The null space method for selecting optimal measurement combinations as controlled variables

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Abstract

The null space method provides a simple way of selecting measurement combinations as controlled variables. The objective is to obtain self-optimizing control, which is when we can achieve near-optimal steady-state operation with constant setpoints for the controlled variables, without the need to re-optimize when new disturbances perturb the plant. For small disturbances, the new method yields optimal controlled variables $\mathbf{c} = \mathbf{H}\mathbf{y}$ that are linear combinations of measurements $\mathbf{y}$. The matrix $\mathbf{H}$ is easily obtained to be in the left null space of the optimal sensitivity matrix. The requirement is that we at least have as many measurements as there are unconstrained degrees of freedom, including disturbances, and that the implementation error can be neglected.

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1 Introduction

Although not widely acknowledged by control theorists, controlling the right variables is a key element in overcoming uncertainty in operation [1;2]. This paper focuses on the

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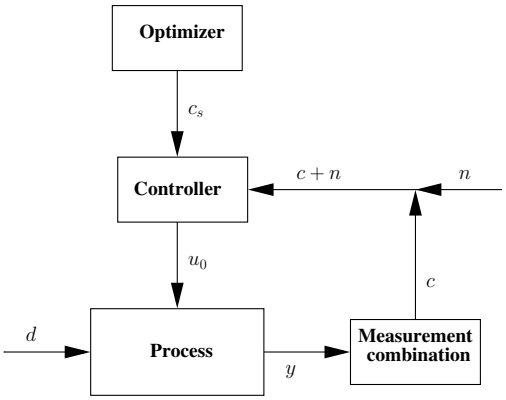

Figure 1: Block diagram of a feedback control structure including an optimizer layer.

interaction between the local optimization layer and the feedback control layer, see Figure 1, and more specifically on the selection of the controlled variables that link these layers. Two sub-problems are important here:

1. *Selection of the controlled variables* **c**: This is a structural decision which is made before implementing the control strategy.

2. *Selection of setpoints* $\mathbf{c}_s$: This is a parametric decision which can be done both online and offline.

Here, we focus on the first, structural problem of finding the controlled variables and we will assume constant nominal optimal setpoints. We will in this paper consider the case of linear combination, that is , $\mathbf{c} = \mathbf{H}\mathbf{y}$ where $\mathbf{H}$ is a constant matrix. From Figure 1 we see that external disturbances ($\mathbf{d}$) and the manipulated inputs ($\mathbf{u}_0$) affect the process, and information is available through the measurements $\mathbf{y}$. Single measurements or functions or combinations of the measurements may be used as controlled variables ($\mathbf{c}$). The objective is to obtain self-optimizing control [2], which is when we can achieve near-optimal steady-state operation with constant setpoints for the controlled variables, without the need to re-optimize when new disturbances perturb the plant. As seen from Figure 1, there are two sources of uncertainty that will make a constant setpoint policy non-optimal:

1. **Disturbances d**: External unmeasured disturbances.

2. **Implementation error n**: The sum of the effect of the measurement error and the control error introduced by the feedback loop.

Ideas related to self-optimizing control have been presented repeatedly in the process control history, but the first quantitative treatment was that of [1]. A quantitative method for selecting controlled variables requires that optimal operation is defined in terms of a scalar cost function J_0 to be minimized. A key point is that the controlled

variables are selected mainly in order to minimize the effect of uncertainty when *implementing* optimal operation. If we keep constant setpoints for the controlled variables, then there will be a loss $L = J_0 - J_0^{\text{opt}}$ compared to the truly optimal operation, and the objective is to find "self-optimizing" variables with a small loss.

Skogestad[2] defined the problem more carefully, linked it to previous work, and was the first to include also the implementation error. He mainly considered the case where single measurements are used as controlled variables, that is, $\mathbf{H}$ is a selection matrix where each row has a single 1 and the rest 0's. The loss for expected disturbances and implementation errors was evaluated using a "brute-force" approach. An important advantage of a brute-force evaluation is that one can also identify controlled variables that may yield infeasability for certain disturbances or implementation errors. This was also considered in more detail by[3] for the Tennessee-Eastman challenge problem and [4] who suggested to adjust the setpoints to achieve feasibility. However, the computational load of the "brute-force" method can be very large, so local methods based on linearizing the behavior around the steady-state are attractive.

Skogestad[2] reviewed previous work on the selection of controlled variables for self-optimizing control.

More recent work includes[2], which introduces the approximate maximum gain rule as a simple method for selecting controlled variables. In the multivariable case, the gain is the minimum singular value of the scaled steady-state transfer matrix from $\mathbf{u}$ to $\mathbf{c}$. A similar simple heuristic rule is presented in[5].

Halvorsen et al.[6] considered the approximate maximum gain method in more detail and also proposed an exact local method which may be used to obtain the optimal measurement combination $\mathbf{H}$. However, this method is also computationally unattractive and in addition somewhat difficult to use.

Hori et al.[7] illustrate the ideas introduces in this paper on indirect control. Indirect control can be formulated as a subproblem of the null space method presented in this paper.

Related work is presented in[8;9;10] on measurement based optimization to enforce the necessary condition of optimality under uncertainty. The ideas are illustrated on batch processes. Francois et al.[11] extends these ideas and focus on steady-state optimal systems, where a clear distinction is made between enforcing active constraints and requiring the sensitivity of the objective to be zero.

