μ-Interaction Measure for Unstable Systems

Vinay Kariwala*, J. Fraser Forbes† and Sigurd Skogestad‡
* Division of Chemical & Biomolecular Engineering, Nanyang Technological University, Singapore 637722
Email: vinay@ntu.edu.sg
† Department of Chemical & Materials Engineering, University of Alberta, Edmonton, Canada T6G 2G6
‡ Department of Chemical Engineering, Norwegian University of Science and Technology, N-7491 Trondheim, Norway

***Abstract*— The requirement that the system and its block diagonal elements have the same unstable poles limits the applicability of μ interaction measure (μ-IM) to stable systems. We propose an extension of the conventional μ-IM to overcome this difficulty. The decentralized controller is designed based on a block diagonal approximation that is different from the block diagonal elements and is selected by minimizing the scaled $\mathcal{L}_\infty$ distance between the system and the approximation. We also present a simple method for designing the decentralized controller based on the approximation. The proposed method is useful for system stabilization using independent designs of the decentralized controller.**

***Keywords*—Decentralized control, Large-scale systems, Structured singular value.**

I. Introduction

This paper deals with system stabilization using independent designs for the decentralized controller. Here, the sub-controllers are designed independently of each other based on a block diagonal approximation that is usually taken as the block diagonal elements of the system. Then, the decentralized controller design problem reduces to design of a number of small dimensional full multivariable controllers. When the interactions are small, such a controller also stabilizes the closed loop system with minimal loss of performance in comparison to the design basis [1]. Although sub-optimal, the controller design is much simpler as compared to other approaches, *i.e.* simultaneous or sequential designs.

Grosdidier and Morari [2] proposed the use of μ interaction measure (μ-IM) to assess the feasibility of system stabilization through independent designs of individual loops. This approach yields sufficient conditions to ensure that the decentralized controller that stabilizes the block diagonal part of the system also stabilizes the system itself. The problem of decentralized controller synthesis through independent designs has also been studied by Limbeer [3] and Ohta *et al.* [4], who used the concepts of generalized block diagonal dominance and quasi-block diagonal dominance, respectively. The use of μ-IM is less conservative than these approaches because the controller structure is taken into account. A connection between these methods based on dominance and μ-IM is established in [5].

The conventional μ-IM requires that the system and its block diagonal part have the same right half plane (RHP) poles. Grosdidier and Morari [2] pointed out that this condition is not satisfied by most of the systems encountered in practice, limiting the applicability of μ-IM to open loop stable systems. Samyudia *et al.* [6] have criticized the μ-IM for this limitation and have instead proposed a method based on ν-gap metric. In this paper, we present a modified μ-IM that easily handles unstable systems. The decentralized controller is designed based on a block diagonal approximation that is different from the block diagonal elements, but has the same number of unstable poles as the system.

Clearly, the number of block diagonal systems with the required number of unstable poles is infinite and the success of the modified μ-IM approach strongly depends on the choice of an appropriate approximation. We express the μ-IM in terms of the closed loop transfer matrix between disturbance and system input (or controller output). This alternate representation shows that the block diagonal approximation can be reasonably selected by minimizing the scaled $\mathcal{L}_\infty$ distance between the system and the approximation. The problem of finding a structured approximation of a full multivariate system has earlier been considered by Li and Zhou [7], but no numerical methods for solving the approximation problem are provided. In this paper, we present a numerical approach, where the approximation problem is first solved at a set of chosen frequencies followed by parametric identification.

Similar to the conventional μ-IM method, the stabilizing decentralized controller can be synthesized using a loop shaping approach based on the block diagonal approximation. An advantage of alternate representation of μ-IM used here is that controller design can be much simplified using the results on input performance limitations [8], [9]. For the sake of brevity, we have omitted the proofs of the results presented in this paper, which can be found in [10].

Notation. We represent matrices by boldface uppercase letters and vectors by boldface lowercase letters. The symbol $\succeq$ denotes partial ordering, *i.e.* $\mathbf{A} \succeq \mathbf{B}$ implies that $\mathbf{A} - \mathbf{B}$ is a positive semi-definite matrix. Let the set $\mathbf{\Delta} \in \mathbb{C}^{n\times m}$ be defined as $\mathbf{\Delta} = \{\mathrm{diag}(\mathbf{\Delta}_i) : \mathbf{\Delta}_i \in \mathbb{C}^{n_i\times m_i}, \bar{\sigma}(\mathbf{\Delta}) \leq 1\}$. Then, $\mu_{\mathbf{\Delta}}(\mathbf{A})$ represents the structured singular value of $\mathbf{A} \in \mathbb{C}^{m\times n}$ calculated with respect to the $\mathbf{\Delta}$; see *e.g.* [11]. Let $\mathcal{D}_L, \mathcal{D}_R$ be the set of matrices that commute with all elements of $\mathbf{\Delta}$ or $\mathbf{D}_L\tilde{\mathbf{\Delta}} = \tilde{\mathbf{\Delta}}\mathbf{D}_R$ for all $\tilde{\mathbf{\Delta}} \in \mathbf{\Delta}, \mathbf{D}_L \in \mathcal{D}_L, \mathbf{D}_R \in \mathcal{D}_R$. Then,

$$\mu_{\mathbf{\Delta}}(\mathbf{A}) \leq \inf_{\mathbf{D}_L\in\mathcal{D}_L, \mathbf{D}_R\in\mathcal{D}_R} \bar{\sigma}(\mathbf{D}_L\mathbf{A}\mathbf{D}_R^{-1}) \quad (1)$$

We denote the upper bound given by (1) as $\bar{\mu}_{\mathbf{\Delta}}(.)$.

