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Abstract: This paper presents practical methods for computation of disturbance rejection measures [Skogestad and Wolff, 1992], which are useful for assessing the dynamic operability of the process. Using L1 optimal control theory, we consider the cases of steady-state, frequency-wise and dynamic systems. In comparison to the available methods [Hovd et al., 2003; Kookos and Perkins, 2003; Hovd and Kookos, 2005], the proposed approach ensures that a linear, causal, feedback-based controller exists that achieves the computed bounds and the method also scales well with problem dimensions.

Reply to reviewers for *$\mathcal{L}_1/Q$ Approach for Efficient Computation of Disturbance Rejection Measures for Feedback Control*

Vinay Kariwala and Sigurd Skogestad

Reviewer 1

Comment: The largest allowable disturbance measure is misunderstood (and of little interest), as explained in Hovd/Ma/Braatz. We are not interested in the largest allowable disturbance, rather the smallest disturbance that causes the maximum allowable offset with the available manipulated variables. For example, consider $G = I_2, G_d = diag(100, 1)$ and an allowable offset of 1 in either controlled variable. Then the largest allowable disturbance is of magnitude just over 2 (d=[0.02;2], u=[-1;-1], y=[1;1]), whereas the smallest disturbance that creates the maximum allowable offset with optimal use of inputs is of magnitude 0.02 (d=[0.02;0];u=[-1;0];y=[1;0]).

Response/Action taken: We agree with the reviewer that the minimax formulation for the finding the largest allowable disturbance was incorrect in the original manuscript. Fortunately, our computations were based on the correct formulation. In the revised manuscript, the formulation has been corrected as:

$$\begin{aligned} \gamma_{d,\max} &= \max \sigma \qquad (3) \\ &\text{s.t.} \quad \max_{\|d\|_\infty \le \sigma} \min_{\|u\|_\infty \le \gamma_u} \|G\, u + G_d\, d\|_\infty \le \gamma_y \end{aligned}$$

which is same as suggested in the previous papers by Skogestad and Wolff (1992), Hovd *et al.* (2003) and Kookos and Perkins (2003). To further aid reader's understanding, we have included the following discussion about the formulation of the optimization problem for finding largest allowable disturbance in the revised manuscript:

The formulation in (3) is equivalent to finding the largest scaling for disturbances such that the achievable output error becomes γ_y. *It may seem that the largest allowable disturbance can be found as*

$$\gamma'_{d,\max} = \max_{\substack{\|G\,u + G_d\,d\|_{\mathcal{L}_1} \leq \gamma_y \\ \|u\|_\infty \leq \gamma_u}} \|d\|_\infty \tag{4}$$

The formulation in (4), however, only finds "a" disturbance having magnitude $\gamma'_{d,\max}$ *such that the bounds on outputs and inputs can be maintained, but does not ensures the outputs and inputs can be kept bounded for "all" disturbances having magnitude smaller than or equal to* $\gamma'_{d,\max}$. *The subtle difference between the two different formulations in (3) and (4) is illustrated by Hovd et al. [3] using an insightful example.*

Comment: This is also the more likely reasons for the discrepancy between the results of this paper and those of Kookos & Perkins in Example 4. Noting that Kookos & Perkins only consider the steady state case, and a linear, constrained model, blaming the apparent discrepancy on K&P allowing for time-varying control seems a bit strange.

Response/Action taken: Though there was an error in the minimax formulation for finding the largest allowable disturbance in the original manuscript, the formulation for the $\mathcal{L}_1/Q$ approach was correct (except one minor typographical error). Kookos & Perkins allow the use of online-optimization based controllers and thus their results show that larger disturbances can be accommodated. This is the main reason behind the discrepancy. The corresponding remark in the revised manuscript has been corrected as:

Note that due to the practical assumption of a linear feedback-based controller, the allowable disturbance magnitude calculated using the $\mathcal{L}_1/Q$ *approach is lower than the minimax formulation, which allows for non-linear and online optimization-based controllers.*

For the steady-state case, when a single disturbance is considered, the minimax formulation considered by Kookos & Perkins reduces to a standard linear program with n_u degrees of freedom. This happens as the worst-case always occurs at the boundary, *i.e.* $d = 1$. Now, for $\mathcal{L}_1/Q$ approach, there is no loss of generality in replacing QG_d by $\bar{Q}$, as Q can be easily recovered later. Thus, the $\mathcal{L}_1/Q$ also results in a linear program with n_u degrees of freedom, which is exactly the same as obtained using the minimax formulation. These arguments show that the allowable value of d_6 reported by Kookos & Perkins is indeed incorrect. That being said, our objective was not to blame anyone, but simply point out that the discrepancy is due to a typographical error. The corresponding remarks have been revised as follows:

We also note that the largest difference between the two approaches is seen, when only disturbance d_6 is considered. It can be shown easily that for single disturbance, the minimax and $\mathcal{L}_1/Q$ formulations are identical for the steady-state case. Then, the apparent difference is due to the typographical errors in [4].

Comment: Another issue is that linear feedback may be too conservative an assumption, considering the proven success of MPC in pushing constraints (the MPC controller becomes nonlinear when constraints are active).

Response/Action taken: For the numerical examples considered in this paper, linear feedback-based controller performs nearly the same as an optimization-based controller. For example, for blown film extruder example, the maximum relative difference is about 7% and for the Tennessee Eastman example, the maximum relative difference is about 2% for the different cases among linear feedback-based and unrestricted controllers. Thus, the assumption of linear feedback-based control is not very restrictive, at least for the examples considered in this paper. That, however; does not rule out the possibility of existence of systems for which an online-optimization based controller can provide significant advantages over linear feedback-based controller. We have added the following comments to the concluding section:

For the various numerical examples considered in this paper, it is found that a linear feedback-based controller can provide nearly the same level of performance as an online-optimization based controller for asymptotic rejection of constant and sinusoidal disturbances. In general, however, the use of a linear feedback-based controller can be be conservative. Future research will focus upon extending the results of this paper to online-optimization based controllers, e.g. *model predictive controllers.*

Response/Action taken: Adressing the advantages of feedforward only in a steady state setting defies reason. Contrary to what is stated in the paper, the controller should - at steady state - not need to estimate the disturbances when there are more disturbances than outputs. We are not interested in the components of the disturbances that have no effect on the output. At steady state, only the magnitude of the available manipulated variables matter, and feedforward and feedback should give the same result. Hypothetically, the controller can wait for the disturbances to reach steady state (in the outputs) and then calculate required inputs based on the observed outputs.

Also on page 16, we should be cautious in replacing QG_d with $\bar{Q}$. Not only must G_d have a left inverse, that inverse should also be stable? (this only applies to dynamical problems, of course).

