_{ii}(j\omega))$, where $\boldsymbol{G}^{sub}_{bd,j\omega}$ is the solution obtained by using $diag(\boldsymbol{G}_{ii}(j\omega))$ as an initial guess. This minor modification ensures that the obtained solution is at least as good as using the diagonal blocks.

4.2 Parametric $\mathcal{L}_\infty$ optimal identification

It would be ideal to directly find $\mathbf{G}_{\rm bd}(s)$ which has the same number of unstable poles as $\mathbf{G}(s)$ and best approximates $\mathbf{G}_{{\rm bd},j\omega}$, but the optimisation problem becomes very difficult when the number of unstable poles is fixed. Thus, we aim at finding $\tilde{\mathbf{G}}_{\rm bd}(s)$ that has *at least* as many unstable poles as $\mathbf{G}(s)$ followed by model order reduction discussed in § 4.3. We minimise the worst case error or the $\mathcal{L}_\infty$ norm of $\mathbf{G}_{{\rm bd},j\omega_i} - \tilde{\mathbf{G}}_{\rm bd}(j\omega_i)$ (*cf.* (17)). Over the past few years, a number of different approaches for worst-case identification have appeared in the literature and an overview of available results can be found in (Chen and Gu, 2000).

In this paper, we parameterise the class of models using transfer functions as compared to the Finite Impulse Response (FIR) models typically used in worst-case identification; (see e.g. Helmicki et al., 1992). An advantage of using the transfer function parametrisation is that low order models can be identified directly in the continuous time domain, the disadvantage being that unlike the FIR parametrisation, no worst case error bounds are available. Nevertheless, practical experience (particularly in $\mathcal{H}_2$ norm minimisation case) suggests that transfer function parametrisation works very well. For simplicity, $\tilde{\mathbf{G}}_{\rm bd}(s)$ is identified element by element, where $[\tilde{\mathbf{G}}_{\rm bd}(s)]_{ij}$ is parameterised as:

$$[\tilde{\mathbf{G}}_{\rm bd}(s)]_{ij} = \frac{a(s)}{b(s)} = \frac{a_m s^m + a_{m-1}s^{m-1} + \cdots a_1 s + a_0}{b_n s^n + b_{n-1}s^{n-1} + \cdots b_1 s + b_0}$$

with $m \leq n$.

In the remaining discussion, we drop the requirement that $\tilde{\mathbf{G}}_{\rm bd}(s)$ has at least as many poles as $\mathbf{G}_{\rm bd}(s)$, as it is easily satisfied by choosing the order of the denominator polynomials sufficiently large. Then, the parameters $a_0, \ldots, a_m, b_0, \ldots, b_n$, are obtained by solving,

$$\min_{a_0,\ldots,a_m,b_0,\ldots,b_n} \left| \frac{a(j\omega_k)}{b(j\omega_k)} - [\mathbf{G}_{{\rm bd},j\omega_k}]_{ij} \right| \qquad k = 1 \cdots n_\omega \tag{22}$$

Note that the objective function in (22) is non-linear, but can be equivalently represented as

$$|b(j\omega_k)|^{-1} \left| a(j\omega_k) - b(j\omega_k)[\mathbf{G}_{{\rm bd},j\omega_k}]_{ij} \right| \tag{23}$$

Now, the following LMI problem can be solved iteratively to minimise (23):

$$\min_{a_0^{(i)},\ldots,a_m^{(i)},b_0^{(i)},\ldots,b_n^{(i)} \in \mathbb{R}} \gamma_1^2 + \gamma_2^2$$

s.t.

$$\begin{aligned} & -\gamma_1^2 |b^{(i-1)}(j\omega_k)| \leq \mathrm{Re}(e(j\omega_k)) \leq \gamma_1^2 |b^{(i-1)}(j\omega_k)| \\ & -\gamma_1^2 |b^{(i-1)}(j\omega_k)| \leq \mathrm{Im}(e(j\omega_k)) \leq \gamma_2^2 |b^{(i-1)}(j\omega_k)| \\ & b_n = 1 \\ & e(j\omega_k) = \left(a^{(i)}(j\omega_k) - b^{(i)}(j\omega_k)[\mathbf{G}_{{\rm bd},j\omega_k}]_{ij} \right) \end{aligned} \tag{24}$$

where $b^{(i-1)}(j\omega_k)$ denotes the identified b polynomial from the previous iteration. In (24), the additional constraint $b_n = 1$ is imposed for numerical stability and in general, fixing any one of the unknown parameters suffices. In the $\mathcal{H}_2$ optimal identification literature, a method similar to (24) is known as Sanathanan and Koerner's method (Pintelon et al., 1994).