Guay and Zhang[12] present similar ideas on measurement based dynamic optimization, no process model is necessary, however, the structure of the objective function must be known in addition to the introduction of an external and known signal in the process in order to find the optimal inputs to use under the influence of uncertainty.

The objective of this paper, is to derive a much simpler local method for selecting the optimal measurement combination $\mathbf{H}$ for the special case with no implementation error. In fact, the method is so simple that the second author (Skogestad) assumed must be wrong when it was originally proposed by the first author (Alstad). We have in this chapter attempted to keep the mathematics as simple as possible. A more detailed comparison with previous results is presented in a forthcoming publication (see also [13]).

2 Problem formulation

We assume that the operational goal is to minimize the cost J_0 while satisfying equality and inequality constraints. The (original) constrained steady state optimization problem can, for a given disturbance $\mathbf{d}$ be formulated as:

$$\min_{\mathbf{x}_0,\mathbf{u}_0} J_0(\mathbf{x}_0, \mathbf{u}_0, \mathbf{d}) \tag{1}$$

subject to

$$\begin{aligned} &\mathbf{f}(\mathbf{x}_0, \mathbf{u}_0, \mathbf{d}) = 0 \\ &\mathbf{g}(\mathbf{x}_0, \mathbf{u}_0, \mathbf{d}) \leq 0 \\ &\mathbf{y} = \mathbf{f}_y(\mathbf{x}_0, \mathbf{u}_0, \mathbf{d}) \end{aligned} \tag{2}$$

where $\mathbf{x} \in \mathbb{R}^{n_x}$, $\mathbf{u}_0 \in \mathbb{R}^{n_{u_0}}$ and $\mathbf{d} \in \mathbb{R}^{n_d}$ are the states, inputs and disturbances, respectively. $\mathbf{f}$ is the set of equality constraints corresponding to the model equation, $\mathbf{g}$ is the set of inequality constraints which limits the operation, e.g. physical limits on temperature measurements or flow constraints and $\mathbf{y}$ the measurements.

We assume here that we control all active constraints (assumption **A3** below). Thus, we split the original input vector $\mathbf{u}_0$ (degrees of freedom) into:

- $\mathbf{u}'$: vector of degrees of freedom used for controlling the active constraints.
- $\mathbf{u}$: vector of remaining degrees of freedom with dimension n_u not used for active constraints.

Remark. It does not actually matter how the original degrees of freedom $\mathbf{u}_0$ are divided into the new subsets of manipulated variables selected for controlling the active constraints ($\mathbf{u}'$) and the "unconstrained" inputs $\mathbf{u}$, as long as the problem remains well posed. If all the inputs are used for controlling the active constraints, $\mathbf{u}' = \mathbf{u}_0$, then implementation is simple by the use of active constraint control [14;15].

We assume that online information about the system behavior is available from the measurements $\mathbf{y}$: The issue in this paper is find a set of n_u controlled variables $\mathbf{c} = \mathbf{H}(\mathbf{y})$ associated with the "unconstrained" degrees of freedom $\mathbf{u}$. In the measurement vector $\mathbf{y}$, we generally include also the input vector $\mathbf{u}_0$, including the inputs $\mathbf{u}'$ that have been selected to the control active constraints. However, the measurements of the active constraints are not included in $\mathbf{y}$, since they are constant and thus provide no information about the operation.

With the active constraints controlled, we can consider the following *unconstrained* optimization problem where the scalar cost function J is to be minimized with respect to the n_u remaining degrees of freedom (inputs) $\mathbf{u}$:

$$\min_{\mathbf{u}} J(\mathbf{u}, \mathbf{d}) \tag{3}$$

Here the equality constraints, including the model equations and active constraints, are implicitly included, so J is generally not a simple function of $\mathbf{u}$ and $\mathbf{d}$.

The loss is defined as the difference between the actual cost and the optimal cost[6].

$$L = J(\mathbf{c}, \mathbf{d}) - J(\mathbf{c^{opt}}(\mathbf{d}), \mathbf{d}) \approx \frac{1}{2}(\mathbf{c} - \mathbf{c}^{opt})^T \mathbf{J}_{cc}(\mathbf{c} - \mathbf{c}^{opt}) \tag{4}$$

where the approximation holds for small deviations from the nominal optimum. We must assume the controlled variables to be independent for the Hessian matrix $\mathbf{J}_{cc}$ to be nonsingular[6]. With a constant setpoint policy, we have $\mathbf{c} = \mathbf{c_s} + \mathbf{n}$ where $\mathbf{n}$ is the implementation error. In this paper, we assume $\mathbf{n} = 0$ (assumption **A4** below) and assume that the setpoint is nominally optimal, $\mathbf{c}_s = \mathbf{c}^{opt}(\mathbf{d}^*)$ where $\mathbf{d}^*$ is the nominal value of the disturbance. The loss for small deviations from the nominal optimum is then

$$L = \frac{1}{2}\left(\mathbf{c}^{opt}(\mathbf{d}^*) - \mathbf{c}^{opt}(\mathbf{d})\right)^T \mathbf{J}_{cc}\left(\mathbf{c}^{opt}(\mathbf{d}^*) - \mathbf{c}^{opt}(\mathbf{d})\right) \tag{5}$$

This gives the following insight (which is not very surprising):

With independent controlled variables $\mathbf{c}$ *and no implementation error, a constant setpoint policy is optimal if* $\mathbf{c}^{\mathrm{opt}}(\mathbf{d})$ *is independent of* $\mathbf{d}$*, i.e.* $\mathbf{c}^{opt}(\mathbf{d}) - \mathbf{c}^{opt}(\mathbf{d}^*) = 0$.