Response/Action taken: The main objective of this paper is to present practical solutions to the three problems introduced by Skogestad and Wolff (199). In view of the objections raised by the reviewer, we realized that the details on the feedforward controller can cause confusion and divert the attention of the reader from the main issues. Thus, we have removed the whole section (and any references to these results) from the revised manuscript. As the issue of feedforward control is only peripheral to the the problem of computing disturbance rejection measures, we believe that removal of these details do not cause any substantial reduction in the contributions of our work.

Comment: The contributions of this paper definitely justify publication, but the authors should carefully address the issues raised above, since otherwise they risk causing more confusion than the contribution of the paper.

Response/Action taken: We thank the reviewer for his critical comments, which helped us in identifying issues that could have caused confusion. The manuscript has been revised thoroughly to address all of these issues.

Reviewer 2

Comment: The manuscript describes a practically-useful approach for efficiently computing disturbance rejection measures for linear time invariant systems. The problems are appropriately formulated, the numerical examples are well-selected, and the paper demonstrates clear advantages over previous formulations.

Response/Action taken: We thank the reviewer for his comments.

Comment: In Remark 2, insert "include" after "explicitly" in the sixth line of Remark 2 on page 15.

Response/Action taken: This typographical error has been corrected.

Comment: The last full sentence of page 17 is not quite correct, since Hovd et al. does not state that an online optimization-based controller is required, only that the optimization includes the possibility of the optimum being an online optimized-based controller. An accurate restatement of the sentence would be "Note that the bounds presented by Hovd et al. (2003) do not specify the structure of the controller(s) that achieves the bound, for example, it could be a nonlinear or online optimization-based controller.

Response/Action taken: In view of the reviewer's comments, the corresponding remark has been revised as follows:

Hovd et al. *[3] do not impose any restrictions on the controller structure and the controller that achieves the bounds presented by them can be a nonlinear or online optimization-based controller.*

Comment: This manuscript should compare its results to that of D. L. Ma, J. G. VanAntwerp, M. Hovd, and R. D. Braatz. Quantifying the potential benefits of constrained control for a large scale system. IEE Proceedings - Control Theory and Applications, 149:423-432, 2002, which also considered controllability measures for steady-state, frequency-wise, and dynamic systems.

Response/Action taken: Ma *et al.* provide explicit expressions for the minimum required input magnitude for perfect control. The novel feature of the results of Ma *et al.* is that their results are also valid for rank deficient

G, when the disturbances are confined to their controllable subspace. We have included the following remark in Section 2.2 of the revised manuscript to show that the $\mathcal{L}_1/Q$ method can be extended to compute the minimum input magnitude required to get perfect control in the controllable subspace of disturbances:

Sometimes, it is of interest to find the minimum input magnitude that provides perfect control of outputs; especially at steady-state or a non-zero frequency. When G is non-singular, an explicit solution to this problem is available in [8]. Ma et al. *[18] have extended these results to rank deficient G by requiring perfect control only for disturbances lying in the controllable subspace. Ma* et al. *[18] define the controllable subspace W as the subspace spanned by the sign-adjusted left singular vectors corresponding to non-zero singular values of G. While no restrictions are imposed on the controller in [18], the minimum input magnitude required to achieve perfect control in the controllable subspace using a linear feedback-based controller can be computed by replacing G and G_d by $W^H G$ and $W^H G_d$ in the optimization problem in (7).*

A further comparison is made for the blown film extruder example, where the linear feedback-based controller requires the same input magnitude for this problem. The corresponding discussion for this example is modified as follows:

For this process, it is not possible to achieve perfect control due to non-invertibility of G, even when arbitrarily large input variations are allowed. For example, for $k = 1, r = 0.7$, the minimum output error calculated using $\mathcal{L}_1/Q$ approach is 0.241*, when $\gamma_u = 3.429$ and increasing γ_u does not reduce γ_y further indicating a fundamental limitation. Though not possible for all disturbance directions, perfect control can be achieved for disturbances confined to their controllable subspace with inputs having magnitude equal to or larger than* 15.303 *[18]. Ma* et al. *[18] do not impose any restrictions on the controller. For this problem, however, a linear feedback-based controller provides same level of performance as the unrestricted controller showing that linear feedback-based controller is optimal; see also Remark 3.*

Reviewer 3

Comment: This paper proposes transforming of three minimax optimization problems of Skogestad & Wolff (1992) into the forms of L1-norm based linear programming. There is some contribution, and it may be made publishable.

Response/Action taken: We thank the reviewer for his comments.

Comment: At the top of page 6, the problem formulation corresponding to the optimization problems should be justified.

Response/Action taken: Among the three cases (steady-state, frequency-wise and dynamic systems) considered in this paper, the case of dynamic systems is clearly most general. The steady-state and frequency-wise versions of the problem are useful for analyzing the asymptotic disturbance rejection capabilities of the system for constant and sinusoidal disturbances, respectively. This has been clearly indicated in Section 2.1 of the revised manuscript as:

Each of these problems may be formulated for the following three cases:

- *(a) Steady-state*
- *(b) Frequency-wise*
- *(c) Dynamic systems*

While the case of dynamic systems is general, the steady-state and frequency-wise formulations are useful for analyzing the capability of the system in asymptotically rejecting constant and sinusoidal disturbances, respectively.

Comment: The first sentence of Section 2.2 is questionable. Note that u=k(r-y-n) is true for a unity feedback control structure (e.g., see (2.15) in Skogestad & Postlewaite, second edition, 2005). Such exercise should be clarified.

Response/Action taken: There is no loss of generality in assuming that the setpoints r are zero. To clarify this issue, the first paragraph of Section 2.2 is modified as follows:

For regulatory control, we consider that $u = -Ky$ (negative feedback). Though the setpoints r are considered to be zero, the results can be extended to include the case of non-zero setpoints by replacing y by $e = y - r$ in the following discussion. Now, the following relationships hold, $\cdots$

The consideration of noise becomes important, when only the measurements corrupted by noise $(y + n)$ are available for feedback control and the performance is measured in terms of y. While this issue is important, the extension of results to include the effect of noise is non-trivial. Note that the effect of noise has not been considered by other related publications as well. That we are using uncorrupted measurements is clear from the expression $u = -Ky$ and no further discussion on this issue is deemed necessary.

Comment: In Section 2.3, the referred feedforward control structure should be illustrated. Note that the following transfer function matrix cannot be derived from a conventional feedforward control structure.

Response/Action taken: Based on the comments of Reviewer 1, Section 2.3 has been removed from the revised manuscript.

Comment: In the numerical examples, each process model should be illustrated with the operation conditions/constraints. The reasons for using your proposed optimization bounds should be given/justified. State also the purposes of showing the two tables.

Response/Action taken: Most of the numerical examples (except the last example) considered in this paper are taken from published literature and the appropriate references are cited. The reader can easily refer to the original papers to get details regarding the operating conditions and constraints and repeating the same information in the present manuscript will unnecessarily increase the length. With this view, the additional details are not included in the revised manuscript.