The sequence of solutions obtained by solving optimisation problem (24) is not guaranteed to converge, but numerical evidence suggests that a reasonable solution can be obtained using a few iterations.

4.3 Optimal Hankel norm approximation

To satisfy the assumption of Proposition 1, we need to find $\mathbf{G}_{\text{bd}}(s)$ which has exactly n unstable poles. We recall that for a stable transfer matrix $\mathbf{H}(s)$ having order k, the optimal kth order model $\hat{\mathbf{H}}^k(s)$ is found by solving (Glover, 1984),

$$\min_{\hat{\mathbf{H}}^k(s)\in\mathcal{RH}_\infty} \|\mathbf{H}(s) - \hat{\mathbf{H}}^k(s)\|_H = \min_{\hat{\mathbf{H}}^k(s),\mathbf{F}^*(-s)\in\mathcal{RH}_\infty} \|\mathbf{H}(s) - \hat{\mathbf{H}}^k(s) - \mathbf{F}(s)\|_\infty \quad (25)$$

where $\|.\|_H$ denotes the Hankel norm given by the largest Hankel singular value of the transfer matrix. Next, we show how (25) can be adapted to handle the given problem, that is, model reduction of the unstable system $\tilde{\mathbf{G}}_{\text{bd}}(s)$.

Let $\tilde{\mathbf{G}}_{\text{bd}}(s) = \mathbf{G}_1(s) + \mathbf{G}_2(s)$ such that $\mathbf{G}_1^*(-s), \mathbf{G}_2(s) \in \mathcal{RH}_\infty$. Without loss of generality, we can parameterise $\mathbf{G}_{\text{bd}}(s)$ as $\mathbf{G}_{\text{bd}}(s) = \mathbf{G}_{\text{bd}}^n(s) + \mathbf{G}_2(s) + \mathbf{J}(s)$ with $\mathbf{J}(s) \in \mathcal{RH}_\infty$, which provides

$$\begin{aligned}
&\|\tilde{\mathbf{G}}_{\text{bd}}(s) - \mathbf{G}_{\text{bd}}(s)\|_\infty \\
&\quad = \|\mathbf{G}_1(s) - \mathbf{G}_{\text{bd}}^n(s) - \mathbf{J}(s)\|_\infty \\
&\quad = \|\mathbf{G}_1^*(-s) - (\mathbf{G}_{\text{bd}}^n(s))^* - \mathbf{J}^*(-s)\|_\infty
\end{aligned}$$

The optimal value for $(\mathbf{G}_{\text{bd}}(s))^* \in \mathcal{RH}_\infty$ is found by solving (*cf.* (25)),

$$\begin{aligned}
&\min_{(\mathbf{G}_{\text{bd}}^n(s))^*,\mathbf{J}(s)\in\mathcal{RH}_\infty} \|\mathbf{G}_1^*(-s) - (\mathbf{G}_{\text{bd}}^n(s))^* - \mathbf{J}^*(-s)\|_\infty \\
&\quad = \|\mathbf{G}_1^*(-s) - (\mathbf{G}_{\text{bd}}^n(s))^*\|_H \qquad (26)
\end{aligned}$$

Since $\mathbf{J}(s)$ and $\mathbf{G}_2(s)$ are stable, $\mathbf{G}_{\text{bd}}(s)$ found by minimising Hankel norm between $\mathbf{G}_1^*(-s)$ and $(\mathbf{G}_{\text{bd}}^n(s))^*$ is the $\mathcal{L}_\infty$ optimal reduced order approximation of $\tilde{\mathbf{G}}_{\text{bd}}(s)$ with n unstable poles.

5 Controller design

With the availability of $\mathbf{G}_{\text{bd}}(s)$ using Algorithm 1, controller design for the modified μ-IM is similar to the conventional μ-IM method. A loop shaping approach can be used to find the stabilising decentralised controller; however, finding a controller using this method to satisfy (8) can be difficult. In this section, we show that with the alternate representation of the μ-IM conditions in terms of $\mathbf{K}_{\text{bd}}\mathbf{S}_{\text{bd}}(s)$, finding $\mathbf{K}_{\text{bd}}(s)$ to satisfy (8) reduces to solving a weighted $\mathcal{H}_\infty$ controller design problem for $\mathbf{G}_{\text{bd}}(s)$.

