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Achievable input performance of linear systems under feedback control†

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In this paper, we characterize the achievable input performance for linear time invariant systems under feedback control. We provide analytical expressions for minimal input requirement for stabilization in both of the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal control frameworks. The achievable input performance primarily depends on the joint controllability and observability of unstable poles. These results are also extended to systems with time delay. It is shown that time delay poses no serious limitations on the achievable input performance for systems with slow instabilities and *vice versa*. The proposed results unify the available results on input performance limitations and are useful for various purposes including selection of variables for the stabilizing layer, process design and formulation of the optimal controller design problem.

1. Introduction

In this paper, we characterize the minimal control effort required for stabilization of linear time invariant systems under feedback control. This problem is important in practice because input saturation is often the main problem for system stabilization. In the $\mathcal{H}_2$ control framework, the problem of control effort minimization is the dual of the well studied minimum variance or cheap control problem (Qiu and Davison 1993, Huang and Shah 1999). It is known that the output performance of the system is limited by its unstable zeros and time delay. Similarly, the unstable poles and time delays pose limitations on the minimal control effort required for stabilization. Here, the minimal control effort required for stabilization is referred as the achievable input performance.

The broad area of fundamental performance limitations has drawn a lot of interest in the past two decades. An overview of the available results and some recent developments in this area can be found in Skogestad and Postlethwaite (1996), Seron *et al*. (1997), Chen and Middleton (2003) and their references. Though the focus has largely been on obtaining bounds on sensitivity and complementary sensitivity functions, which primarily address output performance issues, (see e.g. Chen 2000), some researchers have considered characterizing achievable input performance directly or indirectly.

Glover (1984) studied the robust stability of systems in the presence of unstructured additive uncertainty. With this description of uncertainty, maximizing robust stability is equivalent to minimizing the $\mathcal{H}_\infty$ norm of the transfer matrix from disturbances to inputs. Clearly, the results of Glover (1986) are relevant to the problem considered here, but the disturbance model and frequency dependent weight are assumed to be minimum phase stable. Havre and Skogestad (2001) relaxed the assumption of a minimum phase stable disturbance model and frequency dependent weight, and derived expressions for a lower bound on the achievable input performance. Using a novel approach of pole vectors, Havre and Skogestad (2003) provide exact expressions for rational systems with a single unstable pole driven by measurement noise. Chen *et al*. (2003) studied the optimal regulation problem with input usage penalized for rational unstable systems driven by input disturbances in the $\mathcal{H}_2$ optimal control framework.

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These results can be related to the present problem by an appropriate choice of weights.

It is clear from the previous discussion that, whereas optimal and sub-optimal solutions for different instances of the achievable input performance characterization problem are available, the solution for the general case is lacking. This motivates the present work. In this paper, we characterize the minimal input requirement for stabilization in both of the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal control frameworks. The system is considered to be driven by output disturbances, where the disturbance model may share unstable poles with the system. This representation is not limiting and the case of input disturbances is easily handled by setting the disturbance model same as the system. We further generalize these results to systems with input–output time delay. It is shown that time delay poses no serious limitations on the achievable input performance for systems with slow instabilities and *vice versa*. We consider both single-input single-output (SISO) and multi-input multi-output (MIMO) systems. Naturally, the results presented for MIMO systems are also applicable to SISO systems. The results presented here are useful for (i) selection of input and output variables for stabilization (Havre and Skogestad 2003); (ii) process design considering achievable control performance and; (iii) optimal controller synthesis problem formulation.

For a given system, the control effort required for stabilization can easily be calculated using available numerical techniques for optimal controller design. A limitation of such a numerical approach is that it does not provide any information regarding the factors limiting the input performance. These insights are useful for making appropriate design modifications, when the system cannot be stabilized by constraining the inputs of the system within their maximal allowable ranges. In some special cases, these insights can also provide simple analytic methods for selection of variables for the stabilizing layer (Havre and Skogestad 2003).

The organization of the remaining discussion in this paper is as follows: key results from linear systems theory including optimal control are reviewed in § 2; the problem of designing the optimal controller that minimizes input usage for stabilization is formulated and simplified in § 3; the achievable input performance for SISO and MIMO systems is characterized in § 4 and § 5, respectively; § 6 concludes this paper.

2. Preliminaries

In this section, we standardize notation and collect some general results from linear systems theory, which form the basis for further development in this paper.

2.1. *Notation*

We represent matrices by boldface uppercase letters and vectors by boldface lowercase letters. Given a matrix $\mathbf{A} \in \mathbb{C}^{m \times n}$, $\mathbf{A}'$ is its transpose and $\mathbf{A}^*$ is its complex conjugate transpose. $\mathbf{A}_i$ denotes the ith column of the matrix and accordingly $\mathbf{A}'_i$ represents the ith row. A matrix made of elements $a_{11}, \ldots, a_{1n}, \ldots, a_{mn}$ is represented as $[a_{ij}]$. The maximum and minimum eigenvalues (singular values) are represented as $\bar{\lambda}(\mathbf{A})$ and $\underline{\lambda}(\mathbf{A})$ ($\bar{\sigma}(\mathbf{A})$ and $\underline{\sigma}(\mathbf{A})$) respectively. $\rho(\mathbf{A})$ denotes the spectral radius of the matrix. For $\mathbf{A}, \mathbf{B} \in \mathbb{C}^{m \times n}$, $\mathbf{A} \circ \mathbf{B}$ is the element-wise or Hadamard product.

For a transfer matrix $\mathbf{G}(s)$, $\mathbf{u}_{z_i}$ and $\mathbf{y}_{z_i}$ are called the input and output zero directions, corresponding to the zero z_i, respectively if (Skogestad and Postlethwaite 1996)

$$\mathbf{G}(z_i)\mathbf{u}_{z_i} = \mathbf{0} \quad \text{and} \quad \mathbf{G}(s)\mathbf{u}_{z_i} \neq \mathbf{0} \quad \forall s \neq z_i$$
$$\text{and} \quad \mathbf{y}^*_{z_i}\mathbf{G}(z_i) = \mathbf{0} \quad \text{and} \quad \mathbf{y}^*_{z_i}\mathbf{G}(s) \neq \mathbf{0} \quad \forall s \neq z_i.$$

With a slight abuse of terminology, the poles of $\mathbf{G}(s)$ can be alternatively defined as the zeros of $\mathbf{G}^{-1}(s)$. Then $\mathbf{u}_{p_i}$ and $\mathbf{y}_{p_i}$ are called the input and output pole directions, corresponding to the pole p_i, respectively if (Skogestad and Postlethwaite 1996)

$$\mathbf{u}^*_{p_i}\mathbf{G}^{-1}(p_i) = \mathbf{0} \quad \text{and} \quad \mathbf{u}^*_{p_i}\mathbf{G}^{-1}(s) \neq \mathbf{0} \ \forall s \neq p_i$$
$$\text{and} \quad \mathbf{G}^{-1}(p_i)\mathbf{y}_{p_i} = \mathbf{0} \quad \text{and} \quad \mathbf{G}^{-1}(s)\mathbf{y}_{p_i} \neq \mathbf{0} \ \forall s \neq p_i.$$

The set of all rational stable systems is denoted as $\mathcal{RH}_\infty$. Let $\mathbf{G}(s) = \mathbf{G}_1(s) + \mathbf{G}_2(s)$ such that $\mathbf{G}_1(s) \in \mathcal{RH}_\infty^\perp$ and $\mathbf{G}_2(s) \in \mathcal{RH}_\infty$. Then $\mathbf{G}_1(s)$ is the unstable projection of $\mathbf{G}(s)$ represented as $\mathcal{U}(\mathbf{G}(s))$, where $\mathcal{U}(\mathbf{G}(s)) \in \mathcal{RH}_\infty^\perp$. The $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms of the transfer matrix $\mathbf{G}(s) \in \mathcal{RH}_\infty$ are defined as

$$\|\mathbf{G}(s)\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{tr}(\mathbf{G}(j\omega)^*\mathbf{G}(j\omega))\mathrm{d}\omega$$
$$\|\mathbf{G}(s)\|_\infty = \sup_{\mathrm{Re}(s)>0} \bar{\sigma}(\mathbf{G}(s)) = \sup_{\omega \in \mathbb{R}} \bar{\sigma}(\mathbf{G}(j\omega)).$$

The symbol $\leftrightarrow$ represents the minimal state space realization of a transfer matrix, e.g. $\mathbf{G}(s) \leftrightarrow (\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$. Consider that for $\mathbf{G}(s) \in \mathcal{RH}_\infty$, there exists $\mathbf{X}, \mathbf{Y} \succeq 0$ which solve the following Lyapunov equations

$$\mathbf{AX} + \mathbf{XA}^* + \mathbf{BB}^* = \mathbf{0}$$
$$\mathbf{A}^*\mathbf{Y} + \mathbf{YA} + \mathbf{C}^*\mathbf{C} = \mathbf{0}.$$

Then $\mathbf{X}, \mathbf{Y}$ are called the controllability and observability gramians, respectively. Furthermore, $\sigma_{Hi}(\mathbf{G}(s)) = \lambda_i^{1/2}(\mathbf{XY})$ are the Hankel singular values of $\mathbf{G}(s)$

(Glover 1984, Zhou and Doyle 1998), which are a measure of joint controllability and observability of the poles of the system. As before, the maximum and minimum Hankel singular values are represented as $\bar{\sigma}_H(\mathbf{G}(s))$ and $\underline{\sigma}_H(\mathbf{G}(s))$ respectively. The state space realization of the transfer matrix is said to be balanced, if the $\mathbf{X}$ and $\mathbf{Y}$ that solve the corresponding Lyapunov equations are diagonal and equal. For notational convenience, we drop the frequency argument for a transfer matrix in the remaining discussion, i.e. $\mathbf{G}(s)$ is simply represented as $\mathbf{G}$.