Comment: 1_{d_n} in Eq.(2) and Eq.(4) should be changed to be a row vector.

Response/Action taken: All the vectors are denoted as column vectors in this manuscript, *e.g.* α_v, β_v. Changing the vector of 1's to a row vector will only cause confusion. Similarly, changing all the vectors to row vectors will only make the expressions look messier. Thus, the vectors of 1's are kept as column vectors.

Comment: 'S=(1-GK)-1'in page 7 should be typed as 'S=(I-GK)-1'. So is for other expressions involving the identity matrix.

Response/Action taken: We thank the review for pointing out this typographical error. There were several such errors in Section 2.2, which have been corrected in the revised manuscript.

Comment: The sequential number of simulation examples should begin from 1.

Response/Action taken: We have no control over the counter for examples (and other environments), which is set by the elsevier class file automatically. Hence, no changes are made.

Comment: The readability of References is poor. They should be organized in terms of the journal publication requirements.

Response/Action taken: The bibliography has been revised such that it complies with the journal publication requirements.

Editor

In addition to the changes arising due to the issues raised by the reviewers, the following changes have been made in the revised manuscript:

1. The affiliation of the first author is changes to his present affiliation.
2. In Section 2.1, all the signal norms in the minimax formulation have been corrected to $\|\cdot\|_\infty$, which indicates the peak norm. In the original manuscript, some of these norms were incorrectly denoted as $\|\cdot\|_{\mathcal{L}_1}$ due to typographical errors.
3. There was a typographical error in the optimization problem formulation for finding the largest allowable disturbance using the $\mathcal{L}_1/Q$ approach. The error has been corrected in the revised manuscript as follows:

$$\begin{aligned} \min_{Q} \quad & \sigma \\ \text{s.t.} \quad & \|(I - G\,Q)\,G_d\|_{\mathcal{L}_1} \leq \gamma_y\,\sigma \\ & \|Q\,G_d\|_{\mathcal{L}_1} \leq \gamma_u\,\sigma \end{aligned}$$

   For σ^* solving the above optimization problem, the magnitude of the largest allowable disturbance is given as $1/\sigma^*$.

4. The computational efficiency of the proposed method is better than reported in the original manuscript. For the blown film extruder example, the approach requires at most 3 seconds and not 3 seconds for the different cases. The revised manuscript reads

   We also point out that the $\mathcal{L}_1/Q$ approach requires at most 3 seconds for solving the different cases on a Pentium IV 3.2 GHz PC, showing computational efficiency.

   A similar change is also made for the Tennessee Eastman problem.

$\mathcal{L}_1/Q$ Approach for Efficient Computation of Disturbance Rejection Measures for Feedback Control ⋆

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Abstract

This paper presents practical methods for computation of disturbance rejection measures [1], which are useful for assessing the dynamic operability of the process. Using $\mathcal{L}_1$ optimal control theory, we consider the cases of steady-state, frequency-wise and dynamic systems. In comparison to the available methods [2–4], the proposed approach ensures that a linear, causal, feedback-based controller exists that achieves the computed bounds and the method also scales well with problem dimensions.

Key words: Controllability analysis, Disturbance rejection, Optimal control.

⋆ A preliminary version of this work was presented at the annual meeting of American Institute of Chemical Engineers held in Cincinnati, OH, USA, 2005 [5].

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1 Introduction

The achievable control quality ("controllability") is limited by the plant itself, independent of the controller design algorithm. A key issue in the controllability analysis is to decide upfront if there exists a controller that can reduce the effect of disturbances to an acceptable level with the available manipulated variables. When such a controller exists, the process is said to have "operability" [6]. A closely related problem is that of "flexibility" [7]. Skogestad and Wolff [1] also introduced some measures for judging upon the disturbance rejection capabilities of the process, which is similar to the dynamic operability. All these papers considered the following issue: *Is it possible to keep the outputs within their allowable bounds for the worst possible combination of disturbances, while still keeping the manipulated variables within their physical bounds?*

In this paper, we consider the disturbance rejection measures for linear time-invariant systems. For this case, Skogestad and Wolff [1] introduced the following problems:

(1) What is the minimum output error achievable with the bounded manipulated variables for the worst possible combination of disturbances?

(2) What is the minimum control effort or magnitude of the manipulated variables required to obtain an acceptable output error for the worst possible combination of disturbances?

(3) What is the largest possible magnitude of disturbances such that for the worst possible combination of disturbances upto that magnitude, an acceptable output error is achievable with the bounded manipulated vari-

ables?

These problems have been solved on a frequency-by-frequency basis for SISO systems and also approximately for MIMO systems [8; 9]. Hovd *et al.* [3] provide an exact solution for the steady-state version of this problem, but require solving a non-convex bilinear program. Kookos and Perkins [4] present an integer programming based formulation of the steady-state version of this problem. As pointed by Hovd and Kookos [2], the latter formulation scales better with problem dimensions. Recently, Hovd and Kookos [2] also presented a method for calculating the upper and lower bounds on the minimum output error on a frequency-by-frequency basis. This approach, however, is computationally very demanding. In summary, the available solutions for disturbance rejection problem hold only for restrictive versions of the problem (steady-state or frequency-by-frequency) and are computationally expensive.

In this paper, we consider the same problems under the assumption that the manipulated variables are generated using a linear, causal, feedback-based controller. Note that in the original problem formulation [8; 9], no such assumptions are made. An objective of controllability analysis is to judge upon the existence of controllers that can satisfy the desired performance requirements. In this sense, the restriction on the controller is necessary for practical controllability analysis. Under these assumptions, the calculation of disturbance rejection measures can be treated using $\mathcal{L}_1$-optimal control theory, which results in solving convex programs [10]. This approach yields the optimal controller and also scales well with problem dimensions. In this paper, we consider the steady-state, frequency-wise and dynamic cases in turn.

Notation. We let the linear, causal plant and disturbance models be $G(s)$

and $G_d(s)$, respectively such that

$$y(s) \;=\; G(s)\,u(s) \;+\; G_d(s)\,d(s)$$

where $y(s)$ is the output, $u(s)$ is the input and $d(s)$ is the disturbance. For simplicity, we use the same symbols for the signals and their Laplace transforms. We deal with peak norm of signals defined as

$$\|y(t)\|_\infty \;=\; \max_t \max_i |y_i(t)|$$

and induced $\mathcal{L}_1$-norm (peak to peak) of transfer matrices given as [10]

$$\|G(s)\|_{\mathcal{L}_1} \;=\; \sup_{\|u(t)\|_\infty \neq 0} \frac{\|G\,u(t)\|_\infty}{\|u(t)\|_\infty} \;=\; \sup_{\|u(t)\|_\infty = 1} \|G\,u(t)\|_\infty$$

When dealing with constant real or complex valued matrices, the $\mathcal{L}_1$-norm reduces to the matrix 1-norm, which is defined as maximum absolute row sum. In the following discussion, we drop the frequency argument s and time argument t, where no confusion can arise. For the given matrix, $A \in \mathbb{R}^{m\times n}$, a_v denotes vectorized A as

$$a_v = [A_{11}\;\cdots\;A_{1n}\;\cdots A_{mn}]^T$$

2 Problem Formulation

In this section, we present the mathematical formulation of the problem for calculation of disturbance rejection measures. We first consider the exact problems posed by Skogestad and Wolff [1], which are usually of theoretical interest only. Next, we formulate the same problems under the practical assumption that the controller is rational, causal and feedback-based.