3 The null space method

We consider the unconstrained optimization problem as given by eq. (3), that is, we assume "active constraint control" where all optimally constrained variables are assumed to be kept constant at their optimal values. The goal is to find a linear measurement combination $\mathbf{c} = \mathbf{H}\mathbf{y}$ to be kept at constant setpoints $\mathbf{c}_s$. Here $\mathbf{H}$ is a constant $n_u \times n_y$ matrix and $\mathbf{y}$ is a subset of the available measurements.

In summary, we make the following assumptions:

A1 Steady-state: We consider only steady-state operation. The justification for this is that the economics of operation is primarily determined by the steady-state. Of course, this assumes that we have a control system in place that can quickly bring the plant to its new steady-state.

A2 Disturbances: Only disturbances that affect the steady-state operation are included.

A3 Active constraint control: We assume that the same active constraints remain active for all values of the disturbances and that we control these constraints.

A4 No implementation error: Here the implementation error is the sum of the control error and the effect of the measurement error. The assumption of no steady-state control error is satisfied if we use a controller with integral action. It is a more serious assumption to neglect the measurement error, so the method implicitly assumes that the measurements have been carefully selected.

We then have the following result:

Theorem 3.1 ***The null space method.*** *Assume that we have n_u independent unconstrained free variables* $\mathbf{u}$*, n_d independent disturbances* $\mathbf{d}$*, n_y independent measurements* $\mathbf{y}$*, and we want to obtain $n_c = n_u$ independent controlled variables* $\mathbf{c}$ *that are linear combinations of the measurements*

$$\mathbf{c} = \mathbf{H}\mathbf{y} \tag{6}$$

Let

$$\mathbf{F} = \left(\frac{\partial \mathbf{y}^{opt}}{\partial \mathbf{d}^T}\right)^*$$

be the optimal sensitivity matrix evaluated with constant active constraints. If $n_y \geq n_u + n_d$, it is possible to select the matrix $\mathbf{H}$ *in the left null space of* $\mathbf{F}$*, $\mathbf{H} \in \mathcal{N}(\mathbf{F}^T)$, such that we get*

$$\mathbf{H}\mathbf{F} = 0$$

With this choice for $\mathbf{H}$*, keeping* $\mathbf{c}$ *constant at its nominal optimal value gives zero loss for small disturbance changes* $\mathbf{\Delta d}$*.*

Proof: We first prove that selecting $\mathbf{H}$ such that $\mathbf{HF} = 0$ gives zero disturbance loss. For small disturbances, the optimal change in the measurements to a change in the disturbances can be written

$$\mathbf{y}^{opt}(\mathbf{d}) - \mathbf{y}^{opt}(\mathbf{d}^*) = \mathbf{F}(\mathbf{d} - \mathbf{d}^*) \tag{7}$$

where

$$\mathbf{F} = \begin{bmatrix} \frac{\partial y_1^{opt}}{\partial d_1} & \cdots & \frac{\partial y_1^{opt}}{\partial d_{n_d}} \\ & \ddots & \\ \frac{\partial y_{n_y}^{opt}}{\partial d_1} & \cdots & \frac{\partial y_{n_y}^{opt}}{\partial d_{n_d}} \end{bmatrix} \tag{8}$$

is the optimal sensitivity matrix evaluated at the nominal point *. From eq. (6) the corresponding optimal change in the controlled variables is $\mathbf{c}^{opt}(\mathbf{d}) - \mathbf{c}_{opt}(\mathbf{d}^*) = \mathbf{H}(\mathbf{y}^{opt}(\mathbf{d}) - \mathbf{y}^{opt}(\mathbf{d}^*)$ and by inserting eq. (7) we get

$$\mathbf{c}^{opt}(\mathbf{d}) - \mathbf{c}^{opt}(\mathbf{d}^*) = \mathbf{HF}(\mathbf{d} - \mathbf{d}^*) \tag{9}$$

From the insight stated in the previous section, the constant setpoint policy is optimal if

$$\mathbf{c}^{opt}(\mathbf{d}) - \mathbf{c}^{opt}(\mathbf{d}^*) = 0 \tag{10}$$

which gives the requirement

$$\mathbf{HF}(\mathbf{d} - \mathbf{d}^*) = 0 \tag{11}$$

This needs to be satisfied for any $(\mathbf{d} - \mathbf{d}^*)$ so we must require that

$$\mathbf{HF} = 0 \tag{12}$$

To satisfy this, we need to select $\mathbf{H}$ such that $\mathbf{H} \in \mathcal{N}(\mathbf{F}^T)$, and we next need to prove under which conditions this is possible. The rank of the $n_c \times n_y$ matrix $\mathbf{H}$ is n_u (because $n_y \geq n_c$, $n_c = n_u$ and the controlled variables are independent). The rank of the $n_y \times n_d$ matrix $\mathbf{F}$ is n_d (because $n_y \geq n_d$ and the disturbances are assumed independent). The fundamental theorem of linear algebra [17] says that the left null space of $\mathbf{F}$ ($\mathcal{N}(\mathbf{F}^T)$) has rank $n_y - r$ where $r = n_d$ is the rank of $\mathbf{F}$. To be able to find a H of rank n_u in the left null space of F we must then require, $n_y - n_d \geq n_u$ or equivalently $n_y \geq n_u + n_d$. □

The $n_y \times n_d$ matrix $\mathbf{F}$ may be obtained numerically by perturbing the disturbances $\mathbf{d}$ and re-solving the optimization problem in eq. (3) to obtain new optimal inputs $\mathbf{u}^{\text{opt}}(\mathbf{d})$, assuming that the active constraints are constant. Ganesh and Biegler[16] provide an efficient and rigorous strategy for finding the optimal sensitivity based on a reduced Hessian method. Note that we do not necessarily need an explicit model of the plant as we can find the optimal sensitivity numerically. In addition, many process simulators have built-in optimizers from which the optimal sensitivity $\mathbf{F}$ is easily available.