2.2. *All pass factorization*

A linear system with right half plane (RHP) poles and zeros can be factored into an all-pass factor and a minimum phase or stable part. Such a factorization is useful for manipulation and simplification of expressions arising later in this paper. The two popular approaches for all-pass factorization of linear systems are inner–outer factorization and use of Blaschke products. For SISO systems, both these approaches produce identical results. For MIMO systems, use of Blaschke products provides analytical expressions and is preferred over inner–outer factorization in which solution of algebraic Riccati equations (AREs) is required (Morari and Zafiriou 1989). The idea of using Blaschke products for factorization of RHP poles and zeros was introduced by Wall *et al.* (1980) and has been earlier used for characterization of achievable performance by Chen (2000) and Havre and Skogestad (2001).

Let $\{z_i\}$, $i = 1, \ldots, n_z$ be the non-minimum phase or RHP zeros of $\mathbf{G}$. Then $\mathbf{G}$ can be factored as follows:

$$\mathbf{G} = \mathbf{G}^1 \mathcal{B}_1 \quad \mathcal{B}_1 = \mathbf{I} - \frac{2\mathrm{Re}(z_1)}{s + z_1^*} \hat{\mathbf{u}}_{z_1} \hat{\mathbf{u}}_{z_1}^*, \tag{1}$$

where $\hat{\mathbf{u}}_{z_1}$ is the input zero direction of z_1. With this factorization, z_1 is not a zero of $\mathbf{G}^1$. By repeated application of (1) on $\mathbf{G}^i$, $i = 1, \ldots, n_z - 1$, $\mathbf{G}$ can be factored into a minimum-phase part and an all pass filter as

$$\mathbf{G} = \mathbf{G}_{mi} \mathcal{B}_{zi} \quad \mathcal{B}_{zi} = \prod_{i=1}^{n_z} \left(\mathbf{I} - \frac{2\mathrm{Re}(z_i)}{s + z_i^*} \hat{\mathbf{u}}_{z_i} \hat{\mathbf{u}}_{z_i}^* \right). \tag{2}$$

In (2), $\mathbf{G}_{mi}$ is minimum phase with the RHP zeros of $\mathbf{G}$ mirrored across the imaginary axis and $\mathcal{B}_{zi}$ is an all pass filter. Note that except for the direction associated with the zero factored first, $\hat{\mathbf{u}}_{z_i}$ differs from $\mathbf{u}_{z_i}$, as it is calculated based on $\mathbf{G}^{(i-1)}$ and not $\mathbf{G}$. The RHP zeros can be alternatively factored at system's output similarly

$$\mathbf{G} = \mathcal{B}_{zo} \mathbf{G}_{mo} \quad \mathcal{B}_{zo} = \prod_{i=n_z}^{1} \left(\mathbf{I} - \frac{2\mathrm{Re}(z_i)}{s + z_i^*} \hat{\mathbf{y}}_{z_i} \hat{\mathbf{y}}_{z_i}^* \right). \tag{3}$$

When $\mathbf{G}$ has RHP poles at $\{p_i\}$, $i = 1, \ldots, n_p$, these poles can also be factored into a stable part and an all pass filter on the input and output sides as follows:

$$\mathbf{G} = \mathbf{G}_{si} \mathcal{B}_{pi}^{-1} \quad \mathcal{B}_{pi}^{-1} = \prod_{i=n_p}^{1} \left(\mathbf{I} + \frac{2\mathrm{Re}(p_i)}{s - p_i} \hat{\mathbf{u}}_{p_i} \hat{\mathbf{u}}_{p_i}^* \right) \tag{4}$$

$$\mathbf{G} = \mathcal{B}_{po}^{-1} \mathbf{G}_{so} \quad \mathcal{B}_{po}^{-1} = \prod_{i=1}^{n_p} \left(\mathbf{I} + \frac{2\mathrm{Re}(p_i)}{s - p_i} \hat{\mathbf{y}}_{p_i} \hat{\mathbf{y}}_{p_i}^* \right). \tag{5}$$

For later development in this paper, we derive the balanced state-space realization of the Blaschke product $\mathcal{B}_{pi}^{-1}$. For notational simplicity, we consider that the number of unstable poles $n_p = 2$ and similar results can be derived for systems with $n_p > 2$ by induction. A similar method has been used by Chen (2000) earlier for finding the balanced realization of $\mathcal{B}_{zi}$.

Let $\mathcal{B}_{pi}^{-1} = \mathcal{B}_{p_2}^{-1} \mathcal{B}_{p_1}^{-1}$. Using (4), the balanced realization of $\mathcal{B}_{p_i}^{-1}$ is given as $\mathcal{B}_{p_i}^{-1} \leftrightarrow (\mathbf{A}_i, \mathbf{B}_i, \mathbf{C}_i, \mathbf{D}_i)$, where

$$\mathbf{A}_i = p_i \quad \mathbf{B}_i = \sqrt{2\mathrm{Re}(p_i)}\hat{\mathbf{u}}_{p_i}^* \quad \mathbf{C}_i = \sqrt{2\mathrm{Re}(p_i)}\hat{\mathbf{u}}_{p_i} \quad \mathbf{D}_i = \mathbf{I}. \tag{6}$$

Using (6), the balanced realization of $\mathcal{B}_{pi}^{-1}$ is given as $\mathcal{B}_{pi}^{-1} \leftrightarrow (\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$, where

$$\left.\begin{aligned}
\mathbf{A} &= \begin{bmatrix} \mathbf{A}_2 & \mathbf{B}_2\mathbf{C}_1 \\ \mathbf{0} & \mathbf{A}_1 \end{bmatrix} \\
&= \begin{bmatrix} p_2 & 2\sqrt{\mathrm{Re}(p_1)\,\mathrm{Re}(p_2)}\,\hat{\mathbf{u}}_{p_2}^*\hat{\mathbf{u}}_{p_1} \\ 0 & p_1 \end{bmatrix} \\
\mathbf{B} &= \begin{bmatrix} \mathbf{B}_2\mathbf{D}_1 \\ \mathbf{B}_1 \end{bmatrix} = \begin{bmatrix} \sqrt{2\mathrm{Re}(p_2)}\,\hat{\mathbf{u}}_{p_2}^* \\ \sqrt{2\mathrm{Re}(p_1)}\,\hat{\mathbf{u}}_{p_1}^* \end{bmatrix} \\
\mathbf{C} &= \begin{bmatrix} \mathbf{C}_2 & \mathbf{D}_2\mathbf{C}_1 \end{bmatrix} \\
&= \begin{bmatrix} \sqrt{2\mathrm{Re}(p_2)}\,\hat{\mathbf{u}}_{p_2} & \sqrt{2\mathrm{Re}(p_1)}\,\hat{\mathbf{u}}_{p_1} \end{bmatrix} \\
\mathbf{D} &= \mathbf{D}_2\mathbf{D}_1 = \mathbf{I}.
\end{aligned}\right\} \tag{7}$$

2.3. *Optimal control*

In this paper, we use a state-space approach for characterization of achievable input performance. For this purpose, we briefly review the pioneering results on $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal control due to Doyle *et al.* (1989). Further details can be found in many recently published textbooks dealing with optimal control, e.g. Zhou and Doyle (1998). In later sections, we show how these general results simplify when input performance is maximized.

Let $\mathbf{z}$ and $\mathbf{w}$ denote the exogenous outputs and inputs and, $\mathbf{y}$ and $\mathbf{u}$ be the measured and manipulated variables, respectively. The model of the generalized plant from $\mathbf{w}$ to $\mathbf{z}$ has the following form

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}_w\mathbf{w} + \mathbf{B}\mathbf{u}\\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}_{21}\mathbf{w} \\ \mathbf{z} &= \mathbf{C}_z\mathbf{x} + \mathbf{D}_{12}\mathbf{u}.\end{aligned} \tag{8}$$

Assumption 1: *System* (8) *is assumed to be in the standard form* (Doyle *et al.* 1989)

(a) $(\mathbf{A}, \mathbf{B}_w)$ *is stabilizable and* $(\mathbf{A}, \mathbf{C}_z)$ *is detectable.*
(b) $(\mathbf{A}, \mathbf{B})$ *is stabilizable and* $(\mathbf{A}, \mathbf{C})$ *is detectable.*
(c) $\mathbf{D}_{12}^*\mathbf{D}_{12} = \mathbf{I}$ *and* $\mathbf{D}_{21}^*\mathbf{D}_{21} = \mathbf{I}$.
(d) $\mathbf{D}_{12}^*\mathbf{C}_z = \mathbf{0}$ *and* $\mathbf{D}_{21}^*\mathbf{B}_w = \mathbf{0}$.