We consider that the model has been scaled such that the allowable magnitudes of the peak values of output error, manipulated variables and disturbances are γ_y, γ_u and 1, respectively. A procedure for such scaling has been outlined by Skogestad and Postlethwaite [8]. Then, the three problems introduced by Skogestad and Wolff [1] require solving the following minimax optimization problems [1; 3; 4]

(1) Minimum output error:

$$\gamma_{y,\min} = \max_{\|d\|_\infty \le 1} \min_{\|u\|_\infty \le \gamma_u} \|G\,u + G_d\,d\|_\infty \tag{1}$$

(2) Required input magnitude:

$$\gamma_{u,\min} = \max_{\|d\|_\infty \le 1} \min_{\|G\,u + G_d\,d\|_\infty \le \gamma_y} \|u\|_\infty \tag{2}$$

(3) Largest allowable disturbance:

$$\begin{aligned} \gamma_{d,\max} &= \max\ \sigma \\ &\text{s.t.} \max_{\|d\|_\infty \le \sigma} \min_{\|u\|_\infty \le \gamma_u} \|G\,u + G_d\,d\|_\infty \le \gamma_y \end{aligned} \tag{3}$$

Remark 1 *The formulation in* (3) *is equivalent to finding the largest scaling for disturbances such that the achievable output error becomes* γ_y*. It may seem that the largest allowable disturbance can be found as*

$$\gamma'_{d,\max} = \max_{\substack{\|G\,u + G_d\,d\|_{\mathcal{L}_1} \le \gamma_y \\ \|u\|_\infty \le \gamma_u}} \|d\|_\infty \tag{4}$$

The formulation in (4)*, however, only finds "a" disturbance having magnitude* $\gamma'_{d,\max}$ *such that the bounds on outputs and inputs can be maintained, but does not ensures the outputs and inputs can be kept bounded for "all" disturbances having magnitude smaller than or equal to* $\gamma'_{d,\max}$*. The subtle difference between the two different formulations in* (3) *and* (4) *is illustrated by Hovd et al. [3]*

using an insightful example.

Each of these problems may be formulated for the following three cases:

(a) Steady-state

(b) Frequency-wise

(c) Dynamic systems

While the case of dynamic systems is general, the steady-state and frequency-wise formulations are useful for analyzing the capability of the system in asymptotically rejecting constant and sinusoidal disturbances, respectively. Nevertheless, these problems are difficult to solve due to their minimax nature and different approaches for solving the steady-state [3; 4] and frequency-by-frequency [2] versions have been proposed. In general, these approaches, however, can be computationally very demanding. Also note that these problems pose no restrictions on the controller. Thus achieving these bounds may require non-causal controllers with the knowledge of future disturbances. In this paper, we provide a method for computing the solution to all the problems mentioned above. The proposed methods provide a bound (upper bound on $\gamma_{y,\min}$ and $\gamma_{u,\min}$ and lower bound for $\gamma_{d,\max}$) for the "original" problem with no restrictions on the controller and exact solution for the case of a linear feedback-based causal controller.

Remark 2 *The steady-state and frequency-wise cases only consider the asymptotic behavior of the closed-loop system and the issue of non-causality does not arise. For these cases, the solution obtained using the minimax approach can be practically implemented, but this may require use of online optimization-based controller.*

For regulatory control, we consider that $u = -Ky$ (negative feedback). Though the setpoints r are considered to be zero, the results can be extended to include the case of non-zero setpoints by replacing y by $e = y - r$ in the following discussion. Now, the following relationships hold,

$$\begin{aligned} y &= S\, G_d\, d \\ u &= -K\, S\, G_d\, d \end{aligned}$$

where $S = (I + G\,K)^{-1}$ is the sensitivity function. Next, we use the Youla parametrization of all stabilizing controllers, where G is considered to be stable for simplicity. When the process is unstable, similar coprime factorization based parametrization can be used; see *e.g.* [8]. Parameterizing K as $K = Q\,(I - G\,Q)^{-1}$,

$$\begin{aligned} y &= (I - G\,Q)\, G_d\, d \\ u &= -Q\, G_d\, d \end{aligned}$$

where Q is a stable rational transfer function. Now, the three problems introduced above require solving

(1) Minimum output error:

$$\begin{aligned} &\min_{Q} \quad \|(I - G\,Q)\, G_d\|_{\mathcal{L}_1} \\ &\text{s.t.} \quad \|Q\, G_d\|_{\mathcal{L}_1} \leq \gamma_u \end{aligned} \tag{5}$$

(2) Required input magnitude:

$$\begin{aligned} &\min_{Q} \quad \|Q\, G_d\|_{\mathcal{L}_1} \\ &\text{s.t.} \quad \|(I - G\,Q)\, G_d\|_{\mathcal{L}_1} \leq \gamma_y \end{aligned} \tag{6}$$

(3) Largest allowable disturbance:

$$\begin{aligned} \min_{Q} \quad & \sigma \\ \text{s.t.} \quad & \|(I - G\,Q)\,G_d\|_{\mathcal{L}_1} \leq \gamma_y\,\sigma \\ & \|Q\,G_d\|_{\mathcal{L}_1} \leq \gamma_u\,\sigma \end{aligned} \tag{7}$$

For σ^* solving the optimization problem in (7), the magnitude of the largest allowable disturbance is given as $1/\sigma^*$.

The formulation of these optimization problems using $\mathcal{L}_1$-optimal control theory is along the same lines as done by Dahleh and Diaz-Bobillo [10]. The problem of computing achievable $\|y\|_2$ for specified disturbances (*e.g.* step-type) using the Youla parameterization was also considered by Swartz [11; 12]. The approach taken here considers the time-domain bounds characterized by $\|y\|_\infty$ directly and also allows for the worst possible combination of disturbances, as is relevant for computing disturbance rejection measures.