Lemma 3: *Let $\boldsymbol{G}(s)$ be rational system. Then (Glover, 1986; Kariwala et al., 2005),*

$$\inf_{\boldsymbol{K}(s)} \|\boldsymbol{K}(s)(\boldsymbol{I} + \boldsymbol{GK}(s))^{-1}\|_\infty = \underline{\sigma}_H^{-1}(\mathcal{U}(\boldsymbol{G}(s))^*)$$

where $\mathcal{U}(.)$ denotes the unstable part.

Proposition 4: *Consider that $\boldsymbol{G}(s)$ and $\boldsymbol{G}_{bd}(s)$ have the same number of unstable poles. Let the minimum phase and stable transfer matrix $w(s)$ be chosen such that*

$|w(j\omega)| = \mu_{\boldsymbol{\Delta}}^{-1}(\boldsymbol{G}_I(j\omega))$ for all ω. There exists a block diagonal controller $\boldsymbol{K}_{bd}(s)$ such that $\bar{\sigma}(\boldsymbol{K}_{bd}\boldsymbol{S}_{bd}(j\omega)) < \mu_{\boldsymbol{\Delta}}^{-1}(\boldsymbol{G}_I(j\omega))$ for all $\omega \in \mathbb{R}$ iff

$$\underline{\sigma}_H^{-1}(\mathcal{U}(w^{-1}\boldsymbol{G}_{bd}(s))^*) < 1 \tag{27}$$

Proof: (Sufficiency) Let us define, $\tilde{\mathbf{K}}_{\mathrm{bd}}(s) = w(s)\mathbf{K}_{\mathrm{bd}}(s)$ and $\tilde{\mathbf{G}}_{\mathrm{bd}}(s) = w^{-1}(s)\mathbf{K}_{\mathrm{bd}}(s)$. Then, using Lemma 3, there exists a $\mathbf{K}_{\mathrm{bd}}(s)$ such that,

$$\begin{aligned}
&\inf_{\mathbf{K}_{\mathrm{bd}}(s)} \|w\mathbf{K}_{\mathrm{bd}}\mathbf{S}_{\mathrm{bd}}(s)\|_\infty \\
&\quad = \inf_{\tilde{\mathbf{K}}_{\mathrm{bd}}(s)} \|\tilde{\mathbf{K}}_{\mathrm{bd}}(s)(\mathbf{I} + \tilde{\mathbf{G}}_{\mathrm{bd}}\tilde{\mathbf{K}}_{\mathrm{bd}}(s))^{-1}\|_\infty \\
&\quad = \underline{\sigma}_H^{-1}(\mathcal{U}(w^{-1}\mathbf{G}_{\mathrm{bd}}(s))^*)
\end{aligned}$$

If (27) holds, there exists a $\mathbf{K}_{\mathrm{bd}}(s)$ such that

$$\begin{aligned}
&\|w\mathbf{K}_{\mathrm{bd}}\mathbf{S}_{\mathrm{bd}}(s)\|_\infty < 1 && (28)\\
&\quad \Leftrightarrow \bar{\sigma}(w\mathbf{K}_{\mathrm{bd}}\mathbf{S}_{\mathrm{bd}}(j\omega)) < 1 \qquad \forall\omega \\
&\quad \Leftrightarrow \bar{\sigma}(\mathbf{K}_{\mathrm{bd}}\mathbf{S}_{\mathrm{bd}}(j\omega)) < |w(j\omega)|^{-1} \qquad \forall\omega \\
&\quad \Leftrightarrow \bar{\sigma}(\mathbf{K}_{\mathrm{bd}}\mathbf{S}_{\mathrm{bd}}(j\omega)) < \mu_\Delta^{-1}(\mathbf{G}_I(j\omega)) \qquad \forall\omega
\end{aligned}$$

where the last inequality holds as $|w(j\omega)| = \mu_\Delta(\mathbf{G}_I(j\omega))$ for all ω.