In addition, the assumptions that $\mathbf{D}_{11} = 0$ and $\mathbf{D}_{22} = 0$ are implicit in the realization of the generalized plant (8). The assumption that $\mathbf{D}_{22} = 0$ can be easily satisfied by a linear fractional transformation on the controller $\mathbf{K}$ (Zhou and Doyle 1998, p. 261). $\mathbf{D}_{11} = 0$ is necessary for well-posedness of the $\mathcal{H}_2$ optimal control problem. In general, this assumption can be relaxed for the $\mathcal{H}_\infty$ optimal control problem, but this complicates the formulae substantially. Some additional details on physical interpretation of Assumption 1 and transforming the problem to satisfy them can be found in Skogestad and Postlethwaite (1996, p. 363).

It follows from Assumption 1(a)–(b) that there exist $\mathbf{X}_2, \mathbf{Y}_2 \succeq 0$, which solve the following AREs,

$$\begin{aligned}\mathbf{A}^*\mathbf{X}_2 + \mathbf{X}_2\mathbf{A} - \mathbf{X}_2\mathbf{B}\mathbf{B}^*\mathbf{X}_2 + \mathbf{C}_z^*\mathbf{C}_z = \mathbf{0}\\ \mathbf{A}\mathbf{Y}_2 + \mathbf{Y}_2\mathbf{A}^* - \mathbf{Y}_2\mathbf{C}^*\mathbf{C}\mathbf{Y}_2 + \mathbf{B}_w\mathbf{B}_w^* = \mathbf{0}.\end{aligned}$$

Let $\mathbf{T}_{zw}$ be the closed loop transfer matrix from $\mathbf{w}$ to $\mathbf{z}$. The unique controller minimizing $\|\mathbf{T}_{zw}\|_2$ is given as (Doyle *et al.* 1989):

$$\mathbf{K}_{\text{opt}} = \left[\begin{array}{c|c}\mathbf{A} + \mathbf{B}\mathbf{F}_2 + \mathbf{L}_2\mathbf{C} & -\mathbf{L}_2\\ \hline \mathbf{F}_2 & \mathbf{0}\end{array}\right], \tag{9}$$

where $\mathbf{F}_2 = -\mathbf{B}^*\mathbf{X}_2$, $\mathbf{L}_2 = -\mathbf{Y}_2\mathbf{C}^*$ and optimal cost is (Zhou and Doyle 1998),

$$I_2^2 = \inf_{\mathbf{K}} \|\mathbf{T}_{zw}\|_2^2 = \text{tr}(\mathbf{B}_w^*\mathbf{X}_2\mathbf{B}_w) + \text{tr}(\mathbf{F}_2\mathbf{Y}_2\mathbf{F}_2^*). \tag{10}$$

For the minimization of $\|\mathbf{T}_{zw}\|_\infty$, let $\mathbf{X}_\infty, \mathbf{Y}_\infty \succeq 0$ solve the following AREs,

$$\mathbf{A}^*\mathbf{X}_\infty + \mathbf{X}_\infty\mathbf{A} - \mathbf{X}_\infty(\gamma^{-2}\mathbf{B}_w\mathbf{B}_w^* - \mathbf{B}\mathbf{B}^*)\mathbf{X}_\infty + \mathbf{C}_z^*\mathbf{C}_z = \mathbf{0} \tag{11}$$

$$\mathbf{A}\mathbf{Y}_\infty + \mathbf{Y}_\infty\mathbf{A}^* - \mathbf{Y}_\infty(\gamma^{-2}\mathbf{C}_z^*\mathbf{C}_z - \mathbf{C}^*\mathbf{C})\mathbf{Y}_\infty + \mathbf{B}_w\mathbf{B}_w^* = \mathbf{0}, \tag{12}$$

where $\gamma > 0$. The existence of $\mathbf{X}_\infty, \mathbf{Y}_\infty \succeq 0$ that solve the AREs (11)–(12) is guaranteed, if Assumption 1 holds and $\rho(\mathbf{X}_\infty\mathbf{Y}_\infty) < \gamma^2$. A suboptimal controller achieving $\|\mathbf{T}_{zw}\|_\infty < \gamma$ is (Doyle *et al.* 1989)

$$\mathbf{K}_{\text{sub}} = \left[\begin{array}{c|c}\mathbf{A}_\infty = \mathbf{A} + \gamma^{-2}\mathbf{B}_w\mathbf{B}^*\mathbf{X}_\infty + \mathbf{B}\mathbf{F}_\infty + \mathbf{Z}_\infty\mathbf{L}_\infty\mathbf{C} & -\mathbf{Z}_\infty\mathbf{L}_\infty\\ \hline \mathbf{F}_\infty & \mathbf{0}\end{array}\right], \tag{13}$$

where $\mathbf{F}_\infty = -\mathbf{B}^*\mathbf{X}_\infty$, $\mathbf{L}_\infty = -\mathbf{Y}_\infty\mathbf{C}^*$ and $\mathbf{Z}_\infty = (\mathbf{I} - \gamma^{-2}\rho(\mathbf{X}_\infty\mathbf{Y}_\infty))^{-1}$. The optimal cost is given as

$$I_\infty = \inf_{\mathbf{K}} \|\mathbf{T}_{zw}\|_\infty = \rho^{1/2}(\mathbf{X}_\infty\mathbf{Y}_\infty) \tag{14}$$

3. Problem formulation and simplification

In this section, we formulate an optimal controller design problem that minimizes input usage for stabilization. It is shown how the general results on optimal control can be simplified when only input performance is considered. This simplification in turn enables us to explicitly characterize the achievable input performance.

Consider the system shown in figure 1, where all exogenous inputs have been collected in the block $\mathbf{G}_w$. The closed loop transfer matrix from disturbances to inputs is given as

$$\mathbf{T}_{uw} = \mathbf{W}_u\mathbf{K}(\mathbf{I} + \mathbf{G}\mathbf{K})^{-1}\mathbf{G}_w. \tag{15}$$

The objective is to characterize the minimal input usage required for stabilization expressed in terms of

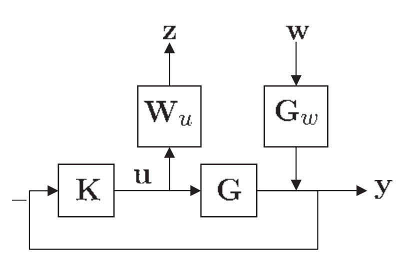

Figure 1. Closed loop system for characterization of achievable input performance. The effect of all exogenous inputs including sensor noise, disturbances and set point changes is collected in $\mathbf{G}_w$.

the norm of $\mathbf{T}_{uw}$ as

$$I_i = \|\mathbf{W}_u\mathbf{K}(\mathbf{I}+\mathbf{G}\mathbf{K})^{-1}\mathbf{G}_w\|_i \quad i = 2, \infty. \tag{16}$$

Assumption 2: *We make the following assumptions*

(a) $\mathbf{G}$ *is strictly proper.*
(b) $\mathbf{W}_u$ *is left invertible and (if unstable) has the same unstable poles as* $\mathbf{G}$ *with the associated input pole directions.*
(c) $\mathbf{G}_w$ *is right-invertible and (if unstable) has the same unstable poles as* $\mathbf{G}$ *with the associated output pole directions.*

Assumption 2(a) is made for notational simplicity and the extension to the general case is simple (see Zhou and Doyle (1998, p. 261) for details). The left and right invertibility of $\mathbf{W}_u$ and $\mathbf{G}_w$, respectively, ensures that the optimal controller design problem is non-singular.

To illustrate the necessity of $\mathbf{W}_u$ and $\mathbf{G}_w$ having the same unstable poles as $\mathbf{G}$ with the associated input and output pole directions respectively, consider that $\mathbf{W}_u = \mathbf{I}$ and $\mathbf{G}_w$ has a single unstable pole p_w such that $\mathbf{G}_w^{-1}(p_w)\mathbf{y}_{p_w} = \mathbf{0}$. Let $p_1, \ldots, p_{n_p}$, $\mathrm{Re}(p_i) \geq 0$ be the unstable poles of $\mathbf{G}$ such that $\mathbf{G}^{-1}(p_i)\mathbf{y}_{p_i} = \mathbf{0}$. For internal stability, the unstable poles of $\mathbf{G}$ and $\mathbf{GK}$ are the same and

$$\begin{aligned}
\mathbf{K}^{-1}\mathbf{G}^{-1}(p_i)\mathbf{y}_{p_i} &= \mathbf{0}\\
\left(\mathbf{I}+\mathbf{K}^{-1}\mathbf{G}^{-1}(p_i)\right)\mathbf{y}_{p_i} &= \mathbf{y}_{p_i}\\
\mathbf{G}\mathbf{K}(p_i)(\mathbf{I}+\mathbf{G}\mathbf{K}(p_i))^{-1}\mathbf{y}_{p_i} &= \mathbf{y}_{p_i}\\
\mathbf{K}(p_i)(\mathbf{I}+\mathbf{G}\mathbf{K}(p_i))^{-1}\mathbf{y}_{p_i} &= \mathbf{G}^{-1}(p_i)\mathbf{y}_{p_i} = \mathbf{0}.
\end{aligned} \tag{17}$$