Numerically, $\mathbf{H}$ may be obtained from a singular value decomposition of $\mathbf{F}^T$. We have $\mathbf{F}^T\mathbf{H}^T = 0$. Thus, selecting $\mathbf{H}^T$ as the input singular vectors of $\mathbf{F}^T$ corresponding to zero singular values in $\mathbf{F}^T$ gives an orthogonal basis.

Selecting $\mathbf{H}$ according to Theorem 3.1 is *exact* for small disturbance perturbations.

Example 3.1 *Consider a simple example with one unconstrained degree of freedom u, $n_u = 1$, and one disturbance d, $n_d = 1$. The cost function to be minimized during operation (for varying d) is*

$$J(u, d) = (u - d)^2$$

Nominally $d^* = 0$. We have advisable two measurements

$$y_1 = 0.9u + 0.1d$$
$$y_2 = 0.5u - d$$

Since $n_y = 2 = n_u + n_d$ and the two measurements are independent it is possible to find a linear measurement combination

$$\mathbf{c} = \mathbf{H}\mathbf{y} = \begin{bmatrix} h_1 & h_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = h_1 y_1 + h_2 y_2$$

where a constant setpoint gives zero disturbance loss, at least locally. We first need to obtain the optimal sensitivity matrix $\mathbf{F}$. Optimality is ensured when $\frac{\partial J}{\partial u} = 2(u^{opt} - d) = 0$ which gives $u^{opt} = d$ and $J^{opt} = 0\ \forall d$. The corresponding optimal outputs are

$$y_1^{opt} = d$$
$$y_2^{opt} = -0.5d$$

and we see that $\mathbf{F}^T = [1\ \ -0.5]$. From the null space method the desired matrix $\mathbf{H}$ must satisfy $\mathbf{HF} = 0$, or

$$h_1 f_1 + h_2 f_2 = 0$$
$$h_1 + h_2(-0.5) = 0 \Rightarrow h_1 = 0.5h_2$$

The solution is non-unique. For example, selecting $h_2 = 1$ gives

$$c = 0.5y_1 + y_2$$

Keeping this controlled variable at its nominally optimal setpoint $c_s = c^{opt}(d^*) = 0$, gives zero disturbance loss, as is easily verified. Generally, the loss will be zero only locally, i.e. for small changes in d, but for this example the cost function is quadratic, and the loss will be zero for any magnitude of the disturbance d.

4 Discussion

4.1 Measurement selection

Intuitively, we would like to avoid using uncertain measurements y_i when forming the combination $\mathbf{c} = \mathbf{H}\mathbf{y}$. The main weakness with the null space method is that it does not consider the measurement error, or more generally the implementation error. If we have extra measurements, that is, $n_y > n_u + n_d$, then we have extra degrees of freedom in selecting $\mathbf{H}$ that should be used to reduce the sensitivity to measurement error. The simplest approach is to select a subset of the "best" measurements such that we get $n_y = n_u + n_d$, but which should these measurements be? This is outside the scope of this paper, and is treated in more detail a forthcoming publication on the extended null space method (see also [13]), but let us provide some results.

Let the linear model be

$$\Delta\mathbf{y} = \mathbf{G}^y\Delta\mathbf{u} + \mathbf{G}_d^y\Delta\mathbf{d} = \tilde{\mathbf{G}}^y \begin{bmatrix}\Delta\mathbf{u}\\ \Delta\mathbf{d}\end{bmatrix} \tag{13}$$

where $\mathbf{y}$ has been scaled with respect to the expected measurement error, and $\mathbf{u}$ and $\mathbf{d}$ have been scaled such that they have similar effects on the cost. It can then be shown that a reasonable approach is to maximize the minimum singular value of the matrix $\tilde{\mathbf{G}}^y = \begin{bmatrix}\mathbf{G}^y & \mathbf{G}_d^y\end{bmatrix}$ from the combined inputs and disturbances to the selected measurements. To understand why this is reasonable, we may imagine using the measurements to determine the inputs and disturbances. For the case with $n_y = n_u + n_d$, $\tilde{\mathbf{G}}^y$ is invertible and we get

$$\begin{bmatrix}\Delta\mathbf{u}\\ \Delta\mathbf{d}\end{bmatrix} = [\tilde{\mathbf{G}}^y]^{-1}\Delta\mathbf{y} \tag{14}$$

In order to avoid sensitivity to measurement errors in $\mathbf{y}$ we want the norm of $[\tilde{\mathbf{G}}^y]^{-1}$ to be small which is equivalent to wanting a large minimum singular value, $\underline{\sigma}(\tilde{\mathbf{G}}^y)$. From (14) we also see why it is reasonable to require $n_y \geq n_u + n_d$ in the null space method, because this is the requirement for being able to uniquely determine from the measurements all independent variables (inputs and disturbances).