The set of all rational stable systems is denoted as $\mathcal{RH}_\infty$. Let $\mathbf{G}(s) = \mathbf{G}_1(s) + \mathbf{G}_2(s)$ such that $\mathbf{G}_1(s) \in \mathcal{RH}_\infty^\perp$ and

$\mathbf{G}_2(s) \in \mathcal{RH}_\infty$. Then $\mathbf{G}_1(s)$ is the unstable projection of $\mathbf{G}(s)$ represented as $\mathcal{U}(\mathbf{G}(s))$, where $\mathcal{U}(\mathbf{G}(s)) \in \mathcal{RH}_\infty^\perp$. The $\sigma_{Hi}(\mathbf{G}(s))$ are the Hankel singular values of $\mathbf{G}(s)$ [12] and $\underline{\sigma}_H(\mathbf{G}(s))$ is the minimum Hankel singular value. The $\mathcal{H}_\infty$ or $\mathcal{L}_\infty$ norm of $\mathbf{G}(s)$ is defined as

$$\|\mathbf{G}(s)\|_\infty = \sup_{\mathrm{Re}(s)>0} \bar{\sigma}(\mathbf{G}(s)) = \sup_{\omega\in\mathbb{R}} \bar{\sigma}(\mathbf{G}(j\omega))$$

II. μ-INTERACTION MEASURE

In this section, we briefly review the available results on μ-IM [2], point its limitation and suggest a modification to overcome the same. Throughout this paper, we assume that the system does not contain any decentralized fixed modes; see *e.g.* [11]. The absence of decentralized fixed modes is both necessary and sufficient for existence of a decentralized stabilizing controller but only necessary, when individual loops are designed independently of each other.

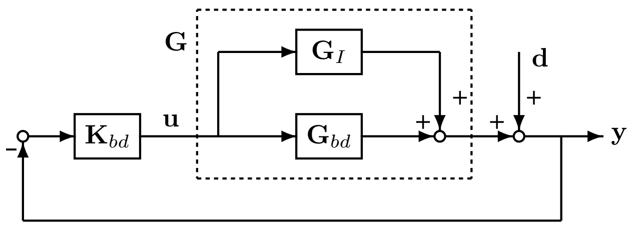

Fig. 1. Partitioning of $\mathbf{G}(s)$ for μ-IM

With reference to Figure 1, let the system $\mathbf{G}(s)$ be partitioned as $\mathbf{G}(s) = \mathbf{G}_{bd}(s) + \mathbf{G}_I(s)$ such that (i) $\mathbf{G}_{bd}(s)$ contains the block-diagonal elements of $\mathbf{G}(s)$ and (ii) $\mathbf{G}_{bd}(s)$ and $\mathbf{G}(s)$ have the same number of RHP poles. Define the transfer matrices $\mathbf{E}(s)$ and $\mathbf{T}_{bd}(s)$ as,

$$\mathbf{T}_{bd}(s) = \mathbf{G}_{bd}\mathbf{K}_{bd}(s)\left(\mathbf{I} + \mathbf{G}_{bd}\mathbf{K}_{bd}(s)\right)^{-1} \quad (2)$$

$$\mathbf{E}(s) = \left(\mathbf{G}(s) - \mathbf{G}_{bd}(s)\right)\mathbf{G}_{bd}(s)^{-1} \quad (3)$$

where $\mathbf{K}_{bd}(s)$ is the block diagonal controller. $\mathbf{T}_{bd}(s)$ can be interpreted as the complementary sensitivity function, if $\mathbf{G}_I(s)$ were zero, and $\mathbf{E}(s)$ as the multiplicative uncertainty in $\mathbf{G}_{bd}(s)$. Let $\mathbf{K}_{bd}(s)$ be designed such that $\mathbf{T}_{bd}(s)$ is stable. The central question remains: Does $\mathbf{K}_{bd}(s)$ also stabilize $\mathbf{G}(s)$? This issue has been addressed by Grosdidier and Morari [2], who proposed the use of μ-IM for this purpose.

Theorem 1: Let $\mathbf{G}(s)$ and $\mathbf{G}_{bd}(s)$ have same number of unstable poles. If $\mathbf{K}_{bd}(s)$ stabilizes $\mathbf{G}_{bd}(s)$, then $\mathbf{K}_{bd}(s)$ also stabilizes $\mathbf{G}(s)$, if

$$\bar{\sigma}\left(\mathbf{T}_{bd}(j\omega)\right) < \mu_{\mathbf{\Delta}}^{-1}\left(\mathbf{E}(j\omega)\right) \qquad \forall\omega\in\mathbb{R} \quad (4)$$

where $\mathbf{\Delta}$ has the same block structure as $\mathbf{G}_{bd}(s)$ and $\mathbf{T}_{bd}(s)$, $\mathbf{E}(s)$ are defined by (2) and (3) respectively.

Theorem 1 was proven by Grosdider and Morari [2] under the requirement that the unstable poles of $\mathbf{G}(s)$ and $\mathbf{G}_{bd}(s)$ be identical; however, the number of unstable poles of $\mathbf{G}(s)$ and $\mathbf{G}_{bd}(s)$ being equal suffices [10]. In either case, design of $\mathbf{K}_{bd}(s)$ solely based on $\mathbf{G}_{bd}(s)$ is equivalent to designing individual loops or control susbsystems independently. The equation (4) is known as the μ-IM. This powerful result allows the designer to impose restrictions on individual controllers, but still be designed solely based on $\mathbf{G}_{bd}(s)$ such that closed loop stability is ensured.

As pointed by Grosdidier and Morari [2] that in practice, $\mathbf{G}(s)$ and $\mathbf{G}_{bd}(s)$ as defined above has same number of RHP poles only for open loop stable systems limiting the applicability of μ-IM. It is noted that this limitation only arises as $\mathbf{G}_{bd}(s)$ is chosen as the block diagonal elements of $\mathbf{G}(s)$ and is easily overcome by relaxing this requirement. The decentralized controller can be designed based on $\mathbf{G}_{bd}(s)$ that is different from the block diagonal elements but has the same number of RHP poles as $\mathbf{G}(s)$. This point is further illustrated using the following simple system:

$$\mathbf{G}(s) = \frac{1}{(s-1)(s-2)}\begin{bmatrix} (s+0.5) & 0.5 \\ (9s-3) & (s+1) \end{bmatrix} \quad (5)$$