Equation (17) is similar to the interpolation or analyticity constraints on sensitivity and complementary sensitivity functions derived by Zames (1981). It follows from (17) that the location and input zero directions of $\mathbf{K}(\mathbf{I}+\mathbf{G}\mathbf{K})^{-1}$ are same as the locations of RHP poles and output pole directions of $\mathbf{G}$. Defining the sensitivity function as $\mathbf{S} = (\mathbf{I}+\mathbf{G}\mathbf{K})^{-1}$ and using the results on Blaschke products (2) and (5),

$$\begin{aligned}
\mathbf{K}\mathbf{S}\mathbf{G}_w &= (\mathbf{KS})_{mi}\,\mathcal{B}_{zi}(\mathbf{KS})\,\mathcal{B}_{po}^{-1}(\mathbf{G}_w)\,(\mathbf{G}_w)_{so}\\
&= (\mathbf{KS})_{mi}\,\mathcal{B}_{po}(\mathbf{G})\,\mathcal{B}_{po}^{-1}(\mathbf{G}_w)\,(\mathbf{G}_w)_{so}.
\end{aligned}$$

If the controller is designed to stabilize $\mathbf{KS}$, the stability of $\mathbf{T}_{uw}$ depends on the stability of $\mathcal{B}_{po}(\mathbf{G})\,\mathcal{B}_{po}^{-1}(\mathbf{G}_w)$. Since the Blaschke products can be calculated for any permutation of poles and zeros, $\mathcal{B}_{po}(\mathbf{G})\mathcal{B}_{po}^{-1}(\mathbf{G}_w)$ is stable if and only if $p_w = p_i$ and $\mathbf{y}_{p_w} = \mathbf{y}_{p_i}$ for some i, $i = 1, \ldots, n_p$. Similar conclusions can be drawn when $\mathbf{G}_w$ has more than one unstable pole or when $\mathbf{W}_u$ is also unstable.

With Assumption 2, let $\mathbf{W}_u$ and $\mathbf{G}_w$ be factorized as

$$\begin{aligned}
\mathbf{W}_u &= \mathcal{B}_{po}^{-1}(\mathbf{W}_u)\mathcal{B}_{zo}(\mathbf{W}_u)(\mathbf{W}_u)_{sm}\\
\mathbf{G}_w &= (\mathbf{G}_w)_{sm}\mathcal{B}_{pi}^{-1}(\mathbf{G}_w)\mathcal{B}_{zi}(\mathbf{G}_w),
\end{aligned}$$

where $(\mathbf{W}_u)_{sm}$ and $(\mathbf{G}_w)_{sm}$ are the stable minimum-phase parts of $\mathbf{W}_u$ and $\mathbf{G}_w$, respectively. Define

$$\begin{aligned}
\hat{\mathbf{G}} &= (\mathbf{G}_w)_{sm}^{-1}\mathbf{G}(\mathbf{W}_u)_{sm}^{-1}\\
\hat{\mathbf{K}} &= (\mathbf{W}_u)_{sm}\mathbf{K}(\mathbf{G}_w)_{sm},
\end{aligned} \tag{18}$$

where $\hat{\mathbf{G}}$ is an $n_y \times n_u$ dimensional transfer matrix. It follows from (15) that

$$\begin{aligned}
\|\mathbf{T}_{uw}\|_i &= \|\mathcal{B}_{po}^{-1}(\mathbf{W}_u)\mathcal{B}_{zo}(\mathbf{W}_u)\hat{\mathbf{K}}(\mathbf{I}+\hat{\mathbf{G}}\hat{\mathbf{K}})^{-1}\mathcal{B}_{pi}^{-1}(\mathbf{G}_w)\mathcal{B}_{zi}(\mathbf{G}_w)\|_i\\
&= \|\hat{\mathbf{K}}(\mathbf{I}+\hat{\mathbf{G}}\hat{\mathbf{K}})^{-1}\|_i \quad i = 2, \infty.
\end{aligned} \tag{19}$$

We point out that in (19), $\mathcal{B}_{po}^{-1}(\mathbf{W}_u)$ and $\mathcal{B}_{pi}^{-1}(\mathbf{G}_w)$ can be factored out without jeopardizing the internal stability, only when Assumptions 2(b)–(c) are satisfied. Now, $\|\mathbf{T}_{uw}\|_i$, $i = 2, \infty$ is minimized by designing an optimal controller for $\hat{\mathbf{G}}$, where the following are equivalent: (a) $\hat{\mathbf{K}}$ stabilizes $\hat{\mathbf{G}}$ and (b) $\mathbf{K}$ stabilizes $\mathbf{G}$. In the remaining discussion, we treat $\hat{\mathbf{G}}$ as the system without loss of generality. These manipulations further allows us to represent the generalized plant as

$$\left.\begin{aligned}
\dot{\hat{\mathbf{x}}} &= \hat{\mathbf{A}}\hat{\mathbf{x}} + \hat{\mathbf{B}}\mathbf{u}\\
\mathbf{y} &= \hat{\mathbf{C}}\hat{\mathbf{x}} + \mathbf{w}\\
\mathbf{z} &= \mathbf{u},
\end{aligned}\right\} \tag{20}$$

where $\hat{\mathbf{G}} \leftrightarrow (\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}})$. Notice that we have transformed a controller design problem where the closed loop system is driven by disturbances filtered through an arbitrary disturbance model to an equivalent problem, in which the closed loop system is driven by measurement noise only. The latter problem is much simpler to solve, as demonstrated later in this section.

For the system (20), let $\hat{\mathbf{X}}_2, \hat{\mathbf{Y}}_2$ and $\hat{\mathbf{X}}_\infty, \hat{\mathbf{Y}}_\infty$ be the solutions of corresponding AREs for the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal controller design problems, respectively (see § 2.3). By comparing (20) with the standard form of generalized plant (8), we notice that for the system (20), the corresponding AREs for the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal controller design problems are the same. It follows that $\hat{\mathbf{X}}_2 = \hat{\mathbf{X}}_\infty = \hat{\mathbf{X}}$ and $\hat{\mathbf{Y}}_2 = \hat{\mathbf{Y}}_\infty = \hat{\mathbf{Y}}$. This observation in turn implies that $\hat{\mathbf{F}}_2 = \hat{\mathbf{F}}_\infty = \hat{\mathbf{F}}$ and $\hat{\mathbf{L}}_2 = \hat{\mathbf{L}}_\infty = \hat{\mathbf{L}}$.

Let $\mathbf{T}$ be a state transformation matrix such that $\mathbf{T}^{-1}\hat{\mathbf{A}}\mathbf{T} = \mathrm{diag}(\mathbf{P}_s, \mathbf{P})$, where $\mathbf{P}_s$ and $\mathbf{P}$ contain all the stable and unstable modes, respectively. Rearranging and partitioning the states of the transformed system

$$\left.\begin{aligned}\dot{\tilde{\mathbf{x}}} &= \mathbf{T}^{-1}\hat{\mathbf{A}}\mathbf{T}\tilde{\mathbf{x}} + \mathbf{T}^{-1}\hat{\mathbf{B}}\mathbf{u} = \begin{bmatrix}\mathbf{P}_s & \mathbf{0}\\ \mathbf{0} & \mathbf{P}\end{bmatrix}\tilde{\mathbf{x}} + \begin{bmatrix}\mathbf{B}_s\\ \mathbf{B}\end{bmatrix}\mathbf{u}\\ \mathbf{y} &= \hat{\mathbf{C}}\mathbf{T}\tilde{\mathbf{x}} + \mathbf{w} = \begin{bmatrix}\mathbf{C}_s & \mathbf{C}\end{bmatrix}\tilde{\mathbf{x}} + \mathbf{w}.\end{aligned}\right\} \quad (21)$$

Let $\tilde{\mathbf{X}} = \mathbf{T}^{-1}\hat{\mathbf{X}}\mathbf{T}$ and $\tilde{\mathbf{Y}} = \mathbf{T}^{-1}\hat{\mathbf{Y}}\mathbf{T}$ solve the corresponding AREs for the transformed system (21). Then, to be non-negative definite, $\tilde{\mathbf{X}}$ and $\tilde{\mathbf{Y}}$ must assume the form

$$\tilde{\mathbf{X}} = \begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{X}\end{bmatrix} \quad \tilde{\mathbf{Y}} = \begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{Y}\end{bmatrix},$$

where $\mathbf{X}, \mathbf{Y} \in \mathbb{C}^{n_p \times n_p} \succ 0$. Then it suffices to solve

$$\mathbf{X}\mathbf{P} + \mathbf{P}^*\mathbf{X} - \mathbf{X}\mathbf{B}\mathbf{B}^*\mathbf{X} = \mathbf{0} \quad (22)$$

$$\mathbf{Y}\mathbf{P}^* + \mathbf{P}\mathbf{Y} - \mathbf{Y}\mathbf{C}^*\mathbf{C}\mathbf{Y} = \mathbf{0}. \quad (23)$$

Let $\hat{\mathbf{G}} = \hat{\mathbf{G}}_1 + \hat{\mathbf{G}}_2$ such that $\hat{\mathbf{G}}_1 = \mathcal{U}(\hat{\mathbf{G}})$ and $\hat{\mathbf{G}}_2 \in \mathcal{RH}_\infty$. Here $\mathcal{U}(\hat{\mathbf{G}})$ denotes the unstable part of $\mathbf{G}$. The triplet $(\mathbf{P}, \mathbf{B}, \mathbf{C})$ can be seen as the realization of $\hat{\mathbf{G}}_1$ and (22)–(23) as the corresponding AREs for $\hat{\mathbf{G}}_1$. Then, the achievable input performance depends only on the unstable part of the system. This is further illustrated by defining $\hat{\mathbf{K}} = \hat{\mathbf{K}}_1(\mathbf{I} - \hat{\mathbf{G}}_2\hat{\mathbf{K}}_1)^{-1}$. With this parametrization of $\hat{\mathbf{K}}$,

$$\hat{\mathbf{K}}(\mathbf{I} - \hat{\mathbf{G}}\hat{\mathbf{K}})^{-1} = \hat{\mathbf{K}}_1(\mathbf{I} - \hat{\mathbf{G}}_1\hat{\mathbf{K}}_1)^{-1}.$$

Thus $\hat{\mathbf{K}}$ exactly cancels the stable part of the system. The different transformations used in this section and their equivalence are shown in figure 2.