(Necessity) We show the necessity of (27) by contradiction. Consider that (27) does not hold, but there exists a $\mathbf{K}_{\mathrm{bd}}(s)$ such that $\bar{\sigma}(\mathbf{K}_{\mathrm{bd}}\mathbf{S}_{\mathrm{bd}}(j\omega)) < \mu_\Delta^{-1}(\mathbf{G}_I(j\omega))\ \forall\omega$. By reversing the series of inequalities used for sufficiency, $\mathbf{K}_{\mathrm{bd}}(s)$ must satisfy (28). The $\underline{\sigma}_H^{-1}(\mathcal{U}(w^{-1}\mathbf{G}_{\mathrm{bd}}(s))^*)$ denotes the least achievable value for $\|w(s)\mathbf{K}_{\mathrm{bd}}\mathbf{S}_{\mathrm{bd}}(s)\|_\infty$ for all LTI controllers. Then, $\|w\mathbf{K}_{\mathrm{bd}}\mathbf{S}_{\mathrm{bd}}(s)\|_\infty < 1$, despite $\underline{\sigma}_H^{-1}(\mathcal{U}(w^{-1}\mathbf{G}_{\mathrm{bd}}(s))^*)$ being less than 1 is a contradiction and the necessity of (27) follows. ■

In Proposition 4, we assumed that $w(s)$ is stable and minimum phase. In general, $w(s)$ can have RHP zeros and RHP poles at the same location as $\mathbf{G}_{\mathrm{bd}}(s)$. Note that

$$\|w(s)\mathbf{K}_{\mathrm{bd}}\mathbf{S}_{\mathrm{bd}}(s)\|_\infty = \|w_{\mathrm{ms}}(s)\mathbf{K}_{\mathrm{bd}}\mathbf{S}_{\mathrm{bd}}(s)\|_\infty$$

where $w_{\mathrm{ms}}(s)$ denotes the minimum phase stable part of $w(s)$. Thus, allowing $w(s)$ to be unstable or non-minimum phase provides no advantage and we can simply replace $w(s)$ by its minimum and stable part in (27). On relaxing the assumption of minimum phase stable $w(s)$, however, a $w(s)$ that achieves $|w(j\omega)| = \mu_\Delta^{-1}(\mathbf{G}_I(j\omega))$ becomes non-unique, where the different instances of $w(s)$ are related by unitary transformations.

Proposition 4 effectively reduces the task of finding a block decentralised controller to satisfy μ-IM condition (8) to finding the minimum phase and stable $w(s)$ such that $|w(j\omega)| = \mu_\Delta^{-1}(\mathbf{G}_I(j\omega))$ and (27) holds. When (27) is satisfied, the standard $\mathcal{H}_\infty$ optimal control design techniques can be used to find the stabilising decentralised controller.

Remark 5: *In practice, it can be difficult to find $w(s)$ that satisfies $|w(j\omega)| = \mu_{\boldsymbol{\Delta}}^{-1}(\boldsymbol{G}_I(j\omega))$ for all $\omega \in \mathbb{R}$. This difficulty can be overcome by recognising that for any $w(s)$ that lower bounds $\mu_{\boldsymbol{\Delta}}(\boldsymbol{G}_I(j\omega))$ at all frequencies, if* (27) *holds,*

$$\begin{aligned}
&\bar{\sigma}(\boldsymbol{K}_{bd}\boldsymbol{S}_{bd}(j\omega)) < |w(j\omega)|^{-1} \\
&\quad \Rightarrow \bar{\sigma}(\boldsymbol{K}_{bd}\boldsymbol{S}_{bd}(j\omega)) < \mu_{\boldsymbol{\Delta}}^{-1}(\boldsymbol{G}_I(j\omega))
\end{aligned}$$

Thus, for a given $\boldsymbol{G}_{bd}(s)$ the existence of a decentralised stabilised controller can be established by verifying (27) *with $w(s)$ that lower bounds $\mu_\Delta(\boldsymbol{G}_I(j\omega))$.*

6 Numerical example

In this section, we demonstrate the efficiency of Algorithm 4 for obtaining optimal block diagonal approximation and the controller design method discussed in the previous sections using a simple example.