Since all the minors of order 1 have $(s-1)(s-2)$ as the denominator and

$$\det(\mathbf{G}(s)) = \frac{s^2-3s+2}{(s-1)^2(s-2)^2} = \frac{1}{(s-1)(s-2)}$$

the system (5) has two unstable poles at 1 and 2 [11]. When $\mathbf{G}_{bd}(s)$ is chosen as the diagonal elements of $\mathbf{G}(s)$,

$$\det(\mathbf{G}_{bd}(s)) = \frac{(s+0.5)(s+1)}{(s-1)^2(s-2)^2}$$

Due to absence of pole-zero cancellation, $\mathbf{G}_{bd}(s)$ has poles at the same locations as $\mathbf{G}(s)$, but repeated twice and the assumptions of μ-IM are violated. Consider that $\mathbf{G}_{bd}(s)$ is chosen as,

$$\mathbf{G}_{bd}(s) = \mathrm{diag}\left(\frac{1}{(s-\alpha_1)}f_1(s), \frac{1}{(s-\alpha_2)}f_2(s)\right)$$

where $\alpha_1, \alpha_2 > 0$ and $f_1(s), f_2(s)$ are arbitrary stable transfer matrices. With this choice, the assumption that $\mathbf{G}_{bd}(s)$ and $\mathbf{G}(s)$ have the same number of unstable poles is easily satisfied. Note that for an arbitrary choice of $\alpha_1, \alpha_2 > 0$, the diagonal blocks of $\mathbf{G}_I(s)$ are not necessarily zero. A similar approach can be used for partitioning any arbitrary system.

Though the generalization used in choosing $\mathbf{G}_{bd}(s)$ extends the practical applicability of μ-IM to unstable systems, the generalization introduces an additional degree of freedom. Clearly, whether the μ-IM condition (4) is satisfied depends on the choice of $\mathbf{G}_{bd}(s)$.

III. ALTERNATE REPRESENTATION OF μ-IM

For given $\mathbf{G}_{bd}(s)$, stabilizing $\mathbf{K}_{bd}(s)$ can be found using a loop shaping approach. In the present case, $\mathbf{G}_{bd}(s)$ can also be treated as a free parameter with the requirement of having the same number of unstable poles as $\mathbf{G}(s)$.

The task of jointly finding the pair $(\mathbf{G}_{bd}(s), \mathbf{K}_{bd}(s))$ such that the closed loop system is stable, is very difficult. We note in (4), both $\bar{\sigma}(\mathbf{T}_{bd}(j\omega))$ and $\mu_{\mathbf{\Delta}}(\mathbf{E}(j\omega))$ depend on $\mathbf{G}_{bd}(j\omega)$, but $\mathbf{E}(j\omega)$ is independent of the controller. Then, a convenient (and not optimal) approach is to find $\mathbf{G}_{bd}(s)$ such that $\mu_{\mathbf{\Delta}}(\mathbf{E}(j\omega))$ is minimized and then design the decentralized controlled based on it to satisfy the μ-IM condition; however, $\mathbf{E}(s)$ is not an affine function of $\mathbf{G}_{bd}(s)$. We next show that this difficulty can be overcome by representing μ-IM alternately in terms of transfer matrix between the disturbances and the inputs.

Proposition 1: Let $\mathbf{G}(s)$ be partitioned as $\mathbf{G}(s) = \mathbf{G}_{bd}(s) + \mathbf{G}_I(s)$ such that $\mathbf{G}_{bd}(s)$ and $\mathbf{G}(s)$ have the same number of RHP poles. Define $\mathbf{S}_{bd}(s) = (\mathbf{I} + \mathbf{G}_{bd}\mathbf{K}_{bd}(s))^{-1}$. Then $\mathbf{K}_{bd}(s)$ stabilizing $\mathbf{G}_{bd}(s)$ also stabilizes $\mathbf{G}(s)$ if [10]

$$\bar{\sigma}(\mathbf{K}_{bd}\mathbf{S}_{bd}(j\omega)) < \mu_{\mathbf{\Delta}}^{-1}(\mathbf{G}_I(j\omega)) \qquad \forall\omega \in \mathbb{R} \tag{6}$$

where $\mathbf{\Delta}$ has the same structure as $\mathbf{G}_{bd}(s)$.

Since the RHS of (6) is affine in $\mathbf{G}_{bd}(s)$, the block diagonal approximation can be sub-optimally selected by minimizing $\mu_{\Delta}(\mathbf{G}_I(j\omega))$. This approach is suboptimal as the LHS of (6) also depends on $\mathbf{G}_{bd}(s)$. For a particular choice of $\mathbf{G}_{bd}(s)$ that optimally minimizes $\mu_{\Delta}(\mathbf{G}_I(j\omega)$, there may not exist any controller satisfying (6) and *vice-versa*. This issue is further discussed later in this paper.

Remark 1: The conditions provided by Theorem 1 and Proposition 1 are only sufficient. This conservativeness of μ-IM arises as the apparent uncertainty set is much larger than the true uncertainty set, which consists of a single element, $\mathbf{G}_I(s)$. The strength of μ-IM is that when (4) or (6) hold, any decentralized controller that stabilizes $\mathbf{G}_{bd}(s)$ also stabilizes $\mathbf{G}(s)$.

Remark 2: We note that in practice, only the upper and lower bounds on μ are computable. Hence, to assess the feasibility of independent designs, one needs to verify

$$\bar{\sigma}(\mathbf{K}_{bd}\mathbf{S}_{bd}(j\omega)) < \bar{\mu}_{\mathbf{\Delta}}^{-1}(\mathbf{G}_I(j\omega)) \qquad \forall\omega \tag{7}$$

where $\bar{\mu}$ represents an upper bound on μ calculated by the D-scaling method with the left and right hand sides scaling matrices being $\mathbf{D}_L(\omega) \in \mathcal{D}_L, \mathbf{D}_R(\omega) \in \mathcal{D}_R$, respectively. Here, $\mathcal{D}_L$ and $\mathcal{D}_R$ are the set of matrices that commute with $\mathbf{G}_{bd}(s)$ or $\mathbf{D}_L\mathbf{G}_{bd}(s) = \mathbf{G}_{bd}(s)\mathbf{D}_R \ \forall\ \mathbf{D}_L \in \mathcal{D}_L, \mathbf{D}_R \in \mathcal{D}_R$.

Grosdidier and Morari [2] pointed out, satisfying μ-IM condition guarantees closed loop stability, but the performance can be arbitrarily poor. In the next proposition, we show that when the μ-IM condition (6) is satisfied, an upper bound on closed loop input performance is always minimized.