For the transformed system (21), the state feedback and the output injection matrices are given as

$$\tilde{\mathbf{F}} = \hat{\mathbf{F}}\mathbf{T} = \begin{bmatrix}\mathbf{0} & \mathbf{F}\end{bmatrix} = \begin{bmatrix}\mathbf{0} & -\mathbf{B}^*\mathbf{X}\end{bmatrix} \quad (24)$$

$$\tilde{\mathbf{L}} = \mathbf{T}^*\hat{\mathbf{L}} = \begin{bmatrix}\mathbf{0} & \mathbf{L}\end{bmatrix}' = \begin{bmatrix}\mathbf{0} & -\mathbf{Y}\mathbf{C}^*\end{bmatrix}'. \quad (25)$$

By substituting for $\tilde{\mathbf{X}}, \tilde{\mathbf{Y}}, \tilde{\mathbf{F}}$ and $\tilde{\mathbf{L}}$ in (10) and (14), the expressions for achievable input performance can be simplified as

$$I_2^2 = \operatorname{tr}(\mathbf{F}\mathbf{Y}\mathbf{F}^*) = \operatorname{tr}(\mathbf{L}^*\mathbf{X}\mathbf{L}) \quad (26)$$

$$I_\infty = \rho^{1/2}(\mathbf{X}\mathbf{Y}). \quad (27)$$

The equations (22) and (23) form the cornerstone for much of the remaining development in this paper. In general, for $\mathcal{H}_\infty$ optimal control, the resulting AREs are dependent on γ and thus need to be solved iteratively. In contrast, the expressions (22)–(23) are independent of γ and can be solved directly. Further note that when (22) and (23) are pre- and post-multiplied by $\mathbf{X}^{-1}$ and $\mathbf{Y}^{-1}$, the resulting expressions are similar to Lyapunov equations. When all of the unstable poles of the system are distinct, closed form solutions of (22)–(23) can be derived, which are expressed in terms of the unstable poles and the matrices $\mathbf{B}$ and $\mathbf{C}$ only, as shown next.

For a system with distinct unstable poles, we can select the state transformation matrix $\mathbf{T}$ such that $\mathbf{P}$ is diagonal and is given as $\mathbf{P} = \operatorname{diag}(p_1, \ldots, p_{n_p})$,

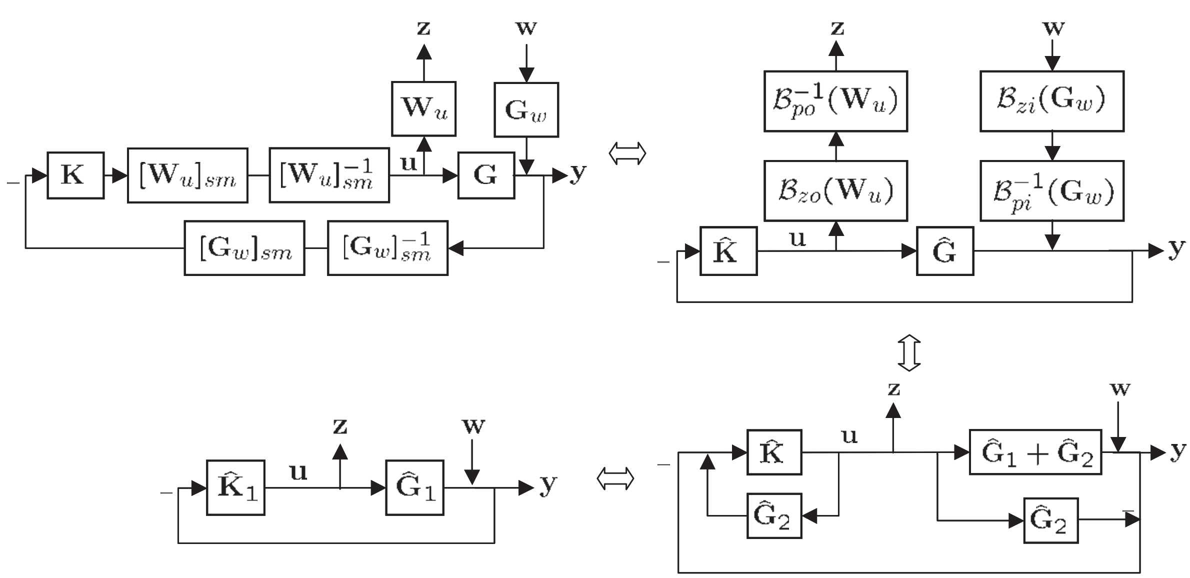

Figure 2. Simplifying transformations on the closed loop system.

$\text{Re}(p_i) > 0$. Let the Hermitian matrix $\mathbf{M} \in \mathbb{C}^{n_p \times n_p}$ be defined as

$$[m_{ij}] = 1/(p_i + p_j^*). \tag{28}$$

Lemma 1: *For a system with distinct unstable poles, let* $\mathbf{X}, \mathbf{Y} \succ 0$ *solve the AREs* (22)–(23) *and* $\mathbf{M}$ *be given by* (28). *Then*

$$\mathbf{X}^{-1} = \sum_{i=1}^{n_u} \text{diag}(\mathbf{B}_i)\, \mathbf{M}\, \text{diag}(\mathbf{B}_i)^* \tag{29}$$

$$\mathbf{Y}^{-1} = \sum_{j=1}^{n_y} \text{diag}(\mathbf{C}'_j)^*\, \mathbf{M}'\, \text{diag}(\mathbf{C}'_j). \tag{30}$$

Proof: Pre- and post multiplying (22) by $\mathbf{X}^{-1}$ gives

$$\mathbf{P}\mathbf{X}^{-1} + \mathbf{X}^{-1}\mathbf{P}^* = \mathbf{B}\mathbf{B}^*. \tag{31}$$

Then $\mathbf{X}^{-1} = \mathbf{M} \circ (\mathbf{B}\mathbf{B}^*)$ (Horn and Johnson 1991). Noting that $\mathbf{B}\mathbf{B}^* = \sum_{i=1}^{n_u} \mathbf{B}_i\mathbf{B}_i^*$,

$$\mathbf{X}^{-1} = \sum_{i=1}^{n_u} \mathbf{M} \circ (\mathbf{B}_i\mathbf{B}_i^*)$$

and (29) follows. Equation (30) follows from a dual argument. □

4. SISO systems

In this section, we quantify achievable input performance for SISO systems with and without time delay. These results are generalized to MIMO systems in the next section, which naturally also hold for SISO systems. SISO systems are considered separately primarily for two reasons: (i) under the minor assumption that the unstable poles of the system are distinct, the expressions for the achievable input performance can be written in terms of the unstable poles and the matrices **B** and **C** only, providing more insight and (ii) they facilitate the derivation of some of the more involved expressions for MIMO systems (particularly for time delay systems).

4.1. *Rational systems*

We derive the expressions for achievable input performance for rational SISO systems and demonstrate their usefulness with a simple design example. These results also form the basis for derivation of similar expressions for SISO systems with time delay.

Lemma 2: *For* $\mathbf{M}$ *defined by* (28), *let* $p_i \neq p_j$ *for all* $i, j = 1, \ldots, n_p$. *Then* $\mathbf{M}^{-1}$ *is given as*

$$[\mathbf{M}^{-1}]_{ij} = \frac{4\text{Re}(p_i)\text{Re}(p_j)}{p_i^* + p_j} \left(\prod_{\substack{k=1\\k\neq i}}^{n_p} \frac{(p_i^* + p_k)}{(p_i^* - p_k^*)} \right) \times \left(\prod_{\substack{k=1\\k\neq j}}^{n_p} \frac{(p_j + p_k^*)}{(p_j - p_k)} \right).$$

Lemma 2 is easily verified by evaluating $\mathbf{M}\mathbf{M}^{-1}$ or $\mathbf{M}^{-1}\mathbf{M}$ and the proof is omitted. Note for SISO systems, $\mathbf{b} = [b_i], \mathbf{c} = [c_j]$.

Proposition 1: *For the rational SISO system* $\hat{g}$ *in* (18) *with distinct unstable poles, let* $\mathcal{U}(\hat{g}) \leftrightarrow (\mathbf{P}, \mathbf{b}, \mathbf{c})$ *such that* $\mathbf{P} = \text{diag}(p_1, \ldots, p_{n_p}), \text{Re}(p_i) > 0$. *Then*

$$I_2^2 = \left[\frac{\mathbf{q}_i^2}{b_i c_i}\right] \mathbf{M} \left[\frac{\mathbf{q}_i^2}{b_i^* c_i^*}\right]' \tag{32}$$

$$I_\infty^2 = \left|\underline{\lambda}^{-1}(\text{diag}(b_i^* c_i^*)\mathbf{M}'\text{diag}(b_i c_i)\mathbf{M})\right|, \tag{33}$$

where $\mathbf{M}$ is defined by (28) and $\mathbf{q}_i$ is the sum of ith column of $\mathbf{M}^{-1}$ or $\mathbf{q} = \mathbf{1}'_{n_p}\mathbf{M}^{-1}$.