Consider the following system:

$$\mathbf{G}(s) = \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 1 & \beta_1 & \beta_1 \\ 0 & 2 & 0 & 0 & \beta_1 & 1 & \beta_1 \\ 0 & 0 & 3 & 0 & \beta_1 & \beta_1 & 1 \\ 0 & 0 & 0 & -4 & 1 & 0.4 & 0.4 \\ \hline 1 & \beta_2 & \beta_2 & 1 & 0 & 0 & 0 \\ \beta_2 & 1 & \beta_2 & 0.6 & 0 & 0 & 0 \\ \beta_2 & \beta_2 & 1 & 0.6 & 0 & 0 & 0 \end{array}\right]$$

where $\beta_1 = 0.5, \beta_2 = 0.1$.

A set of equally spaced frequencies in the range $0 - 10$ is chosen and the locally optimal diagonal approximation is obtained using the following steps:

- we use frequency-wise minimisation to achieve 3 decimal digits of accuracy from the locally optimal solution in 2 iterations.
- we fit 4 or lower order models for the frequency data using the formulation (24) with 2 iterations.
- the identified model has 4 unstable poles, which is reduced to a model with 3 unstable poles using the Hankel norm approximation method discussed in Section 4.3.

The $\mathbf{G}_{\text{bd}}^{\text{sub}}(s)$, as obtained following these steps, is given as:

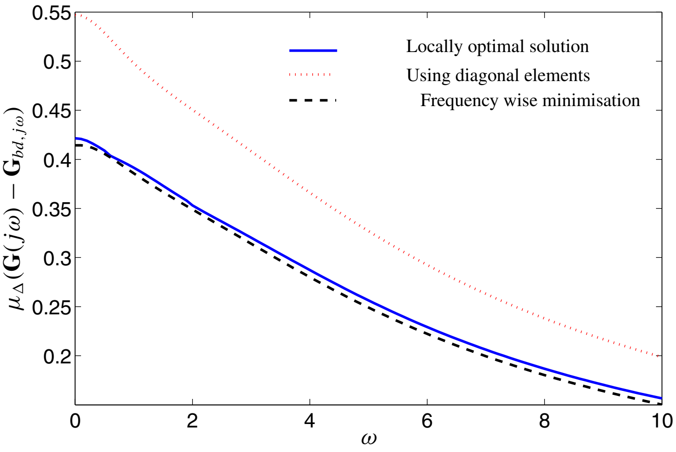
$$\text{diag}\Big(\frac{-0.002s^2 + 2.22s + 3.42}{s^2 + 2.92s - 3.96}, \frac{-0.015s^2 + 2.04s + 6.02}{s^2 + 2.57s - 9.76}, \frac{-0.0153s^2 + 1.85s + 4.97}{s^2 + 1.75s - 8.97}\Big)$$

For comparison purposes, we also calculate the sub-optimal solution $\mathbf{G}_{\text{bd}}^{\text{diag}}(s)$ by reducing the order of diagonal elements of $\mathbf{G}$. In this case, five Hankel singular values of the stable part of $\mathbf{G}_{\text{bd}}^{\text{diag}}(s)$ are negligible, which are removed to get a reduced order model given as:

$$\text{diag}\Big(\frac{2.08s + 3.27}{s^2 + 2.96s - 4.16}, \frac{1.33s + 3.90}{s^2 + 2.06s - 7.76}, \frac{-0.006s^2 + 1.26s + 3.53}{s^2 + 1.42s - 10.31}\Big)$$

To show the advantage of Algorithm 1 over using diagonal elements, $\gamma^{\text{sub}} = \mu_\Delta(\mathbf{G}(j\omega) - \mathbf{G}_{\text{bd}}^{\text{sub}}(j\omega))$ and $\gamma^{\text{diag}} = \mu_\Delta(\mathbf{G}(j\omega) - \mathbf{G}_{\text{bd}}^{\text{diag}}(j\omega))$ are compared in Figure 2. The relative difference between γ^{diag} and γ^{sub} is 0.23 at the zero frequency, which monotonically reduces to 0.21 for $\omega = 10$. This significant reduction in the approximation error is useful for finding the stabilising controller easily. Figure 2 also shows that the γ^{sub} closely matches the approximation error obtained using frequency wise minimisation. Thus, (at least for this example), the conservativeness in using the two-step approach for identifying a model, with same number of unstable poles as the system, is minimal.