Remark 3 *Sometimes, it is of interest to find the minimum input magnitude that provides perfect control of outputs; especially at steady-state or a non-zero frequency. When G is non-singular, an explicit solution to this problem is available in [8]. Ma* et al. *[18] have extended these results to rank deficient G by requiring perfect control only for disturbances lying in the controllable subspace. Ma* et al. *[18] define the controllable subspace W as the subspace spanned by the sign-adjusted left singular vectors corresponding to non-zero singular values of G. While no restrictions are imposed on the controller in [18], the minimum input magnitude required to achieve perfect control in the controllable subspace using a linear feedback-based controller can be computed by replacing G and G_d by $W^H G$ and $W^H G_d$ in the optimization problem in* (6)*.*

In the following discussion, we only consider the minimum output error problem (problem 1) in detail and the formulations for the remaining two problems can be obtained similarly. Before dealing with the dynamic systems, we first deal with the steady-state and frequency-dependent versions of disturbance rejection problems. The reason for detailed discussion of the steady-state version of the problem is that its formulation is similar to the corresponding formulation for discrete-time dynamic systems, which facilitates the introduction of more involved expressions.

2.2.1 Steady-state

We recall that for constant matrices, the $\mathcal{L}_1$-norm reduces to the matrix 1-norm. Now, let Q be vectorized as

$$q_v = \left[Q_{11} \cdots Q_{1n_y} \cdots Q_{n_u n_y}\right]^T$$

Then, $\|Q\, G_d\|_1 \leq \gamma_u$ is equivalent to

$$-\alpha_v \leq (I_{n_u} \otimes G_d^T)\, q_v \leq \alpha_v \tag{8}$$
$$(I_{n_u} \otimes 1_{n_d})\, \alpha_v \leq \gamma_u \cdot 1_{n_u} \tag{9}$$

where $\otimes$ is the Kronecker tensor product and 1_{n_u} is an n_u dimensional column vector of 1's. Similarly, $\|G_d - G\, Q\, G_d\|_1 \leq \gamma_y$ is equivalent to

$$-\beta_v \leq (G_d)_v \, - \, (G \otimes G_d^T)\, q_v \leq \beta_v \tag{10}$$
$$(I_{n_y} \otimes 1_{n_d})\, \beta_v \leq \gamma_y \cdot 1_{n_y} \tag{11}$$

In (8) and (10), $\alpha_v \in \mathbb{R}^{n_u \cdot n_d}$ and $\beta_v \in \mathbb{R}^{n_y \cdot n_d}$ bound the absolute values of the elements of $Q\, G_d$ and $(G_d - G\, Q\, G_d)$, respectively. Similarly, in (9) and (11), the sum of the absolute values of the elements of each row (arising due to matrix

1-norm) of $Q\,G_d$ and $(G_d - G\,Q\,G_d)$ are bounded by γ_u and γ_y, respectively. Define $x = [q_v^T \quad \alpha_v^T \quad \beta_v^T \quad \gamma_y]^T$. Then, in the standard linear program form the minimum output error is determined by solving

$$
\begin{aligned}
\min_x \quad & \begin{bmatrix} 0 \; 0 \; 0 \; 1 \end{bmatrix} x \\
\text{s.t.} \quad & \begin{bmatrix} I_{n_u} \otimes G_d^T & -I & 0 & 0 \\ -I_{n_u} \otimes G_d^T & -I & 0 & 0 \\ 0 & I_{n_u} \otimes 1_{n_d} & 0 & 0 \\ -G \otimes G_d^T & 0 & -I & 0 \\ G \otimes G_d^T & 0 & -I & 0 \\ 0 & 0 & I_{n_y} \otimes 1_{n_d} & -1_{n_y} \end{bmatrix} x \leq \begin{bmatrix} 0 \\ 0 \\ \gamma_u \cdot 1_{n_u} \\ -(G_d)_v \\ (G_d)_v \\ 0 \end{bmatrix} \\
& \begin{bmatrix} -\infty \; 0 \; 0 \; 0 \end{bmatrix} \leq x \leq \infty
\end{aligned}
$$

The above linear program is sparse with $(n_u\,n_y + n_u\,n_d + n_y\,n_d + 1)$ variables and $(2\,(n_u\,n_d + n_y\,n_d) + n_u + n_y)$ constraints. In this paper, we use Tomlab/CPlex [13] for solving this program. Note that for finding the required input magnitude, one only needs to change the roles of γ_u and γ_y in the above formulation.

2.2.2 Frequency-wise

We next consider the calculation of minimum output error on a frequency-by-frequency basis. As compared to the steady-state case, the additional complication is that the matrix 1-norm requires calculation of absolute values, which is non-linear for complex scalars in terms of its real and imaginary parts. To overcome this difficulty, Hovd and Kookos [2] suggest under and overestimating the peak norms of various signals using polyhedral approximations. Such an approximation, however, increases the computational requirements considerably, especially when the approximation error is required to be small. In the following discussion, we show that under the $\mathcal{L}_1/Q$ approach, the calculation of minimum output error can be posed as a convex program. The formulation is based on the observation that though non-linear, the absolute value of a complex scalar can be bounded using a linear matrix inequality (LMI) [14].

We recall that the vectorized format of $Q\,G_d$ is given as $(I_{n_u} \otimes G_d^T)q_v$. Let $[(I_{n_u} \otimes G_d^T)]_{i*}$ denote the i^{th} row of $(I_{n_u} \otimes G_d^T)$. Then the magnitude of the elements of $Q\,G_d$ can be bounded as

$$\begin{aligned}
&\left|[(I_{n_u} \otimes G_d^T)]_{i*}\, q_v\right| \leq [\alpha_v]_i \\
\Leftrightarrow\, &|a_i + j\, b_i| \leq [\alpha_v]_i \\
\Leftrightarrow\, &\begin{bmatrix} a_i & b_i \end{bmatrix} \begin{bmatrix} a_i & b_i \end{bmatrix}^T \leq [\alpha_v]_i^2 \\
\Leftrightarrow\, &\begin{bmatrix} [\alpha_v]_i & a_i & b_i \\ a_i & [\alpha_v]_i & 0 \\ b_i & 0 & [\alpha_v]_i \end{bmatrix} \geq 0
\end{aligned} \tag{12}$$

where the last equivalence is obtained using Schur complement lemma [14]. Here

$$\begin{aligned}
a_i &= \mathrm{Re}\,[(I_{n_u} \otimes G_d^T)]_{i*}\, \mathrm{Re}\, q_v - \mathrm{Im}\,[(I_{n_u} \otimes G_d^T)]_{i*}\, \mathrm{Im}\, q_v \\
b_i &= \mathrm{Re}\,[(I_{n_u} \otimes G_d^T)]_{i*}\, \mathrm{Im}\, q_v + \mathrm{Im}\,[(I_{n_u} \otimes G_d^T)]_{i*}\, \mathrm{Re}\, q_v
\end{aligned}$$