Proposition 2: Assume that $\mathbf{G}(s)$ and $\mathbf{G}_{bd}(s)$ have the same number of RHP poles and (7) holds. If $\mathbf{D}_L(\omega) \in \mathcal{D}_L$, $\mathbf{D}_R(\omega) \in \mathcal{D}_R$ are chosen to maximize $\bar{\sigma}\left(\mathbf{D}_L(\omega)\mathbf{G}_I(j\omega)\mathbf{D}_R^{-1}(\omega)\right)$ [10]

$$\bar{\sigma}(\mathbf{K}_{bd}\mathbf{S}(j\omega)) \leq \frac{\kappa(\mathbf{D}_L(\omega))}{\bar{\sigma}^{-1}\left(\mathbf{K}_{bd}\mathbf{S}_{bd}(j\omega)\right) - \bar{\mu}_{\mathbf{\Delta}}(\mathbf{G}_I(j\omega))} \forall\omega \in \mathbb{R}$$

where $\mathbf{\Delta}$ has same structure as $\mathbf{G}_{bd}$ and κ denotes the Euclidean condition number.

We point out that the bound on the closed loop performance is very loose in general. When the performance requirements are specified in terms of a frequency dependent weight, it can be very difficult to satisfy these requirements by minimizing the upper bound.

IV. Block diagonal approximation

In this section, we consider the problem of finding an optimal block diagonal approximation $\mathbf{G}_{bd}(s)$ for the given system $\mathbf{G}(s)$ such that $\mu_{\Delta}(\mathbf{G}(j\omega) - \mathbf{G}_{bd}(j\omega))$ is minimized. Since only $\bar{\mu}_{\Delta}(.)$ is computable in practice, the block diagonal $\mathbf{G}_{bd}(s)$ is instead chosen by solving,

$$\min_{\mathbf{G}_{bd}(j\omega)} \bar{\sigma}\left(\mathbf{D}_L(\omega)\big(\mathbf{G}(j\omega) - \mathbf{G}_{bd}(j\omega)\big)\mathbf{D}_R^{-1}(\omega)\right) \tag{8}$$

where the number of unstable poles of $\mathbf{G}_{bd}(s)$ and $\mathbf{G}(s)$ is same.

As mentioned earlier, the block diagonal elements of the system usually have more unstable poles than the system itself. Intuitively, a suboptimal solution to the optimization problem (8) can be obtained by simply reducing the order of the block diagonal elements of $\mathbf{G}(s)$. In fact, for systems decomposed into 2 blocks, the solution obtained by order reduction of the diagonal elements is optimal, as shown below.

Proposition 3: For $\mathbf{G}$ partitioned into two blocks $\mathbf{G}_{bd} = \mathrm{diag}(\mathbf{G}_{11}, \mathbf{G}_{22})$ minimizes $\mu_{\mathbf{\Delta}}(\mathbf{G} - \mathbf{G}_{bd})$, where $\mathbf{G}_{bd}$ and $\mathbf{\Delta}$ have the same structure as $\mathrm{diag}(\mathbf{G}_{11}, \mathbf{G}_{22})$ and [10]

$$\min_{\mathbf{G}_{bd}} \mu_{\mathbf{\Delta}}(\mathbf{G} - \mathbf{G}_{bd}) = \sqrt{\bar{\sigma}(\mathbf{G}_{12})\bar{\sigma}(\mathbf{G}_{21})} \tag{9}$$

Unfortunately, the attractive result in Proposition 3 does not hold for matrices partitioned into more than 2 blocks and the solution can be very poor; see [10] for numerical experiments. We next present an algorithm that provides a locally optimal solution for the optimization problem (8).

Algorithm 1: For a given system $\mathbf{G}(s)$ with n unstable poles, a locally optimal solution to the block diagonal approximation problem is obtained by the following steps:

1) Solve the optimization problem (8) at a set of chosen frequencies to yield $\mathbf{G}_{bd,j\omega}$.
2) Solve a parametric optimization problem to find $\tilde{\mathbf{G}}_{bd}(s)$ that has at least n unstable poles and minimizes the worst case error between $\tilde{\mathbf{G}}_{bd}(j\omega)$ and $\mathbf{G}_{bd,j\omega}$.
3) If $\tilde{\mathbf{G}}_{bd}(s)$ has more than n unstable poles, the order of $\tilde{\mathbf{G}}_{bd}(s)$ is reduced to n through optimal Hankel norm approximation to get $\mathbf{G}_{bd}(s)$.

The role of these steps becomes clear by noting,

$$\begin{aligned}\mu_{\mathbf{\Delta}}(\mathbf{G}(j\omega) - \mathbf{G}_{bd}(j\omega)) &\leq \mu_{\mathbf{\Delta}}(\mathbf{G}(j\omega) - \mathbf{G}_{bd,j\omega}) \\ &+ \bar{\sigma}(\mathbf{G}_{bd,j\omega} - \tilde{\mathbf{G}}_{bd}(j\omega)) + \bar{\sigma}(\tilde{\mathbf{G}}_{bd}(j\omega) - \mathbf{G}_{bd}(j\omega))\end{aligned} \tag{10}$$

Note that every step in the proposed method minimizes the contribution of one of the terms on RHS of (10) to the total approximation error.