4.1.1 Freedom in selecting H

Even for the case $n_y = n_u + n_d$, there are an infinite number of matrices $\mathbf{H}$ that satisfy $\mathbf{H}\mathbf{F} = 0$. This stems from the freedom of selecting basis vectors for the null space [17]. Let $\mathbf{H}_0$ be one such matrix, i.e. $\mathbf{H}_0\mathbf{F} = 0$. For example, $\mathbf{H}_0$ may consist of the one set of basis vectors that span the null space of $\mathbf{F}^T$. Then $\mathbf{H} = \mathbf{C}\mathbf{H}_0$ also satisfies $\mathbf{H}\mathbf{F} = 0$ provided the $n_c \times n_c$ matrix $\mathbf{C}$ is non-singular.

Actually, the degrees of freedom in selecting $\mathbf{C}$ (and $\mathbf{H}$) are the same as the degrees of freedom that are used in steady-state decoupling (or similar) in control. The linear model for the selected controlled variables can be written

$$\Delta\mathbf{c} = \mathbf{H}\Delta\mathbf{y} = \mathbf{H}\mathbf{G}^y\Delta\mathbf{u} + \mathbf{H}\mathbf{G}_d^y\Delta\mathbf{d} = \mathbf{G}\Delta\mathbf{u} + \mathbf{G}_d\Delta\mathbf{d} \tag{15}$$

and the degrees of freedom in the matrix $\mathbf{C}$ may be used to affect $\mathbf{G} = \mathbf{H}\mathbf{G}^y$ and $\mathbf{G}_d = \mathbf{H}\mathbf{G}_d^y$. For example, it is possible to select $\mathbf{H}$ such that $\mathbf{G} = \mathbf{I}$, and we have a decoupled steady-state response from $\mathbf{u}$ to $\mathbf{c}$.

4.2 Disturbance elimination

The required number of measurements in the null space method, $n_y \geq n_u + n_d$, may be a large if we have many disturbances (n_d large). In practical applications, it is therefore desirable to reduce the number of disturbances. Unfortunately, there does not seems to be any simple rigorous procedure for eliminating unimportant disturbances, although some approaches are discussed in Chapter 5 of [13].

It is obvious that we may eliminate disturbances d_i with

1. no steady-state effect on the measurements ($\mathbf{y}$ is independent of d_i, i.e., $\mathbf{G}_{d_i}^y = 0$) *and*
2. no steady-state effect on the optimal operation ($\mathbf{u}^{opt}$ is independent of d_i).

Note that we need to satisfy both conditions.

It could be argued that we may eliminate all "unobservable" disturbances that satisfy condition 1, because we have no way of detecting them and thus correcting for them. However, such disturbances may affect the optimal operation and result in large losses, so an analysis based on neglecting them may be highly misleading. To achieve acceptable operation in such cases, we need to obtain additional measurements, for example, of the disturbance itself. One example would be a price change as is discussed in more detail below. Also, we cannot eliminate all disturbance that have no effect on optimal operation and thus satisfy condition 2. This is because the disturbance may effect a measurement, and controlling this measurement will then result in a loss.

In practice, with too few measurements, one may eliminate some disturbances and obtain the controlled variables $\mathbf{c} = \mathbf{H}\mathbf{y}$ using the null space method, but one should afterwards analyze the loss with all disturbances included. Alternatively, one may be obtain the optimal combination numerically using the exact local method of Halvorsen et al.[6] or the extended null space method presented in a forthcoming publication (see Rule 4 in Chapter 5 of [13]).

4.2.1 Physical interpretation

The proposed null space method yields controlled variables that are linear combinations of the available measurements. A disadvantage is that the physical interpretation of what we control is usually lost. This is by no means a fundamental limitation, since in principle we can control any signal from the process as long as they are independent. Thus, if all measurements are regarded as signals, the concept of controlling a combination of signals may be easier to grasp. If possible, one can choose to combine measurements of one type, for instance only temperatures (e.g. in a distillation column) or only mass flows. In any case, we can scale variables such that the resulting measurements are dimensionless, which is common in practice.

4.2.2 Change in active constraints

A new set of optimal controlled variables (**H**) needs to be found for each set of active constraints. If the active constraints change, this needs to be identified and some logic is involved in order to switch to a new set of controlled variables. Thus, for a process with a small operating window, where the active constraints shift with the disturbances, other methods may be better suited for optimizing control, e.g. real-time optimization (RTO) combined with Model predictive control (MPC). Alternatively, we could use the ideas of[15] on how to handle varying active constraints.

4.2.3 Non-observable disturbances and price changes

Self-optimizing control is based on using feedback to detect disturbances and optimally adjust the inputs so as to achieve near-optimal operation. Thus, one must require that the disturbances are observable (visible) in the measurements **y**. One example of a "disturbance" that is not visible in the measurements is prices. However, prices p_i do enter in the objective function, because typically $J = \sum_i p_i x_i$, and price changes will change the optimal point of operation.

To handle price changes (or more generally disturbances that are not observable in the measurements **y**), one must assume that the price (disturbances) is known (measured). Price changes can then be handled in two ways:

1. Adjust the setpoints in a feedforward manner. Then, for a price change $\Delta\mathbf{p}$ we have that
$$\mathbf{c}_s = \mathbf{c}_s(p^*) + \mathbf{H}\mathbf{F}_p\Delta\mathbf{p} \tag{16}$$
where $\mathbf{F}_p = \left(\frac{\mathrm{d}\mathbf{y}^{opt}}{\mathrm{d}\mathbf{p}^T}\right)$ is the optimal sensitivity from the prices to the measurements.
2. Include the prices as extra measurements in y and use the regular procedure of selecting self-optimizing control variables as above.