Similarly, the the magnitude of the elements of $(G_d - G\,Q\,G_d)$ can be bounded using the following LMI

$$\begin{bmatrix} [\beta_v]_j & c_j & d_j \\ c_j & [\beta_v]_j & 0 \\ d_j & 0 & [\beta_v]_j \end{bmatrix} \geq 0 \qquad (13)$$

where

$$\begin{aligned} c_j &= \operatorname{Re}[(G_d)_v]_j - \operatorname{Re}[(G \otimes G_d^T)]_{j*} \operatorname{Re} q_v + \operatorname{Im}[(G \otimes G_d^T)]_{j*} \operatorname{Im} q_v \\ d_j &= \operatorname{Im}[(G_d)_v]_j - \operatorname{Re}[(G \otimes G_d^T)]_{j*} \operatorname{Im} q_v - \operatorname{Im}[(G \otimes G_d^T)]_{j*} \operatorname{Re} q_v \end{aligned}$$

Now, by defining $z = [\operatorname{Re} q_v^T \quad \operatorname{Im} q_v^T \quad \alpha_v^T \quad \beta_v^T \quad \gamma_y]^T$, the problem requires solving

$$\begin{aligned} \min_z \quad & \begin{bmatrix} 0 \; 0 \; 0 \; 0 \; 1 \end{bmatrix} z \\ \text{s.t.} \quad & \begin{bmatrix} 0 \; 0 \; I_{n_u} \otimes 1_{n_d} & 0 & 0 \\ 0 \; 0 \quad 0 & I_{n_y} \otimes 1_{n_d} & -1_{n_y} \end{bmatrix} z \leq \begin{bmatrix} \gamma_u \cdot 1_{n_u} \\ 0 \end{bmatrix} \\ & (12) \text{ for } i = 1, 2, \cdots n_u\, n_d \\ & (13) \text{ for } j = 1, 2, \cdots n_y\, n_d \\ & \begin{bmatrix} -\infty \; -\infty \; 0 \; 0 \; 0 \end{bmatrix} \leq z \leq \infty \end{aligned}$$

In this paper, we use the software package Tomlab/PenSDP [13] using the interface Yalmip [15] for solving this semi-definite program.

For continuous-time systems, the computation of $\mathcal{L}_1$-norm is difficult and the problems involving this norm are almost exclusively solved using discretized models. For the discrete-time univariable system $g(z^{-1})$, the $\mathcal{L}_1$-norm is given as

$$\|g(z^{-1})\|_{\mathcal{L}_1} = \sum_{i=1}^{\infty} |g_i|$$

where g_i is the i^{th} impulse response coefficient and z^{-1} is the backshift operator. In practice, finite impulse response (FIR) models are used and a method for selecting the order of the FIR model is given by Dahleh and Diaz-Bobillo [10]. For the multivariable system $G(z^{-1})$ with n_y outputs and n_u inputs, the $\mathcal{L}_1$-norm is

$$\|G(z^{-1})\|_{\mathcal{L}_1} = \left\| \begin{matrix} \|G_{11}(z^{-1})\|_{\mathcal{L}_1} & \cdots & \|G_{1n_u}(z^{-1})\|_{\mathcal{L}_1} \\ \vdots & \cdots & \vdots \\ \|G_{n_y1}(z^{-1})\|_{\mathcal{L}_1} & \cdots & \|G_{n_yn_u}(z^{-1})\|_{\mathcal{L}_1} \end{matrix} \right\|_1$$

With this minor detour, we next formulate the linear programming problem that can be used for calculating the minimum output error. We consider that for $G(z^{-1})$ and $G_d(z^{-1})$, FIR models of order N are used, whereas the order of FIR model of $Q(z^{-1})$ is N_Q with $N_Q \leq N$. Note that the order of the decision variable $Q(z^{-1})$ is difficult to determine *a priori* and in practice, N_Q can be increased sequentially, until convergence.

For posing this problem as a standard linear program, we need to vectorize the impulse response coefficients of $Q(z^{-1})\,G_d(z^{-1})$ and $G_d(z^{-1}) - G(z^{-1})\,Q(z^{-1})\,G_d(z^{-1})$.

For this purpose, it is useful to represent these impulse response coefficients using matrix notation. The impulse response coefficients of $Q(z^{-1})\,G_d(z^{-1})$ are given as

$$\begin{bmatrix} (Q\,G_d)_1 \\ (Q\,G_d)_2 \\ \vdots \\ (Q\,G_d)_N \end{bmatrix} = \begin{bmatrix} Q_1 & 0 & \cdots & \cdots & 0 \\ Q_2 & Q_1 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 \\ 0 & \cdots & Q_{N_Q} & \cdots & Q_1 \end{bmatrix} \begin{bmatrix} G_{d,1} \\ G_{d,2} \\ \vdots \\ G_{d,N} \end{bmatrix}$$

which can be vectorized as

$$\begin{bmatrix} I_{n_u} \otimes G_{d,1}^T & 0 & \cdots & 0 \\ I_{n_u} \otimes G_{d,2}^T & I_{n_u} \otimes G_{d,1}^T & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ I_{n_u} \otimes G_{d,N}^T & \cdots & \cdots & I_{n_u} \otimes G_{d,(N-N_Q+1)}^T \end{bmatrix} \begin{bmatrix} (Q_1)_v \\ (Q_2)_v \\ \vdots \\ (Q_{N_Q})_v \end{bmatrix} \tag{14}$$

where

$$(Q_i)_v = \left[Q_{i,11} \,\cdots\, Q_{i,1n_y} \,\cdots Q_{i,n_u n_y}\right]^T$$

Similarly, impulse response coefficients of $G_d(z^{-1}) - G(z^{-1})\,Q(z^{-1})\,G_d(z^{-1})$

can be vectorized as

$$\begin{bmatrix} (G_{d,1})_v \\ (G_{d,2})_v \\ \vdots \\ (G_{d,N})_v \end{bmatrix} - \begin{bmatrix} G_1 \otimes G_{d,1}^T & 0 & \cdots & 0 \\ G_1 \otimes G_{d,2}^T + G_2 \otimes G_{d,1}^T & G_1 \otimes G_{d,1}^T & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ \sum_{i=1}^{N} G_i \otimes G_{d,(N-i+1)}^T & \cdots & \cdots & \sum_{i=1}^{N-N_Q+1} G_i \otimes G_{d,(N-N_Q-i+2)}^T \end{bmatrix} \begin{bmatrix} (Q_1)_v \\ (Q_2)_v \\ \vdots \\ (Q_{N_Q})_v \end{bmatrix} \tag{15}$$

Using the vectorized impulse response coefficients in (14) and (15), and manipulations similar to (8)-(11), the problem of calculating the minimum output error for dynamic systems is same as the steady-state case with $N\,(n_u\,n_d + n_y\,n_d) + N_Q\,n_u\,n_y + 1$ variables and $(2\,N\,(n_u\,n_d + n_y\,n_d) + n_u + n_y)$ constraints. The steady-state case was handled in Section 2.2.1 and the details are not repeated.