A. Frequency wise approximation

The first step of Algorithm 1 consists of minimizing (8) at a set of chosen frequencies. In the remaining discussion, the frequency argument of the scaling matrices is dropped for notational convenience. Using similar arguments as used in calculating $\bar{\mu}(.)$ by solving a linear matrix inequality (LMI) in [13],

$$\bar{\sigma}(\mathbf{D}_L(\mathbf{G}(j\omega) - \mathbf{G}_{bd,j\omega})\mathbf{D}_R^{-1}) \quad \leq \quad \gamma$$

$$\mathbf{D}_R^{-*}(\mathbf{G}(j\omega)-\mathbf{G}_{bd,j\omega})^*\mathbf{D}_L^*\mathbf{D}_L(\mathbf{G}(j\omega)-\mathbf{G}_{bd,j\omega})\mathbf{D}_R^{-1} \preceq \gamma^2\mathbf{I} \quad (11)$$

$$(\mathbf{G}(j\omega)-\mathbf{G}_{bd,j\omega})^*\mathbf{P}_L(\mathbf{G}(j\omega)-\mathbf{G}_{bd,j\omega}) \preceq \gamma^2\mathbf{P}_R \quad (12)$$

where $\mathbf{P}_L = \mathbf{D}_L^*\mathbf{D}_L \in \mathcal{D}_L$, $\mathbf{P}_R = \mathbf{D}_R^*\mathbf{D}_R \in \mathcal{D}_R$ and $\mathbf{P}_L, \mathbf{P}_R \succ 0$. Note that unlike the calculation of $\bar{\mu}(.)$ [13], (12) is a bilinear matrix inequality (BMI) and thus not affine in the decision variables $\mathbf{G}_{bd,j\omega}, \mathbf{P}_L$ and $\mathbf{P}_R$; however, a locally optimal solution can be found using an iterative approach.

Using the Schur complement lemma [13], (11) can be equivalently expressed as,

$$\begin{bmatrix} -\gamma\mathbf{I} & \mathbf{D}_R^{-*}(\mathbf{G}(j\omega)-\mathbf{G}_{bd,j\omega})^*\mathbf{D}_L^* \\ \mathbf{D}_L(\mathbf{G}(j\omega)-\mathbf{G}_{bd,j\omega})\mathbf{D}_R^{-1} & -\gamma\mathbf{I} \end{bmatrix} \preceq 0 \quad (13)$$

Note that for fixed $\mathbf{D}_L, \mathbf{D}_R$, (13) is an LMI in $\mathbf{G}_{bd,j\omega}$. Now, a locally optimal solution for the frequency wise approximation problem can be found by first solving (13) for $\mathbf{G}_{bd,j\omega}$ by fixing $\mathbf{D}_L, \mathbf{D}_R$. Then, (12) can be solved for $\mathbf{P}_L, \mathbf{P}_R$ using a bisection search method by fixing $\mathbf{G}_{bd,j\omega}$. This procedure is repeated until convergence. Note that unlike a general BMI problem, the sequence of solutions obtained using this iterative procedure is guaranteed to converge [10].

***Remark** 3:* Since the approximation problem has multiple local minima, the iterative procedure can converge to a minimum that is worse than using the diagonal blocks. This difficulty is overcome by using $\mathrm{diag}(\mathbf{G}_{ii}(j\omega))$ as an initial guess. Then, the modified procedure always obtains a solution that is at least as good as using the diagonal blocks.

B. Parametric $\mathcal{L}_\infty$ optimal identification

It would be ideal to directly find $\mathbf{G}_{bd}(s)$ which has the same number of unstable poles as $\mathbf{G}(s)$ and best approximates $\mathbf{G}_{bd,j\omega}$, but the optimization problem becomes very difficult when the number of unstable poles is fixed. Thus, we aim at finding $\tilde{\mathbf{G}}_{bd}(s)$ that has *at least* as many unstable poles as $\mathbf{G}(s)$ followed by model order reduction discussed in § IV-C. We minimize the worst case error or the $\mathcal{L}_\infty$ norm of $\mathbf{G}_{bd,j\omega_i} - \tilde{\mathbf{G}}_{bd}(j\omega_i)$ (*cf.* (10)). Over the past few years, a number of different approaches for worst-case identification have appeared in the literature and the current state of the art can be found in [14].

In this paper, we parameterize the class of models using transfer functions as compared to the finite impulse response (FIR) models typically used in worst-case identification; see *e.g.* [15]. An advantage of using the transfer function parametrization is that low order models can be identified, the disadvantage being that unlike the FIR parametrization, no worst case error bounds are available. Nevertheless, practical experience suggests that transfer function parametrization works very well. For simplicity, $\tilde{\mathbf{G}}_{bd}(s)$ is identified element by element, where $[\tilde{\mathbf{G}}_{bd}(s)]_{ij}$ is parameterized as:

$$[\tilde{\mathbf{G}}_{bd}(s)]_{ij} = \frac{a(s)}{b(s)} = \frac{a_m s^m + a_{m-1}s^{m-1} + \cdots a_1 s + a_0}{b_n s^n + b_{n-1}s^{n-1} + \cdots b_1 s + b_0}$$

where $m \leq n$. In the remaining discussion, we drop the requirement that $\tilde{\mathbf{G}}_{bd}(s)$ has at least as many poles as $\mathbf{G}(s)$, as it is easily satisfied by choosing the order of the denominator polynomials sufficiently large. Then, the parameters $a_0 \cdots a_m, b_0 \cdots b_n$, are obtained by solving,

$$\min_{a_0\cdots a_m, b_0\cdots b_n} \left| \frac{a(j\omega_k)}{b(j\omega_k)} - [\mathbf{G}_{bd,j\omega_k}]_{ij} \right| \qquad k = 1\cdots n_\omega \quad (14)$$

Note that the objective function in (14) is nonlinear, but can be equivalently represented as

$$|b(j\omega_k)|^{-1}\,|a(j\omega_k) - b(j\omega_k)[\mathbf{G}_{bd,j\omega_k}]_{ij}| \quad (15)$$

The following optimization problem can be solved iteratively to minimize (15),

$$\begin{aligned} &\min_{a_0^{(i)}\cdots a_m^{(i)}, b_0^{(i)}\cdots b_n^{(i)} \in \mathbb{R}} \gamma_1^2 + \gamma_2^2 \\ &-\gamma_1^2|b^{(i-1)}(j\omega_k)| \leq \mathrm{Re}\left(a^{(i)}(j\omega_k) - b^{(i)}(j\omega_k)[\mathbf{G}_{bd,j\omega_k}]_{ij}\right) \leq \gamma_1^2|b^{(i-1)}(j\omega_k)| \\ &-\gamma_1^2|b^{(i-1)}(j\omega_k)| \leq \mathrm{Im}\left(a^{(i)}(j\omega_k) - b^{(i)}(j\omega_k)[\mathbf{G}_{bd,j\omega_k}]_{ij}\right) \leq \gamma_2^2|b^{(i-1)}(j\omega_k)| \\ &b_n = 1 \end{aligned} \quad (16)$$

where $b^{(i-1)}(j\omega_k)$ denotes the identified b polynomial from the previous iteration. In (16), the additional constraint $b_n = 1$ is imposed for numerical stability and in general, fixing any one of the unknown parameters suffices. The sequence of solutions obtained by solving optimization problem (16) is not guaranteed to converge, but numerical evidence suggests that a reasonable solution can be obtained using a few iterations.