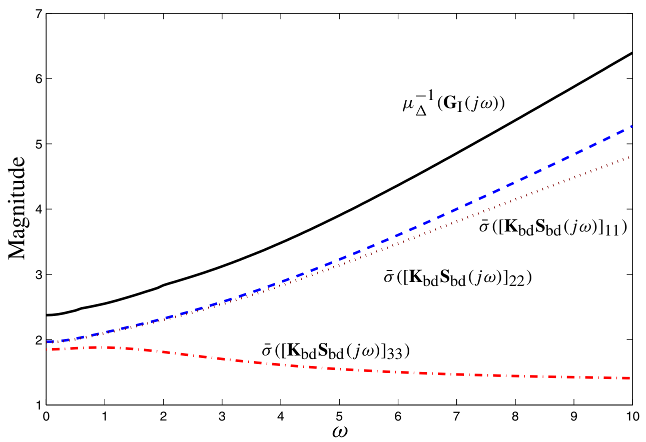
Figure 2 Efficiency of the proposed method for finding optimal block diagonal approximation



Next, we consider the controller design. For the locally optimal diagonal approximation, the following weight lower bounds $\mu_\Delta(\mathbf{G}_\mathrm{I}(j\omega))$ closely,

$$w_1(s) = \frac{0.0123s^2 + 1.71s + 1.88}{s^2 + 5.495s + 4.52}$$

Figure 3 Validation of modified μ-IM for stabilising decentralised controller designed using independent designs



Using this $w_1(s)$, $\underline{\sigma}_\mathrm{H}(\mathcal{U}(w_1^{-1}\mathbf{G}_\mathrm{bd}^\mathrm{sub}(s))^*) = 1.22 > 1$ and standard $\mathcal{H}_\infty$ optimal controller design technique is used to find a decentralised stabilising controller. The plots of $\mu_\Delta^{-1}(\mathbf{G}_\mathrm{I}(j\omega))$ and $\bar{\sigma}([\mathbf{K}_\mathrm{bd}\mathbf{S}_\mathrm{bd}(j\omega)]_{ii})$, $i = 1, 2, 3$ are shown in Figure 3, where

$\mu_{\Delta}^{-1}(\mathbf{G}_{\mathrm{I}}(j\omega)) > \bar{\sigma}([\mathbf{K}_{\mathrm{bd}}\mathbf{S}_{\mathrm{bd}}(j\omega)]_{ii})$, as expected. On the other hand, for the suboptimal solution obtained using the diagonal elements, the weight that lower bounds $\mu_{\Delta}(\mathbf{G}_{\mathrm{I}}(j\omega))$ closely is

$$w_2(s) = \frac{0.05s^2 + 2.165s + 2.38}{s^2 + 5.404s + 4.44}$$

and $\underline{\sigma}_H(\mathcal{U}(w_2^{-1}(s)\mathbf{G}_{\mathrm{bd}}^{\mathrm{diag}}(s))^*) = 0.59 < 1$. Then, the conservativeness of using the diagonal elements to find a suboptimal solution is emphasised.

7 Conclusions

In this paper, we extended the practical applicability of μ-IM to unstable systems. The decentralised controller is designed based on a block diagonal approximation that is different from the block diagonal elements, but has same number of unstable poles as the system. By expressing the μ-IM in terms of transfer matrix from disturbances to inputs, it is shown that:

- the block diagonal approximation can be (suboptimally) chosen by minimising the scaled $\mathcal{L}_\infty$ distance between the system and the approximation.
- the task of designing the controller based on the block diagonal approximation can be reduced to solving a weighted $\mathcal{H}_\infty$ optimal controller design problem.

We have shown that when the system is partitioned into 2 blocks, the optimal block diagonal approximation can be obtained by order reduction of diagonal blocks. For the general case, a step-wise numerical approach is presented for finding the locally optimal solution to the block diagonal approximation problem. The proposed approach involves solving the approximation problem at a set of frequencies followed by $\mathcal{L}_\infty$ optimal identification.

The primary limitation of choosing the block diagonal approximation by minimising the scaled $\mathcal{L}_\infty$ distance is that the properties of the approximation are not taken into account. As shown in this paper, whether the stabilising controller can be easily found depends on the minimum Hankel singular value of the approximation. A better approach is to use a multiobjective optimisation framework, where the $\mathcal{L}_\infty$ distance between the system and the approximation is minimised and simultaneously the minimum Hankel singular of the approximation is maximised. This non-trivial problem is a topic for future work.

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