The first approach is probably the simplest and most transparent.

4.3 Controllability

All derivations here are based on steady-state models, and we must later check that the candidate structure has acceptable controllability. If not, we may go back and look for other measurements to use in the combination. Next we illustrate the null space method on a gasoline blending example.

5 Gasoline blending example

A simple gasoline blending example is included to illustrate the null space method. In practice, such a system is characterized by frequent price changes and thereby changes in the active constraints, and better suited for an online-optimization approach. Nevertheless, we want to use a constant setpoint policy and compare the control structure synthesized using the null space method with other candidate controlled variables. The system is illustrated in Figure 2 and consists of four gasoline feed streams with varying

octane number and benzene concentration. The nominal data is given in Table 1. The operational objective is to minimize the cost of the feed streams

$$\min_{\mathbf{u}_0} J = \sum_{i=1}^{i=4} p_i \dot{m}_i \tag{17}$$

where p_i and $\dot{m}_i$ are the price and mass flowrate for stream i respectively, while satisfying the following constraints

- Produce $\dot{m}_p = 1$ kg/s of gasoline with
- minimum octane number of 98 ($C_p^o \geq 98$).
- the product stream should not contain more that 1 wt% benzene ($C_p^b \leq 1$ *wt%*).
- the maximum flow rate for stream 4 is 0.4 kg/s, i.e. $\dot{m}_4 \leq 0.4$.

For the octane number we assume "linear mixing" on weight basis. The three mass balances for total mass, benzene mass and octane are then all linear

$$\sum_i \dot{m}_i = \dot{m}_p$$

$$\sum_i C_i^b \dot{m}_i = \dot{m}_p C_p^b$$

Here C_i^o and C_i^b denote the octane and benzene concentrations for stream i, respectively. The full set of manipulated variables are:

$$\mathbf{u}_0^T = [\dot{m}_1 \; \dot{m}_2 \; \dot{m}_3 \; \dot{m}_4]$$

We assume that the available measurements are all the flows plus the octane and benzene contents of the product:

$$\mathbf{y}_0^T = [\dot{m}_1 \; \dot{m}_2 \; \dot{m}_3 \; \dot{m}_4 \; \dot{m}_p \; C_p^o \; C_p^b]$$

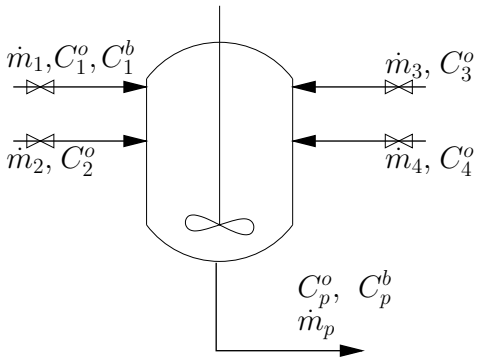

Figure 2: Illustration of the gasoline blending process, where four gasoline feeds are blended to produce the product.

Table 1: Nominal data for the gasoline blending example

Stream i	Octane (C_i^o)	Benzene (C_i^b) [wt %]	Price p_i [Unit/kg]
1	99	2	1.85
2	105	0	2
3	95	0	1.20
4	99	0	$(1 + \dot{m}_4)$

We consider a single disturbance, the octane of feed stream 3,

$$d = C_3^o$$

The feed streams have different prices as shown in the last column of Table 1. For streams $(1 - 3)$ the prices are independent of the flow rate, while for stream 4 the price depends on the flow rate. The latter is required in order to get an "interesting" problem with unconstrained degrees of freedom. With a fixed price also for stream 4, we would have linear problem, and from the theory of linear programming it would optimal to use all degrees of freedom to satisfy active constraints, and there would be no need to use the null space method.

The optimal solution for the nominal disturbance with an octane number of 95 is:

$$\mathbf{u}_0^{opt^T}(d^* = 95) = \begin{bmatrix} 0.000 & 0.196 & 0.544 & 0.260 \end{bmatrix} \tag{18}$$

The minimum cost is $J^{opt}(d^* = 95) = 1.3724$. For a disturbance $d = 97$ (octane number 97 in stream 3) the optimal inputs are

$$\mathbf{u}_0^{opt^T}(d = 97) = \begin{bmatrix} 0 & 0.075 & 0.725 & 0.20 \end{bmatrix} \tag{19}$$

with a cost of $J^{opt}(d = 97) = 1.2600$. We note that the following constraints are active for all disturbances

$$\dot{m}_1 = 0 \quad C_p^o = 98 \quad \dot{m}_p = 1 \tag{20}$$

With "active constraint control" this gives three controlled variables and we are left with one unconstrained degree of freedom ($n_u = 1$). To find the optimal measurement combination to control using this unconstrained degree of freedom, we use the null space method. Since measurements that are optimally constrained give no information, we are left with the following $n_y = 4$ measurements

$$\mathbf{y}^T = [\dot{m}_2 \; \dot{m}_3 \; \dot{m}_4 \; C_p^b]$$

The minimum number of measurements for the null space method is $n_u + n_d = 2$, so there exists an infinite number of combinations $\mathbf{c} = \mathbf{H}\mathbf{y}$ with zero disturbance loss. The simplest is to use only 2 of the 4 measurements. These should be selected such that the influence of measurement error is minimized. We assume that the benzene measurement of the product, C_p^b is unreliable, and that for some reason the measurement of feed flow 3 is uncertain. We then want to use the following measurements

$$\mathbf{y}^T = [\dot{m}_2 \; \dot{m}_4]$$

The optimal sensitivity at the nominal point is $\Delta \mathbf{y}^{\text{opt}} = \mathbf{F}\Delta d$ and we get:

$$\begin{bmatrix} \Delta \dot{m}_2^{opt} \\ \Delta \dot{m}_4^{opt} \end{bmatrix} = \begin{bmatrix} -0.0448 \\ -0.0240 \end{bmatrix} \Delta C_3^o \tag{21}$$