C. Optimal Hankel norm approximation

To satisfy the assumption of Proposition 1, we need to find $\mathbf{G}_{bd}(s)$ which has exactly n unstable poles. We recall that for a stable transfer matrix $\mathbf{H}(s)$ having order n, the optimal k^{th} order model $\hat{\mathbf{H}}^k(s)$ is found by solving [12],

$$\min_{\hat{\mathbf{H}}^k(s)\in\mathcal{RH}_\infty} \|\mathbf{H}(s) - \hat{\mathbf{H}}^k(s)\|_H = \min_{\hat{\mathbf{H}}^k(s),\mathbf{F}^*(-s)\in\mathcal{RH}_\infty} \|\mathbf{H}(s) - \hat{\mathbf{H}}^k(s) - \mathbf{F}(s)\|_\infty \quad (17)$$

where $\|.\|_H$ denotes the Hankel norm given by the largest Hankel singular value of the transfer matrix. Next, we show how (17) can be adapted to handle the given problem, *i.e.* model reduction of the unstable system $\tilde{\mathbf{G}}_{bd}(s)$.

Let $\tilde{\mathbf{G}}_{bd}(s) = \mathbf{G}_1(s) + \mathbf{G}_2(s)$ such that $\mathbf{G}_1^*(-s), \mathbf{G}_2(s) \in \mathcal{RH}_\infty$. Without loss of generality, we can parameterize $\mathbf{G}_{bd}(s)$ as $\mathbf{G}_{bd}(s) = \mathbf{G}_{bd}^n(s) + \mathbf{G}_2(s)$, which provides

$$\begin{aligned} \|\tilde{\mathbf{G}}_{bd}(s) - \mathbf{G}_{bd}(s)\|_\infty &= \|\mathbf{G}_1(s) - \mathbf{G}_{bd}^n(s)\|_\infty \\ &= \|\mathbf{G}_1^*(-s) - (\mathbf{G}_{bd}^n(s))^*\|_\infty \end{aligned}$$

The optimal value for $(\mathbf{G}_{bd}(s))^* \in \mathcal{RH}_\infty$ is found by solving (*cf.* (17)),

$$\min_{(\mathbf{G}_{bd}^n(s))^*,\mathbf{F}^*(-s)\in\mathcal{RH}_\infty} \|\mathbf{G}_1^*(-s) - (\mathbf{G}_{bd}^n(s))^* - \mathbf{F}(s)\|_\infty$$

Then, the optimal value of $\mathbf{G}_{bd}(s)$ is given as $\mathbf{G}_{bd}(s) = \mathbf{G}_{bd}^n(s) + \mathbf{F}^*(-s) + \mathbf{G}_2(s)$. Since $\mathbf{F}^*(-s)$ and $\mathbf{G}_2(s)$ are stable, $\mathbf{G}_{bd}(s)$ is the $\mathcal{L}_\infty$ optimal reduced order approximation of $\tilde{\mathbf{G}}_{bd}(s)$ with n unstable poles.

V. Controller design

With the availability of $\mathbf{G}_{bd}(s)$ using Algorithm 1, controller design for the modified μ-IM is similar to the conventional μ-IM method. A loop shaping approach can be used to find the stabilizing decentralized controller; however, finding a controller using this method to satisfy (6) can be difficult. In this section, we show that with the alternate representation of the μ-IM conditions in terms of $\mathbf{K}_{bd}\mathbf{S}_{bd}(s)$, finding $\mathbf{K}_{bd}(s)$ to satisfy (6) reduces to solving a weighted $\mathcal{H}_\infty$ controller design problem for $\mathbf{G}_{bd}(s)$.

Proposition 4: Consider that $\mathbf{G}(s)$ and $\mathbf{G}_{bd}(s)$ have the same number of unstable poles. Let the minimum phase and stable transfer matrix $w(s)$ be chosen such that $|w(j\omega)| = \mu_{\mathbf{\Delta}}(\mathbf{G}_I(j\omega))$ for all ω. There exists a block diagonal controller $\mathbf{K}_{bd}(s)$ such that $\bar{\sigma}(\mathbf{K}_{bd}\mathbf{S}_{bd}(j\omega)) < \mu_{\mathbf{\Delta}}^{-1}(\mathbf{G}_I(j\omega))$ for all $\omega \in \mathbb{R}$ iff [10]

$$\underline{\sigma}_H^{-1}(\mathcal{U}(w^{-1}\mathbf{G}_{bd}(s))^*) < 1 \tag{18}$$

where $\mathcal{U}(.)$ denotes the unstable part.

The proof of Proposition 4 primarily utilizes the fact that the minimal achievable value of $\|\mathbf{K}(s)(\mathbf{I}+\mathbf{G}\mathbf{K}(s))^{-1}\|_\infty$ is given as [8], [9]

$$\inf_{\mathbf{K}(s)} \|\mathbf{K}(s)(\mathbf{I}+\mathbf{G}\mathbf{K}(s))^{-1}\|_\infty = \underline{\sigma}_H^{-1}(\mathcal{U}(\mathbf{G}(s))^*)$$

In Proposition 4, we assumed that $w(s)$ is stable and minimum phase. In general, $w(s)$ can have RHP zeros and RHP poles at same the location as $\mathbf{G}_{bd}(s)$. Note that

$$\|w(s)\mathbf{K}_{bd}\mathbf{S}_{bd}(s)\|_\infty = \|w_{ms}(s)\mathbf{K}_{bd}\mathbf{S}_{bd}(s)\|_\infty$$

where $w_{ms}(s)$ denotes the minimum phase stable version of $w(s)$. Thus, allowing $w(s)$ to be unstable or non-minimum phase provides no advantage and we can simply replace $w(s)$ by its minimum and stable version in (18). On relaxing the assumption of minimum phase stable $w(s)$, however, a $w(s)$ that achieves $|w(j\omega)| = \mu_{\mathbf{\Delta}}(\mathbf{G}_I(j\omega))$ becomes non-unique, where the different instances of $w(s)$ are related by a unitary transformation.