Proof

(1) For (32), substituting for **X** and **Y** in the expression for I_2 (26) using Lemma 1,

$$I_2^2 = \mathbf{f}\mathbf{Y}\mathbf{f}^* = \mathbf{b}^*\mathbf{X}\mathbf{Y}\mathbf{X}\mathbf{b}$$
$$= \mathbf{1}'_{n_p}\mathbf{M}^{-1}(\text{diag}(\mathbf{b})\text{diag}(\mathbf{c}))^{-1}(\mathbf{M}')^{-1} \times (\text{diag}(\mathbf{b}^*)\text{diag}(\mathbf{c}^*))^{-1}\mathbf{M}^{-1}\mathbf{1}_{n_p}. \tag{34}$$

Based on Lemma 2,

$$\mathbf{q}_i = \sum_{j=1}^{n_p} [\mathbf{M}^{-1}]_{ij} = 2\,\text{Re}(p_i) \prod_{\substack{k=1\\k\neq i}}^{n_p} \frac{(p_i + p_k^*)}{(p_i - p_k)}; \quad i = 1, \ldots, n_p \tag{35}$$

and $\mathbf{M}^{-1} = \text{diag}(\mathbf{q}^*)\mathbf{M}'\text{diag}(\mathbf{q})$. By substituting for $\mathbf{M}^{-1}$ and $\mathbf{q}$, (34) can be simplified as,

$$I_2^2 = \mathbf{q}(\text{diag}(\mathbf{b})\text{diag}(\mathbf{c}))^{-1}\text{diag}(\mathbf{q})\mathbf{M}\text{diag}(\mathbf{q}^*) \times (\text{diag}(\mathbf{b}^*)\text{diag}(\mathbf{c}^*))^{-1}\mathbf{q}^*.$$

Equation (32) can be obtained by simplifying the above expression.

(2) For (33),

$$I_\infty^2 = \rho(\mathbf{X}\mathbf{Y}) = \left|\underline{\lambda}^{-1}(\mathbf{Y}^{-1}\mathbf{X}^{-1})\right|$$

By substituting for $\mathbf{X}^{-1}$ and $\mathbf{Y}^{-1}$ using Lemma 1

$$\begin{aligned} I_\infty^2 &= \left|\underline{\lambda}^{-1}(\text{diag}(\mathbf{c}^*)\,\mathbf{M}'\,\text{diag}(\mathbf{c})\text{diag}(\mathbf{b})\,\mathbf{M}\,\text{diag}(\mathbf{b}^*))\right| \\ &= \left|\underline{\lambda}^{-1}(\text{diag}(\mathbf{b}^*)\text{diag}(\mathbf{c}^*)\,\mathbf{M}'\,\text{diag}(\mathbf{c})\text{diag}(\mathbf{b})\,\mathbf{M})\right| \\ &= \left|\underline{\lambda}^{-1}(\text{diag}(b_i^* c_i^*)\,\mathbf{M}'\text{diag}(b_i c_i)\mathbf{M})\right|. \end{aligned}$$ □

Using Proposition 1, the optimal value of z can be characterized explicitly. As $z \to p_i$, the joint controllability and observability of p_i reduces monotonically, increasing the input requirement. Using (32) and (36),

$$I_2^2 = \frac{8(p_1+p_2)^3\left[p_1^2(p_2-z)^2 + p_2^2(p_1-z)^2 + p_1p_2(3z^2 - p_1p_2)\right]}{(p_1-z)^2(p_2-z)^2} \quad (37)$$

$$I_\infty = \frac{4p_1p_2(p_1+p_2)}{z(p_1+p_2) - \left[p_1^2(2p_2-z)^2 + p_2^2(2p_1-z)^2 + 2p_1p_2(3z^2 - 2p_1p_2)\right]^{0.5}}. \quad (38)$$

In the realization, $\mathcal{U}(\hat{g}) \leftrightarrow (\mathbf{P}, \mathbf{b}, \mathbf{c})$, when $\hat{g}$ has only real unstable poles, $\text{diag}(b_i^* c_i^*) = \text{diag}(b_i c_i)$ and $\mathbf{M}' = \mathbf{M}$. In this case, the expression for I_∞ (33) can be further simplified as,

$$\begin{aligned} I_\infty^2 &= \left|\underline{\lambda}^{-1}\left((\text{diag}(b_i c_i)\mathbf{M})^2\right)\right| \\ I_\infty &= \left|\underline{\lambda}^{-1}(\text{diag}(b_i c_i)\mathbf{M})\right|. \end{aligned} \quad (36)$$

The expression for $\mathbf{q}$ in (35) appears to suggest that in general, $I_2 \to \infty$ as $p_i \to p_j$ for some i, j, which is clearly not true. Since $b_i c_i = [\hat{g}(s)(s-p_i)]_{s=p_i}$, $b_i c_i \to \infty$, as $p_i \to p_j$, which negates the effect of $\mathbf{q}$. When the system has an RHP zero close to RHP poles, $b_i c_i$ fails to increase monotonically and stabilization can be difficult. For example, consider

$$\hat{g} = \frac{(s-p)}{(s-p+\epsilon)(s-p-\epsilon)}.$$

As $\epsilon \to 0$, the RHP poles approach the RHP zero. Due to near cancellation of the unstable pole by the unstable zero, $I_2, I_\infty \to \infty$ as $\epsilon \to 0$.

Example 1: In order to demonstrate the utility of Proposition 1 for process design purposes, consider a rational SISO system with two distinct unstable poles $p_1, p_2 \in \mathbb{R}$, $p_1 < p_2$ and a RHP zero z. The location of z can be influenced by process or operating point changes. Such a system can arise, when different systems are connected in parallel. The objective is to choose z in the range $p_1 < z < p_2$, such that input usage for stabilization is minimal. A purely numerical approach requires solving the following nested optimization problem

$$\min_z \inf_k \|k(1+gk)^{-1}\|_i \quad i = 2, \infty.$$

The *locally* optimal value of z in the range $p_1 < z < p_2$ can be obtained by evaluating the stationary points of (37) and (38),

$$z_{\mathcal{H}_2,\text{opt}} = \frac{p_1p_2(3(p_1+p_2) + \sqrt{5p_1^2 + 5p_2^2 + 6p_1p_2})}{2(p_1^2 + p_2^2 + 3p_1p_2)} \quad (39)$$

$$z_{\mathcal{H}_\infty,\text{sub}} = \frac{4p_1p_2(p_1+p_2)}{p_1^2 + p_2^2 + 6p_1p_2}. \quad (40)$$

As an example, for $\hat{g} = (s-z)/((s-p_1)(s-p_2))$ with $p_1 = 1$ and $p_2 = 2$, the variation of I_2 with z is shown in figure 3. The locally optimal zero location in the range $1 < z < 2$ is $z_{\mathcal{H}_2,\text{opt}} = 1.37$, which can also be confirmed using (39).

It is also noted that unlike the output performance, unstable zeros do not limit the achievable input performance, except when located close to unstable poles resulting in near pole-zero cancellation. As an example, consider two SISO systems $g_1 = (s-z)/((s-1)(s-2))$ and $g_2 = (s+z)/((s-1)(s-2))$, $z > 0$. Note that g_1 and g_2 have same unstable poles, but

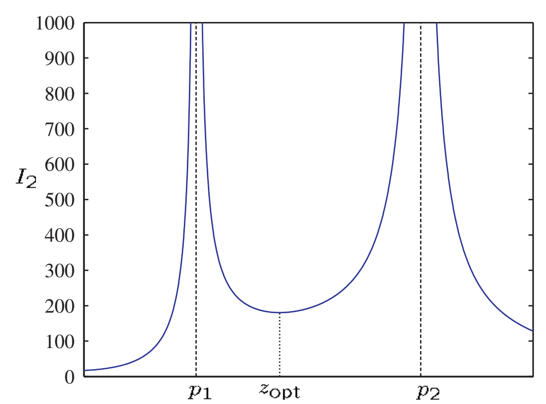

Figure 3. Variation of I_2 with z for $\hat{g} = ((s-z)/(s-p_1)(s-p_2))$ with $p_1 = 1$ and $p_2 = 2$. The locally optimal zero location in the range $p_1 < z < p_2$ is $z_{\mathcal{H}_2,\text{opt}} = 1.37$.

have zeros at z and $-z$, respectively. Using (37), it can be shown that in the range, $0 \leq z \leq 0.25$, the achievable $\mathcal{H}_2$ optimal input performance for these two systems is nearly the same (for example, $I_2 \approx 14.5$ for $z = 0.25$ in either case). When z approaches 1, however, the input requirement for the non-minimum phase system becomes much larger than its minimum-phase counterpart due to near pole-zero cancellation.