From the null space method, we select $\mathbf{H}$ such that $\mathbf{HF} = 0$ or equivalently $\mathbf{H} = \mathcal{N}(\mathbf{F}^T)$. The null space of $\mathbf{F}^T$ may be obtained numerically from a singular value decomposition of $\mathbf{F}^T$. We have $\mathbf{F}^T\mathbf{H}^T = 0$, so selecting $\mathbf{H}^T$ as the input singular vectors of $\mathbf{F}^T$ corresponding to zero singular values gives an orthogonal basis. The input singular vector of $\mathbf{F}^T$ corresponding to the zero singular value is $[-0.472\ 0.882]^T$, so we get

$$\mathbf{H} = \mathcal{N}(\mathbf{F}^T) = [-0.472\ 0.882] \tag{22}$$

The matrix $\mathbf{H}$ is non-unique. We choose to scale $\mathbf{H}$ by a constant factor such that the second element is 1, resulting in the following optimal fourth controlled variable (ns stands for null space)

$$c_{ns} = \mathbf{Hy} = -0.53\dot{m}_2 + \dot{m}_4 \tag{23}$$

To compare, we also consider the cases where one of the three unconstrained feed flows ($\dot{m}_2$, $\dot{m}_3$ or $\dot{m}_4$) is selected as the fourth unconstrained controlled variable (that is, it is kept constant, whereas the other two feed flows must vary to satisfy the active constraints). The losses for a disturbance $d = C_{3,0}^o$ from 95 to 97 octane are shown

Table 2: Loss for the different control structures for a disturbance in C_3^o from 95 to 97

CS #	c_1	c_2	c_3	c_4	Setpoint ($c_{4,s}$)	L	L[%]
1	$\dot{m}_1$	C_p^o	$\dot{m}_p$	$c_{ns} = -0.53\dot{m}_2 + \dot{m}_4$	0.1550	0	0
2	$\dot{m}_1$	C_p^o	$\dot{m}_p$	$\dot{m}_4$	0.2600	0.0036	0.2857
3	$\dot{m}_1$	C_p^o	$\dot{m}_p$	$\dot{m}_3$	0.5440	0.0582	4.6224
4	$\dot{m}_1$	C_p^o	$\dot{m}_p$	$\dot{m}_2$	0.1960	infeasible	infeasible

for the four structures in Table 2. The loss for the null space candidate (c_{ns}) is zero and shows perfect self-optimizing properties. Candidate $\dot{m}_4$ also has small loss, and is also a good candidate for self-optimizing control. Candidate $\dot{m}_3$ shows a loss of approximately 5% while $\dot{m}_2$ gives infeasible operation. Thus, the best candidate for self-optimizing control is c_{ns}.

One possible implementation of the control structure is shown in Figure 3. This is shown for illustration as the pairing of variables does not influence the steady-state.

Note that the self-optimizing controlled variable is a combined measurements of two inputs. The correction in the input when the disturbance enters, is done implicitly through the control of the active constraints. That is, when the octane number of stream 3 (C_3^o) increases, this is measured in the product stream (the octane number of the product stream increases) which in turn, depending on the control structure, may lead to a reduction in the flowrate for stream 2. When reducing the flow of stream 2, the flow of stream 3 needs to be increased, since the total flowrate is now less than 1. At the same time, the flow rate of stream 4 is adjusted so that c_{ns} is kept at the nominal setpoint.

We only considered one disturbance, but since there are four available unconstrained measurements (three feed flows plus the product benzene contents) we may

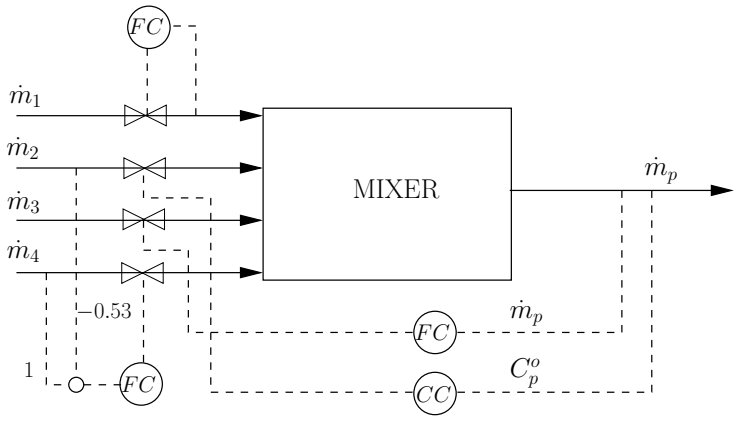

Figure 3: Possible implementation with $c_{ns} = -0.53m_2 + m_4$ as the self-optimizing controlled variable.

combine these into a controlled variable and in theory get zero loss with a constant setpoint for up to three independent disturbances. We could, for example, handle octane number variations in two additional feed streams.