Proposition 4 effectively reduces the task of finding a block decentralized controller to satisfy μ-IM condition (6) to finding the minimum phase and stable $w(s)$ such that $|w(j\omega)| = \mu_{\mathbf{\Delta}}(\mathbf{G}_I(j\omega))$ and (18) holds. When (18) is satisfied, the standard $\mathcal{H}_\infty$ optimal control design techniques can be used to find the stabilizing decentralized controller, *i.e.* the optimal controller that minimizes $\|w(s)\mathbf{K}_{bd}\mathbf{S}_{bd}(s)\|_\infty$ also stabilizes $\mathbf{G}(s)$.

Remark 4: In practice, it can be difficult to find $w(s)$ that satisfies $|w(j\omega)| = \mu_{\mathbf{\Delta}}(\mathbf{G}_I(j\omega))$ for all $\omega \in \mathbb{R}$. This difficulty can be overcome by finding $w(s)$ such that $|w(j\omega)| < \mu_{\mathbf{\Delta}}(\mathbf{G}_I(j\omega))$ at all frequencies. Then, if (18) holds, for the given $\mathbf{G}_{bd}(s)$ the existence of a decentralized stabilized controller is established.

VI. Numerical Example

In this section, we demonstrate the efficiency of Algorithm 1 for obtaining optimal block diagonal approximation and the controller design method discussed in the previous sections using the following simple system:

$$\mathbf{G}(s) = \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 1 & \beta_1 & \beta_1 \\ 0 & 2 & 0 & 0 & \beta_1 & 1 & \beta_1 \\ 0 & 0 & 3 & 0 & \beta_1 & \beta_1 & 1 \\ 0 & 0 & 0 & -4 & 1 & 0.4 & 0.4 \\ \hline 1 & \beta_2 & \beta_2 & 1 & 0 & 0 & 0 \\ \beta_2 & 1 & \beta_2 & 0.6 & 0 & 0 & 0 \\ \beta_2 & \beta_2 & 1 & 0.6 & 0 & 0 & 0 \end{array}\right]$$

where $\beta_1 = 0.5, \beta_2 = 0.1$. A set of equally spaced frequencies in the range $0-10$ is chosen and the locally optimal diagonal approximation is obtained using the following steps:

- We use an iterative procedure for frequency-wise minimization, which converges in 2 iterations.
- We fit 4^{th} or lower order models for the frequency data using the formulation (16) with 2 iterations.
- The identified model has 4 unstable poles, which is reduced to a model with 3 unstable poles using the Hankel norm approximation method.

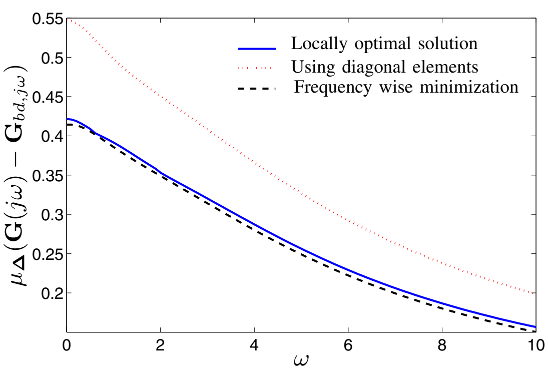

Fig. 2. Efficiency of the proposed method for finding optimal block diagonal approximation

Following these steps, $\mathbf{G}_{bd}^{sub}(s)$ is obtained as

$$\text{diag}\Big(\frac{-0.002s^2+2.22s+3.42}{s^2+2.92s-3.96}, \frac{-0.015s^2+2.04s+6.02}{s^2+2.57s-9.76}, \frac{-0.0153s^2+1.85s+4.97}{s^2+1.75s-8.97}\Big)$$

For comparison purposes, we also calculate the sub-optimal solution $\mathbf{G}_{bd}^{diag}(s)$ by reducing the order of diagonal elements of $\mathbf{G}$. In this case, five Hankel singular values of the stable part of $\mathbf{G}_{bd}^{diag}(s)$ are negligible, which are removed to get a reduced order model given as:

$$\text{diag}\Big(\frac{2.08s+3.27}{s^2+2.96s-4.16}, \frac{1.33s+3.90}{s^2+2.06s-7.76}, \frac{-0.006s^2+1.26s+3.53}{s^2+1.42s-10.31}\Big)$$

To show the advantage of Algorithm 1 over using diagonal elements, $\gamma^{sub} = \mu_{\mathbf{\Delta}}(\mathbf{G}(j\omega) - \mathbf{G}_{bd}^{sub}(j\omega))$ and $\gamma^{diag} = \mu_{\mathbf{\Delta}}(\mathbf{G}(j\omega) - \mathbf{G}_{bd}^{diag}(j\omega))$ are compared in Figure 2. The relative difference between γ^{diag} and γ^{sub} is 0.23 at the zero frequency, which monotonically reduces to 0.21 for $\omega = 10$. This significant reduction in the approximation error is useful for finding the stabilizing controller easily. Figure 2 also shows that the γ^{sub} closely matches the approximation error obtained using frequency wise minimization. Thus, (at least for this example), the conservativeness in using the two-step approach for identifying a model, with same number of unstable poles as the system, is minimal.