4.2. *Time delay systems*

Many systems arising in practice contain time delay. These irrational systems cannot be handled directly in the optimal control framework discussed in § 2.3. A common approach for optimal control for such systems is to design the controller based on a rational approximation (e.g. Padè approximation) of the time delay system. In this paper, we use this approach and the achievable performance is characterized by letting the order of approximation approach infinity in the limit.

To extend Proposition 1 to systems with a finite time delay, let $\hat{g}$ in (18) be expressed as

$$\hat{g} = \tilde{g}e^{-\theta s}, \tag{41}$$

where $\tilde{g}$ is the delay-free part of the system. If g_w also contains delay, the delay can be factored as an all-pass factor and thus $\hat{g}$ remains causal (cf. (18)).

Lemma 3: Consider $\mathbf{H} \leftrightarrow (\mathbf{P}, \mathbf{B}, \mathbf{C})$ such that $\mathbf{P} = \mathrm{diag}(p_1, \ldots, p_{n_p})$, $\mathrm{Re}(p_i) > 0$, $p_i \neq p_j$. Let $\mathbf{H}_1 \in \mathcal{RH}_\infty$ with no zeros at p_i. Then

$$\mathcal{U}(\mathbf{H}_1\mathbf{H}) = \sum_{i=1}^{n_p} \frac{1}{s - p_i} \mathbf{H}_1(p_i)\mathbf{C}_i\mathbf{B}_i'. \tag{42}$$

Proof: Using a dyadic expansion of $\mathbf{H}$,

$$\mathbf{H} = \sum_{i=1}^{n_p} \frac{1}{s - p_i} \mathbf{C}_i\mathbf{B}_i'.$$

Let $\mathcal{U}(\mathbf{H}_1\mathbf{H}) \leftrightarrow (\tilde{\mathbf{P}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}})$. Since $\mathbf{H}_1$ does not cancel the RHP poles of $\mathbf{H}$, $\tilde{\mathbf{P}} = \mathbf{P}$. Now, $\tilde{\mathbf{C}}_i\tilde{\mathbf{B}}_i' = [\mathbf{H}_1\mathbf{H} \cdot (s - p_i)]_{s=p_i}$ and (42) follows. □

Note that the applicability of Lemma 3 is not limited to the case where $\mathbf{H}$ only has unstable poles, since $\mathcal{U}(\mathbf{H}_1\mathbf{H}) = \mathcal{U}(\mathbf{H}_1\mathcal{U}(\mathbf{H}))$.

Proposition 2: *Let the SISO system expressed by* (41) *have distinct unstable poles and* $\mathcal{U}(\tilde{g}) \leftrightarrow (\mathbf{P}, \tilde{\mathbf{b}}, \tilde{\mathbf{c}})$ *such that* $\mathbf{P} = \mathrm{diag}(p_1, \ldots, p_{n_p})$, $\mathrm{Re}(p_i) > 0$ *and* $\boldsymbol{\Gamma} = \mathrm{diag}(e^{\theta p_1}, \ldots, e^{\theta p_{n_p}})$. *Then*

$$I_2^2 = \left[\frac{\mathbf{q}_i^2}{\tilde{b}_i\tilde{c}_i}\right]\boldsymbol{\Gamma}\,\mathbf{M}\,\boldsymbol{\Gamma}^*\left[\frac{\mathbf{q}_i^2}{\tilde{b}_i^*\tilde{c}_i^*}\right]' \quad i = 1, \ldots, n_p \tag{43}$$

$$I_\infty^2 = \left|\underline{\lambda}^{-1}(\boldsymbol{\Gamma}^{-*}\mathrm{diag}(\tilde{b}_i^*\tilde{c}_i^*)\mathbf{M}'\boldsymbol{\Gamma}^{-1}\mathrm{diag}(\tilde{b}_i\tilde{c}_i)\mathbf{M})\right|, \tag{44}$$

where $\mathbf{M}$ *is defined by* (28) *and* $\mathbf{q} = \mathbf{1}_{n_p}'\mathbf{M}^{-1}$.

Proof: Let $f(\theta s, n)$ be the nth order rational approximation of $e^{-\theta s}$ (e.g. Padè approximation). For any n, if a RHP zero of $f(\theta s, n)$ cancels a RHP pole of $\tilde{g}(s)$, the system is not stabilizable due to presence of hidden unstable modes; however, as $n \to \infty$, the magnitude of RHP zeros of $f(\theta s, n)$ approaches infinity. Thus, for an FDLTI system with poles at finite locations, such cancellation of RHP pole of $\tilde{g}(s)$ by an RHP zero of $f(\theta s, n)$ does not occur for all $n \geq N$ for sufficiently large N.

(1) For (43), using (42), $b_ic_i \approx \tilde{b}_i\tilde{c}_i f(\theta p_i, n), n \geq N$ and

$$I_2^2(n) = \left[\frac{\mathbf{q}_i^2}{\tilde{b}_i\tilde{c}_i f(\theta p_i, n)}\right]\mathbf{M}\left[\frac{\mathbf{q}_i^2}{\tilde{b}_i^*\tilde{c}_i^* f(\theta p_i, n)}\right]'$$
$$= \sum_{i=1}^{n_p}\sum_{j=1}^{n_p} \frac{\mathbf{q}_i^2}{\tilde{b}_i\tilde{c}_i}\frac{\mathbf{q}_j^2}{\tilde{b}_j\tilde{c}_j} m_{ij} f^{-1}(\theta p_i, n) f^{-1}(\theta p_j, n). \tag{45}$$

As $n \to \infty$, the Padè approximation is convergent (Parington 2004). Thus, $\lim_{n\to\infty} f^{-1}(\theta p_i, n) = e^{\theta p_i}$ and $\lim_{n\to\infty} f^{-1}(\theta p_i, n) f^{-1}(\theta p_j, n) = e^{\theta p_i}e^{\theta p_j}$. Noting that except the bilinear term $f^{-1}(\theta p_i, n) f^{-1}(\theta p_j, n)$, all other terms in (45) are independent of n, we conclude that $\lim_{n\to\infty} I_2^2(n)$ exists and is given by (43).

(2) For (44), using similar arguments as before and following the proof of Proposition 1,

$$I_\infty^2(n) = \Big|\underline{\lambda}^{-1}(\mathrm{diag}(f(\theta p_i, n)^*)^{-1}\mathrm{diag}(\tilde{b}_i^*\tilde{c}_i^*)\mathbf{M}' \times \mathrm{diag}(f(\theta p_i, n))^{-1}\mathrm{diag}(\tilde{b}_i\tilde{c}_i)\mathbf{M})\Big|$$

The eigenvalues are roots of a polynomial equation, whose coefficients are functions of $f^{-1}(\theta p_i, n)$. As $n \to \infty$, these coefficients and thus the roots converge. Hence, $\lim_{n\to\infty} I_\infty^2(n)$ exists and is given by (44). □

Similar to (36), for a system with real unstable poles only, (44) can be simplified to

$$I_\infty^2 = \left|\underline{\lambda}^{-1}(\Gamma^{-1}\mathrm{diag}(b_ic_i)\mathbf{M})\right|.$$

By differentiating (43) with respect to θ,

$$\frac{\mathrm{d}I_2^2}{\mathrm{d}\theta} = \sum_{i=1}^{n_p}\sum_{j=1}^{n_p} p_i p_j \frac{\mathbf{q}_i^2}{\tilde{b}_i\tilde{c}_i}\frac{\mathbf{q}_j^2}{\tilde{b}_j\tilde{c}_j} m_{ij} e^{p_i\theta} e^{p_j\theta} \geq \min_i p_i^2 I_2^2.$$

The last inequality follows since $e^{p_i\theta} > 1$ for positive p_i and θ. Thus, $\mathrm{d}I_2/\mathrm{d}\theta > 0$ for all θ. Similar conclusions can be drawn by differentiating I_∞ with respect to θ. This shows that for SISO systems, the input usage cannot be decreased by introducing additional lag in the system. Surprisingly, for MIMO systems, such intuitive conclusion does not hold, as is shown later.

Corollary 1: *Under same conditions as Proposition* 2, *let* $g_p \leftrightarrow (\mathbf{P}, \mathbf{\Gamma}^{-1}\tilde{\mathbf{b}}, \tilde{\mathbf{c}})$ or $(\mathbf{P}, \tilde{\mathbf{b}}, \tilde{\mathbf{c}}\mathbf{\Gamma}^{-1})$. *Then* $I_2(\hat{g}) = I_2(g_p)$ *and* $I_\infty(\hat{g}) = I_\infty(g_p)$.

It follows from Corollary 1 that I_2 and I_∞ for a time delay system depend on its unstable projection, which is rational.

Corollary 2: *For a SISO system with a single real unstable pole p*,

$$I_2^2 = \frac{8p^3 e^{2p\theta}}{\tilde{b}^2\tilde{c}^2} \quad I_\infty = \frac{2pe^{p\theta}}{|\tilde{b}\tilde{c}|}. \tag{46}$$

Corollary 2 can be shown to be true by considering (43) and noting that in this case $\tilde{\mathbf{b}}, \tilde{\mathbf{c}}$ are scalars and $\mathbf{M} = 1/2p$. For delay-free systems, Havre and Skogestad (2003) obtained expressions similar to (46). Propositions 1 and 2 can be seen as the generalizations of the results of Havre and Skogestad (2003) to SISO systems with multiple unstable poles and time delay. We point out that the expression for I_∞ in (46) can alternatively be obtained using the approach of Havre and Skogestad (2001).