Remark 4 *When the process has unstable zeros, these zeros give rise to interpolation constraints. The interpolation constraints ensure that there are no unstable pole-zero cancelations and thus internal stability is maintained. When the closed-loop system is internally unstable, some of the signals become unbounded (*e.g. *u), while others may remain within bounds (*e.g. *y). In this paper, we do not explicitly include the interpolation constraints, as the signals* *are bounded by finite* γ_y *and* γ_u*. In some limiting cases of theoretical interest (*e.g. *cheap control), there is no finite upper bound on the magnitudes of some signals and interpolation constraints need to be taken into account. This problem can also be approximately handled by using large but finite values for* γ_y *or* γ_u *in the proposed approach.*

3 Examples

In this section, we consider a number of process examples taken from the literature to illustrate the concepts discussed in this paper. Some of these examples were earlier considered using the minimax formulation in [2–4].

Example 5 *We first discuss the calculation of minimum output error for blown film extruder earlier considered by Hovd* et al. *[3]. This process has* 15 *inputs,* 15 *outputs and* 15 *disturbances, where the steady-state gain matrices of* G *and* G_d *are circulant matrices with* G *being rank deficient. The disturbance model* G_d *is parameterized by* k, r*, which defines the spatial correlation among different variables.*

[Table 1 about here.]

The minimum output error calculated using the bilinear formulation by Hovd et al. *[3] and the* $\mathcal{L}_1/Q$ *approach are shown in Table 1, where* $\gamma_u = 1$*. Hovd* et al. *[3] do not impose any restrictions on the controller structure and the controller that achieves the bounds presented by them can be a nonlinear or on-line optimization-based controller. In comparison, the* $\mathcal{L}_1/Q$ *approach provides the optimal linear controller that achieves the practical bounds. The results in Table 1 show that there is no significant performance loss in using a linear* *feedback-based controller as compared to an online-optimization based or non-linear controller for this process, at least for asymptotic rejection of constant disturbances. We also point out that the $\mathcal{L}_1/Q$ approach requires at most* 3 *seconds for solving the different cases on a Pentium IV 3.2 GHz PC, showing computational efficiency.*

For this process, it is not possible to achieve perfect control due to non-invertibility of G, even when arbitrarily large input variations are allowed. For example, for $k = 1, r = 0.7$, the minimum output error calculated using $\mathcal{L}_1/Q$ approach is 0.241*, when $\gamma_u = 3.429$ and increasing γ_u does not reduce γ_y further indicating a fundamental limitation. Though not possible for all disturbance directions, perfect control can be achieved for disturbances confined to their controllable subspace with inputs having magnitude equal to or larger than* 15.303 *[18]. Ma* et al. *[18] do not impose any restrictions on the controller. For this problem, however, a linear feedback-based controller provides same level of performance as the unrestricted controller showing that linear feedback-based controller is optimal; see also Remark 3.*

Example 6 *Next, we consider the calculation of largest allowable disturbance for the Tennessee Eastman process [16]. This example was earlier considered by Kookos and Perkins [4], where the original process was stabilized using a subset of variables. The stabilized process has* 5 *outputs,* 5 *inputs and* 7 *disturbances. The steady-state gain matrices for the stabilized process are available in [4].*

[Table 2 about here.]

The magnitude of the largest allowable disturbances computed using the integer programming formulation [4] are compared with the corresponding values calculated using the $\mathcal{L}_1/Q$ approach for different combinations of disturbances *in Table 2. Note that due to the practical assumption of a linear feedback-based controller, the allowable disturbance magnitude calculated using the $\mathcal{L}_1/Q$ approach is lower than the minimax formulation, which allows for non-linear and online optimization-based controllers. In all cases, the disturbance magnitude calculated using the two alternate approaches is reasonably close. The integer programming formulation requires about 0.5 seconds for solving this problem for the different disturbance scenarios [4]. In comparison, the $\mathcal{L}_1/Q$ approach requires at most* 0.03 *seconds for the different cases on a Pentium IV 3.2 GHz PC, showing computational efficiency and better scalability. We also note that the largest difference between the two approaches is seen, when only disturbance* d_6 *is considered. It can be shown easily that for single disturbance, the minimax and $\mathcal{L}_1/Q$ formulations are identical for the steady-state case. Then, the apparent difference is due to the typographical errors in [4].*

Example 7 *The previous two examples dealt with steady-state case only. Here, the usefulness of formulations for frequency-wise computation and dynamic systems is illustrated using fluid catalytic cracker (FCC) process earlier considered by Hovd and Kookos [2]. The unscaled dynamic model for this process is given by Wolff [9]. In this paper, we use the following scaling matrices such that the allowed disturbance magnitude is* 1.

$$D_y = \text{diag}\begin{pmatrix} 3 & 2 & 3 \end{pmatrix}; \quad D_u = \text{diag}\begin{pmatrix} 3 & 30 & 4.75\times 10^{-4} \end{pmatrix}; \quad D_d = \text{diag}\begin{pmatrix} 5 & 5 & 4 \end{pmatrix}$$

For this process, perfect control is possible at steady-state using a linear rational controller. Hovd and Kookos [2] made a similar observation using an integer programming formulation. Thus, there is no limitation in using a linear controller, at least at steady-state.

[Fig. 1 about here.]

Next, we consider the frequency-wise computation of minimum output error. Hovd and Kookos [2] presented lower and upper bounds on minimum output error using polyhedral approximations. The minimum output error calculated using the $\mathcal{L}_1/Q$ *approach, and the lower and upper bounds computed by Hovd and Kookos [2] are shown in Figure 1, where the close proximity of the solution obtained using* $\mathcal{L}_1/Q$ *approach and Hovd and Kookos's lower bound should be noted. Note that the* $\mathcal{L}_1/Q$ *approach gives an exact value if we require the controller to be linear, causal and feedback-based, but it provides an upper bound on the minimum output error in comparison to the minimax formulation. This happens as in* $\mathcal{L}_1/Q$ *approach, the controller and hence the manipulated variables are restricted, but the disturbances are still allowed to take all possible values (as the minimax formulation). This shows that for the FCC process, the lower bound computed by Hovd and Kookos [2] is tighter in comparison to the upper bound.*

[Fig. 2 about here.]