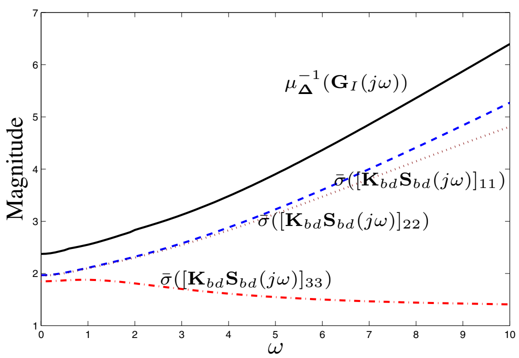

Fig. 3. Validation of modified μ-IM for stabilizing decentralized controller designed using independent designs

Next, we consider the controller design. For the locally optimal diagonal approximation, the following weight lower bounds $\mu_{\mathbf{\Delta}}(\mathbf{G}_I(j\omega))$ closely,

$$w_1(s) = \frac{0.0123s^2 + 1.71s + 1.88}{s^2 + 5.495s + 4.52}$$

Using this $w(s)$, $\underline{\sigma}_H(\mathcal{U}(w_1^{-1}\mathbf{G}_{bd}^{sub}(s))^*) = 1.22 > 1$ and standard $\mathcal{H}_\infty$ optimal controller design technique is used to find a decentralized stabilizing controller. The plots of $\mu_{\mathbf{\Delta}}^{-1}(\mathbf{G}_I(j\omega))$ and $\bar{\sigma}([\mathbf{K}_{bd}\mathbf{S}_{bd}(j\omega)]_{ii})$, $i = 1, 2, 3$ are shown in Figure 3, where $\mu_{\mathbf{\Delta}}^{-1}(\mathbf{G}_I(j\omega)) > \bar{\sigma}([\mathbf{K}_{bd}\mathbf{S}_{bd}(j\omega)]_{ii})$, as expected. On the other hand, for the suboptimal solution obtained using the diagonal elements, the weight that lower bounds $\mu_{\mathbf{\Delta}}(\mathbf{G}_I(j\omega))$ closely is

$$w_2(s) = \frac{0.05s^2 + 2.165s + 2.38}{s^2 + 5.404s + 4.44}$$

and $\underline{\sigma}_H(\mathcal{U}(w_2^{-1}(s)\mathbf{G}_{bd}^{diag}(s))^*) = 0.59 < 1$. Then, the conservativeness of using the diagonal elements to find a suboptimal solution is emphasized.

VII. Conclusions

In this paper, we extended the practical applicability of μ-IM to unstable systems. The decentralized controller is designed based on a block diagonal approximation that is different from the block diagonal elements, but has same number of unstable poles as the system. By expressing the μ-IM in terms of transfer matrix from disturbances to inputs, it is shown that the block diagonal approximation can be (sub-optimally) chosen by minimizing the scaled $\mathcal{L}_\infty$ distance between the system and the approximation. Further, the task of designing the controller based on the block diagonal approximation can be reduced to solving a weighted $\mathcal{H}_\infty$ optimal controller design problem.

A step-wise numerical approach is presented for finding the locally optimal solution to the block diagonal approximation problem. The proposed approach involves solving the approximation problem at a set of frequencies followed by $\mathcal{L}_\infty$ optimal identification. The primary limitation of choosing the block diagonal approximation by minimizing the scaled $\mathcal{L}_\infty$ distance is that the properties of the approximation are not taken into account. As shown in this paper, whether the stabilizing controller can be easily found depends on the minimum Hankel singular value of the approximation. A better approach is to use a multi-objective optimization framework, where the $\mathcal{L}_\infty$ distance between the system and the approximation is minimized and simultaneously the minimum Hankel singular of the approximation is maximized. This non-trivial problem is a topic for future work.

References

[1] S. Skogestad and M. Morari, "Robust performance of decentralized control systems by independent designs," *Automatica*, vol. 25, no. 1, pp. 119–125, 1989.

[2] P. Grosdidier and M. Morari, "Interaction measures for systems under decentralized control," *Automatica*, vol. 22, no. 3, pp. 309–319, 1986.

[3] D. J. N. Limbeer, "The application of generalized diagonal dominance to linear system stability theory," *Int. J. of Control*, vol. 36, no. 2, pp. 185–212, 1982.

[4] Y. Ohta, D. D. Siljek, and T. Matsumoto, "Decentralized control using quasi-block diagonal dominance of transfer function matrices," *IEEE Trans. Automat. Contr.*, vol. 31, no. 5, pp. 420–429, 1986.

[5] V. Kariwala, J. F. Forbes, and E. S. Meadows, "Block relative gain: Properties and pairing rules," *Ind. Eng. Chem. Res.*, vol. 42, no. 20, pp. 4564–4574, 2003.

[6] Y. Samyudia, P. L. Lee, and I. T. Cameron, "A new approach to decentralized control design," *Chem. Eng. Sci.*, vol. 50, no. 11, pp. 1695–1706, 1995.

[7] L. Li and K. Zhou, "An approximation approach to decentralized $\mathcal{H}_\infty$ control," in *Proceedings of 4^{th} World Congress on Intelligent Control and Automation*, Shanghai, China, June 2002.

[8] K. Glover, "Robust stabilization of linear multivariable systems: Relations to approximation," *Int. J. of Control*, vol. 43, no. 3, pp. 741–766, 1986.

[9] V. Kariwala, S. Skogestad, J. F. Forbes, and E. S. Meadows, "Achievable input performance of linear systems under feedback control," *Int. J. of Control*, vol. 78, no. 16, pp. 1327–1341, 2005.

[10] V. Kariwala, "Multi-loop controller synthesis and performance analysis," Ph.D. Thesis, University of Alberta, Edmonton, Canada, July 2004, Available at http://www.nt.ntnu.no/users/skoge/publications/thesis/more/kariwala_PhD04/.

[11] S. Skogestad and I. Postlethwaite, *Multivariable Feedback Control: Analysis and Design*, 2nd ed. Chichester, UK: John Wiley & Sons, 2005.

[12] K. Glover, "All optimal hankel-norm approximations of linear multivariable systems and their $\mathcal{L}^\infty$ bounds," *Int. J. of Control*, vol. 39, no. 6, pp. 1115–1193, 1984.

[13] S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*. Philadelphia: SIAM, 1994.

[14] J. Chen and G. Gu, *Control-Oriented System Identification: An $\mathcal{H}_\infty$ Approach*. Mississauga, ON, Canada: John Wiley & Sons, 2000.

[15] A. J. Helmicki, C. A. Jacobson, and C. N. Nett, "Worst-case/deterministic approach in $\mathcal{H}_\infty$: The continuous-time case," *IEEE Trans. Automat. Contr.*, vol. 37, no. 5, pp. 604–610, 1992.