In general, the linear variable combination $\mathbf{c} = \mathbf{H}\mathbf{y}$ obtained with the null space method is optimal only locally, that is, for small disturbances. For this particular example, however, it happens that the model equations are linear and the cost function is quadratic, so a linear variable combination is optimal as long as the optimal active constraints remain do not change.

In this example, the prices were assumed constant. If the prices change, then we may easily correct for this by changing the setpoint for c_{ns}; see Skogestad[18] where a similar example is discussed. However, this assumes that the active constraints remain constant. For larger price changes, the optimal constraints may change, and we would also need to change the controlled variable, c_{ns}. This may be done off-line by performing an analysis similar to the one given above for each region. However, the implementation of this requires logic, so most likely an online optimizing strategy based on solving the optimization problem at each time step will be preferred in practice.

6 Conclusions

This paper has introduced the null space method for selecting controlled variables $\mathbf{c}$. We consider a constant setpoint policy, where the controlled variables are kept at constant setpoints $\mathbf{c}_s$. We propose to select self-optimizing controlled variables as linear combinations $\mathbf{c} = \mathbf{H}\mathbf{y}$ of a subset of the available measurements $\mathbf{y}$. With no implementation error, it is optimal to select $\mathbf{H}$ such that $\mathbf{HF} = 0$, where $\mathbf{F} = (\mathrm{d}\mathbf{y}^{opt}/\mathrm{d}\mathbf{d}^T)$ is the optimal sensitivity with respect to disturbance $\mathbf{d}$. The method has been illustrated on a simple gasoline blending example where we find that the null space method yields a controlled variable that has zero loss.

References

[1] Morari, M., Stephanopoulos, G., and Arkun, Y. "Studies in the synthesis of control structures for chemical processes. Part I: Formulation of the problem, process decomposition and the classification of the controller task. Analysis of the optimizingcontrol structures", *AIChE Journal*, **1980**, 26(2).

[2] Skogestad, S. "Plantwide control: The search for the self-optimizing control structure." *J. Proc. Control*, **2000**, 10.

[3] Larsson, T., Hestetun, K., Hovland, E. and Skogestad, S. "Self-Optimizing Control of a Large-Scale Plant: The Tennessee Eastman Process", *Ind. Eng. Chem. Res.*, **2001**, 40 (22).

[4] Govatsmark, M.S. and Skogestad,S. "Selection of controlled variables and robust setpoints", *Ind.Eng.Chem.Res*, **2005**, 44 (7), 2207-2217.

[5] Mahajanam, R., Zheng, A., and Douglas, J. "A shortcut method for controlled variable selection and its application to the butane alkylation process", *Ind. Eng. Chem. Res.*, **2001**, 40(14).

[6] Halvorsen, I.J., Skogestad, S., Morud, J.C. and Alstad, V. "Optimal selection of controlled variables", *Ind. Eng. Chem. Res.*, **2003**, 42 (14), 3273-3284.

[7] Hori, E.S., Skogestad, S. and V. Alstad "Perfect steady-state indirect control", *Ind.Eng.Chem.Res*, **2005**, 44 (4), 863-867.

[8] Srinivasan, B., Primus, CJ., Bonvin, D., et al. "Run-to-run optimization via control of generalized constraints",*Control Eng. Prac*, **2001**, 9(8).

[9] Srinivasan B, Bonvin D, Visser E. "Dynamic optimization of batch processes - I. Characterization of the nominal solution", *Computers and chemical engineering*, **2003**, 27(1).

[10] Srinivasan B, Bonvin D, Visser E, et al. "Dynamic optimization of batch processes - II. Role of measurements in handling uncertainty", *Computers and chemical engineering*, **2003**, 27(1).

[11] Francois, G., Srinivasan, B., Bonvin, D. "Use of measurements for enforcing the necessary conditions of optimality in the presence of constraints and uncertainty", *J. Proc. Control*, **2005**, 15 (6).

[12] Guay, M. and Zhang, T. "Adaptive extremum seeking control of nonlinear dynamic systems with parametric uncertainties", *Automatica*, 39, 2003

[13] Alstad, V. "Studies on the Selection of Controlled Variables", *PhD Thesis, Norwegian University of Science and Technology*, available at WWW: `http://www.nt.ntnu.no/users/skoge/publications/thesis/2005_alstad/alstad_thesis_webversion.pdf`, **2005**.

[14] Maarleveld, A. and Rijnsdorp, J.E., "Constraint control on distillation columns", *Automatica*, **1970**, 6(1), 51-58.

[15] Arkun, Y. and Stephanopoulos, G. "Studies in the Synthesis of Control Structures for Chemical Processes: Part IV. Design of Steady-State Optimizing Control Structures for Chemical Process Units", *AIChE Journal*, **1980**, 26(6), 975-991.

[16] Ganesh, N. and Biegler, L.T. "A reduced Hessian strategy for sensitivity analysis of optimal flow sheets", *AIChE Journal*, **1987**, 33(2), 282-296.

[17] Strang, G. "Linear algebra and its applications", 3'rd edition, *Harcourt Brace & Company*, **1988**.

[18] Skogestad, S. "Near-optimal operation by self-optimizing control: From process control to marathon running and business systems", *Computers and Chemical Engineering*, **2004**, 29 (1), 127-137.