Correction of the proof of Theorem 1 in “Limit cycles with imperfect valves: Implications for controllability of processes with large gains”

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1 Appendix. Proof of Theorem 1 Revised

From Figure 2:

$$u(s) = K(s)[r(s) - y(s)], \tag{1}$$

where $K(s)$ is given by eq 13, $r(s)$ is a step change in reference ($r(s) = \frac{r_0}{s}$), and $y(s) = K(s)G(s)u_q(s)$, where $G(s)$ is given by eq 12.

In the limit when $t \to \infty$, the quantizer behaves exactly as the relay depicted in Figure 10 and assuming that q_1 and q_2 are arbitrary values, the first four terms of u_q are:

$$u_q(s) = \frac{q_2}{s} + \frac{q_1 - q_2}{s}(e^{-t_0 s} - e^{-(t_0+t_1)s} + e^{-(t_0+t_1+t_2)s}) \tag{2}$$

Consider a PI-controller. Substituting (2) into (1) and inverting it to the time domain, the following equation for $u(t)$ is observed:

$$\begin{aligned} u(t) \;=\; & \frac{K_c}{\tau_I}\{r_0(t+\tau_I) - kq_2[(\tau_I - \tau)(1 - e^{-(t-\theta)/\tau}) + (t-\theta)] - \\ & k(q_1 - q_2)[(\tau_I - \tau)(1 - e^{-(t-t_0-\theta)/\tau}) + (t - t_0 - \theta)] + \\ & k(q_1 - q_2)[(\tau_I - \tau)(1 - e^{-(t-t_0-t_1-\theta)/\tau}) + (t - t_0 - t_1 - \theta)] - \\ & k(q_1 - q_2)[(\tau_I - \tau)(1 - e^{-(t-t_0-t_1-t_2-\theta)/\tau}) + (t - t_0 - t_1 - t_2 - \theta)]\} \end{aligned} \tag{3}$$

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Assuming, without loss of generality, that $t_0 = 0$,

$$u(t) = \frac{K_c}{\tau_I}\{r_0(t+\tau_I) - kq_1[(\tau_I-\tau)(1-e^{-(t-\theta)/\tau}) + (t-\theta)] + k(q_1-q_2)[(\tau_I-\tau)(1-e^{-(t-t_1-\theta)/\tau}) + (t-t_1-\theta)] - k(q_1-q_2)[(\tau_I-\tau)(1-e^{-(t-t_1-t_2-\theta)/\tau}) + (t-t_1-t_2-\theta)]\} \quad (4)$$

Now, for the interval $\theta \leq t < t_1 + \theta$, $u(t)$ is given by

$$u(t) = \frac{K_c}{\tau_I}\{r_0(t+\tau_I) - kq_1[(\tau_I-\tau)(1-e^{-(t-\theta)/\tau}) + (t-\theta)]\} \quad (5)$$

For the interval $\theta + t_1 \leq t < t_1 + t_2 + \theta$, $u(t)$ is given by

$$u(t) = \frac{K_c}{\tau_I}\{r_0(t+\tau_I) - kq_1[(\tau_I-\tau)(1-e^{-(t-\theta)/\tau}) + (t-\theta)] + k(q_1-q_2)[(\tau_I-\tau)(1-e^{-(t-t_1-\theta)/\tau}) + (t-t_1-\theta)]\} \quad (6)$$

Finally, for the interval $t \geq t_0 + t_1 + t_2 + \theta$, we have that $u(t)$ is

$$u(t) = \frac{K_c}{\tau_I}\{r_0(t+\tau_I) - kq_1[(\tau_I-\tau)(1-e^{-(t-\theta)/\tau}) + (t-\theta)] + k(q_1-q_2)[(\tau_I-\tau)(1-e^{-(t-t_1-\theta)/\tau}) + (t-t_1-\theta)] - k(q_1-q_2)[(\tau_I-\tau)(1-e^{-(t-t_1-t_2-\theta)/\tau}) + (t-t_1-t_2-\theta)]\} \quad (7)$$

So far, no assumptions on the controller settings (K_c and τ_I) have been made. The expressions (5)-(7) drastically simplify if the integral time is selected as $\tau_I = \tau$, which is an appropriate setting for many plants [9].

Furthermore, for a relay without hysteresis its output ($u_q(t)$) changes as its input ($u(t)$) equals to zero and since the quantizer behaves as a relay when $t \to \infty$, the following equations give relations for t_1 and t_2.

For $t = 0$:

$$r_0\tau_I = -kq_1\theta \quad (8)$$

For $t = t_1$:

$$r_0(t_1+\tau_I) = kq_1(t_1-\theta) - k(q_1-q_2)\theta \quad (9)$$

For $t = t_1 + t_2$:

$$r_0(t_1+t_2+\tau_I) = kq_1(t_1+t_2-\theta) - k(q_1-q_2)(t_2-\theta) - k(q_1-q_2)\theta \quad (10)$$

Combining (8)-(10), the following expressions give the period T of the oscillations:

$$t_1 = \frac{k(q_1 - q_2)\theta}{kq_1 - r_0} \tag{11}$$

$$t_2 = \frac{k(q_1 - q_2)\theta}{r_0 - kq_2} \tag{12}$$

$$T = t_1 + t_2 \tag{13}$$

On average, the input must equal the steady-state value $u_{ss} = \frac{y_{ss}}{G(0)} = \frac{r_0}{k}$ (where $k = G(0)$), and if this does not happen to exactly correspond to one of the quantizer level, the quantized input u_q will cycle between the two neighboring quantizer levels, q_1 and q_2. Let f and $(1-f)$ denote the fraction of time spent at each level. Then, at steady state $u_{ss} = \frac{r_0}{k} = fq_1 + (1-f)q_2$ and from this expression f is found to be

$$f = \frac{r_0 - kq_2}{k(q_1 - q_2)} \tag{14}$$

From (14),

$$t_1 = \frac{\theta}{1-f} \tag{15}$$

$$T = \theta\Big(\frac{1}{1-f} + \frac{1}{f}\Big),$$

which completes the proof.