Finally, we consider the dynamic case. The continuous-time model is discretized using a sampling time of 2 *minutes, for which FIR models having order* $N = 150$ *suffice. The order of the Youla parameter* Q *is increased sequentially and no further improvements are seen for* $N_Q \geq 18$*. This results into a sparse linear program with* 2863 *variables and* 5406 *constraints. The variation of minimum output error for different values of* γ_u *is shown in Figure 2. It is interesting to note that the minimum output error reduces sharply for small increments in* γ_u *intially, but requires much larger increments in* γ_u *for similar reductions, as we get closer to the perfect control case. For exam-*

ple, when γ_u *is increased from* 0.5 *to* 1*, minimum output error decreases from* 7.4 *to* 0.55*. However, decreasing the minimum output error from* 0.1 *to* 0.05 *requires increasing* γ_u *from* 112.61 *to* 138.34 *(not shown in Figure 2).*

Example 8 *To demonstrate the effect of non-minimum phase zeros on the minimum output error, we consider*

$$G(z^{-1}) = \frac{0.05}{1+a}\,\frac{1+a\,z^{-1}}{1+0.5\,z^{-1}+0.25z^{-2}}; \quad G_d(z^{-1}) = \frac{0.5\,z^{-1}}{1+0.5\,z^{-1}} \tag{16}$$

where $G(z^{-1})$ *has a zero at* $z = -a$ *(non-minimum phase for* $a \geq 1$*). The process gain has been scaled by the factor* $(1+a)$ *such that it remains constant for all values of* a*. For this process, we use* $N = 300$ *and* $N_Q = 25$*. The variation of minimum output error with the location of the zero for* $\gamma_u = 1$ *is shown in Figure 3. For this case, the minimum output error only shows minor variations with zero location indicating that the non-minimum phase zero puts no serious limitations and the performance is primarily limited by the bound on the manipulated variable.*

[Fig. 3 about here.]

When γ_u *is increased to* 100*, the minimum output error remains close to* 0.5 *for minimum phase* $G(z^{-1})$*. In this case, the performance is limited by the unit time delay. By canceling the controller-dependent terms, as is usually done in minimum variance control literature [17], it can be analytically shown that* 0.5 *is the optimal value for minimum output error for the cheap control case. For non-minimum phase* $G(z^{-1})$*, the minimum output error is much larger as compared to the minimum phase* $G(z^{-1})$ *indicating that the limitation is due to unstable zero. It shall also be noted that when the zero recedes away from the unit disc, the limitation due to unstable zero decreases, as is usually the* *case [8].*

4 Conclusions

We used a Youla parametrization and $\mathcal{L}_1$ optimal control based ($\mathcal{L}_1/Q$) approach for practical and efficient computation of the disturbance rejection measures proposed by Skogestad and Wolff [1]. The approach taken in this paper is numerical and explicit (and possibly approximate) characterization of the limitations on the achievable output performance with bounded inputs is an issue for future research. To this end, the reader is referred to [18], where explicit conditions for judging the feasibility of perfect control are derived.

For the various numerical examples considered in this paper, it is found that a linear feedback-based controller can provide nearly the same level of performance as an online-optimization based controller for asymptotic rejection of constant and sinusoidal disturbances. In general, however, the use of a linear feedback-based controller can be conservative. Future research will focus upon extending the results of this paper to online-optimization based controllers, *e.g.* model predictive controllers.

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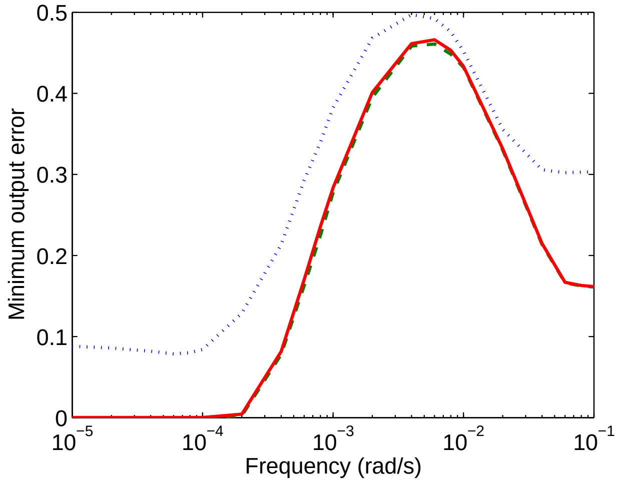

Fig. 1. Comparison of $\mathcal{L}_1/Q$ approach (solid) with upper (dots) and lower (dashed) bounds calculated by Hovd and Kookos [2] for FCC process (frequency-wise)

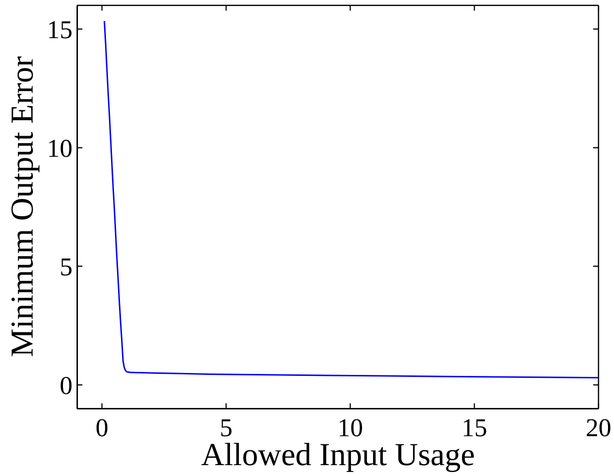

Fig. 2. Trade-off between $\|y\|_\infty$ and $\|u\|_\infty$ for the FCC process (dynamic case)

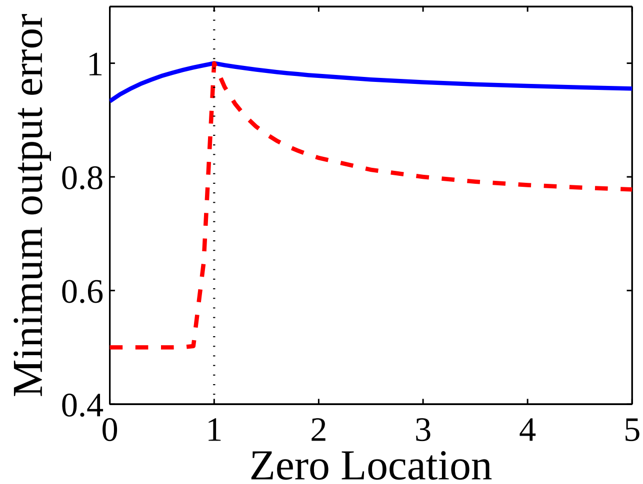

Fig. 3. Minimum output error as a function of zero location for non-minimum phase process with $\|u\|_\infty \leq 1$ (solid) and $\|u\|_\infty \leq 100$ (dash)

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Table 1
Comparison for minimum output error for blown film extruder (steady-state)

Case	Bilinear [3] (Online optimization)	$\mathcal{L}_1/Q$ approach (This work) (Linear feedback)
$k = 1, r = 0.7$	0.783	0.783
$k = 1, r = 0.3$	0.894	0.935
$k = 0.5, r = 0.3$	0.382	0.409

Table 2
Comparison for largest allowable disturbances for Tennessee Eastman process (steady-state)

Case	Integer Programming [4] (Online optimization)	$\mathcal{L}_1/Q$ approach (This work) (Linear feedback)
$d_1 - d_7$	0.392	0.385
d_1	0.601	0.601
d_2	1.273	1.273
d_6	1.302	1.231
d_7	3.368	3.368
d_1, d_2, d_7	0.393	0.386