Remark 1: The time-delay enters (43)–(44) assuming the form $e^{\theta p_i}$ and thus does not pose any serious limitations on input performance for systems with slow instabilities and *vice versa*. This happens as $e^{\theta p_i} \approx 1 + \theta p_i$, when $|\theta p_i|$ much smaller than 1 and thus the achievable input performance is nearly the same for time delay and delay-free systems. It follows from Corollary 1 that time delay essentially reduces the controllability (or observability) of poles and the faster the instability, the weaker the controllability (or observability) of the pole is, as compared to the delay-free system.

Example 2: To illustrate the findings of this section, consider

$$\hat{g} = \frac{2\,(s+10)}{(s-2)\,(s+0.4)}\, e^{-\theta s}. \tag{47}$$

Here, $\tilde{b}\tilde{c} = [(s-2)g]_{s=2} = 10$ and using (46), $I_\infty = 0.4e^{2\theta}$. Thus, for $\theta = 0$, 0.05 and 0.5, $I_\infty = 0.4$, 0.44 and 1.08, respectively. It should be noted that the additional limitation on the achievable input performance due to a small delay is minimal (see also Remark 1).

For any practical system, the manipulated variables are physically bounded and input saturation is a major concern for stabilization. Input saturation is avoided ($|u(t)| < u_{\max}$ for all t), if $\|k(1+gk)^{-1}\|_{\mathcal{L}_1} < u_{\max}$, where $\|.\|_{\mathcal{L}_1}$ is the induced $\mathcal{L}_1$-norm. This implies that stabilization without input saturation is not possible, if $I_\infty \geq u_{\max}$, since $\|.\|_\infty \leq \|.\|_{\mathcal{L}_1}$ (see e.g. Zhou and Doyle (1998)). Then, for g in (47) with $\theta = 0.5$, the physical limits on u must be larger than 1.08 to avoid input saturation.

The lower bound on the physical limits on u, as derived above, inherently assumes that the inputs can be manipulated arbitrarily fast and thus is somewhat unrealistic. To take the finite bandwidth of real systems into account, we consider the frequency-dependent weight $w_u = \alpha + s/\omega_B$, $\alpha > 0$ where it is desired that $\|w_u k(1+gk)^{-1}\|_\infty \leq 1$. This weight requires that $|k(1+gk)^{-1}| \leq 1/\alpha$ for $\omega \in \{0, \omega_B\}$ and then approaches 0 with a slope of -1 on a log–log plot, as $\omega \to \infty$. Here, α is closely related to the allowable peak value of input ($u_{\max} \approx 1/\alpha$) and ω_B is the available bandwidth. Though the weight is improper, the regularity assumptions can be easily satisfied by adding a stable pole at high frequency. In the present case, we can treat $w_u^{-1}g$ as the generalized system and thus the requirement $\|w_u k(1+gk)^{-1}\|_\infty \leq 1$ implies $0.4e^{2\theta}(\alpha + 2/\omega_B) \leq 1$. A rearrangement of this expression reveals the trade-off between ω_B and α

$$\alpha \leq 2.5e^{-2\theta} - 2/\omega_B. \tag{48}$$

For $\theta = 0.5$, (48) requires that $u_{\max} \approx 1/\alpha \geq 1.39$ and 1.92, when $\omega_B = 10$ and 5 rad/s, respectively. For $\omega_B \leq 2.175$ rad/s, the inequality (48) becomes infeasible. This is expected as unstable systems require fast control and $\omega_B \approx p$ (Skogestad and Postlethwaite 1996) is the lower limit on the required bandwidth for practical stabilization, even when manipulated input are allowed to have arbitrarily large variations.

5. MIMO systems

In this section, we generalize the results of the last section to MIMO systems. It is shown that the achievable input performance primarily depends on the joint controllability and observability of the unstable poles of the system. These results can be directly used for selection of the subset of controlled and manipulated variables for stabilization.

5.1. *Rational systems*

Similar to SISO systems, the achievable input performance is first characterized for rational systems. Later in this section, these results are extended to MIMO systems with time delay. To obtain expressions for I_2 and I_∞ for MIMO systems, we relate $\mathbf{X}$ and $\mathbf{Y}$ solving the AREs (11)–(12) to the Hankel singular values of $\mathcal{U}(\hat{\mathbf{G}})^*$. When $\hat{\mathbf{G}}$ has distinct unstable poles, the next lemma also provides an alternate expression for the Hankel singular values of $\mathcal{U}(\hat{\mathbf{G}})^*$, which may also be of independent interest.

Lemma 4: *Let $\hat{\mathbf{G}}$ be a rational system and $\mathbf{X}, \mathbf{Y} \succ 0$ solve the corresponding AREs* (22)–(23). *Then*,

$$\sigma_{Hi}^2(\mathcal{U}(\hat{\mathbf{G}})^*) = \lambda_i(\mathbf{X}^{-1}\mathbf{Y}^{-1}) \quad i = 1, \ldots, n_p. \qquad (49)$$

Further, if $\hat{\mathbf{G}}$ has distinct unstable poles, let $\mathcal{U}(\hat{\mathbf{G}}) \leftrightarrow (\mathbf{P}, \mathbf{B}, \mathbf{C})$, such that $\mathbf{P} = \mathrm{diag}(p_1, \ldots, p_{n_p})$, $\mathrm{Re}(p_i) > 0$. Then $\sigma_{Hi}(\mathcal{U}(\hat{\mathbf{G}})^*)$ is given as,

$$\sigma_{Hi}(\mathcal{U}(\hat{\mathbf{G}})^*) = \lambda_i^{1/2}\big[\big((\mathbf{B}\mathbf{B}^*) \circ \mathbf{M}\big)\big((\mathbf{C}^*\mathbf{C}) \circ \mathbf{M}'\big)\big], \qquad (50)$$

where $\mathcal{U}(\cdot)$ denotes the unstable part and $\mathbf{M}$ is defined by (28).

Proof: Pre- and post-multiplying (31) by $\mathbf{T}_1$ and $\mathbf{T}_1^*$ respectively, where $\mathbf{T}_1$ is a state transformation matrix,

$$\begin{aligned} \mathbf{T}_1\mathbf{P}\mathbf{X}^{-1}\mathbf{T}_1^* + \mathbf{T}_1\mathbf{X}^{-1}\mathbf{P}^*\mathbf{T}_1^* &= \mathbf{T}_1\mathbf{B}\mathbf{B}^*\mathbf{T}_1^* \\ \Leftrightarrow \bar{\mathbf{P}}\bar{\mathbf{X}}^{-1} + \bar{\mathbf{X}}^{-1}\bar{\mathbf{P}}^* &= \bar{\mathbf{B}}\bar{\mathbf{B}}^*, \end{aligned} \qquad (51)$$

where $\bar{\mathbf{P}} = \mathbf{T}_1\mathbf{P}\mathbf{T}_1^{-1}$, $\bar{\mathbf{B}} = \mathbf{T}_1\mathbf{B}$ and $\bar{\mathbf{X}} = \mathbf{T}_1^{-*}\mathbf{X}\mathbf{T}_1^{-1}$. Similarly, by setting $\bar{\mathbf{C}} = \mathbf{C}\mathbf{T}_1^{-1}$ and $\bar{\mathbf{Y}} = \mathbf{T}_1\mathbf{Y}\mathbf{T}_1^*$,

$$\bar{\mathbf{P}}^*\bar{\mathbf{Y}}^{-1} + \bar{\mathbf{Y}}^{-1}\bar{\mathbf{P}} = \bar{\mathbf{C}}^*\bar{\mathbf{C}}. \qquad (52)$$

Note that $\bar{\mathbf{Y}}^{-1}$ and $\bar{\mathbf{X}}^{-1}$ are the controllability and observability gramians of stable system $\mathcal{U}(\hat{\mathbf{G}})^* \leftrightarrow (-\bar{\mathbf{P}}^*, \bar{\mathbf{C}}^*, \bar{\mathbf{B}}^*)$ and (51)–(52) are the corresponding Lyapunov equations. If $\mathbf{T}_1$ is chosen such that $(-\bar{\mathbf{P}}^*, \bar{\mathbf{C}}^*, \bar{\mathbf{B}}^*)$ is a balanced realization, then $\bar{\mathbf{X}}^{-1} = \bar{\mathbf{Y}}^{-1} = \mathrm{diag}(\sigma_{Hi}(\mathcal{U}(\hat{\mathbf{G}})^*))$ (Zhou and Doyle 1998) and

$$\begin{aligned} \sigma_{Hi}^2(\mathcal{U}(\hat{\mathbf{G}})^*) &= \lambda_i(\bar{\mathbf{X}}^{-1}\bar{\mathbf{Y}}^{-1}) = \lambda_i(\mathbf{T}_1^{-*}\mathbf{X}^{-1}\mathbf{Y}^{-1}\mathbf{T}_1^*) \\ &= \lambda_i(\mathbf{X}^{-1}\mathbf{Y}^{-1}). \end{aligned}$$

When $\hat{\mathbf{G}}$ has distinct unstable poles, the alternate expression for the Hankel singular values of $\mathcal{U}(\hat{\mathbf{G}})^*$ (50) can be obtained by substituting for $\mathbf{X}^{-1}$ and $\mathbf{Y}^{-1}$ in (49) using Lemma 1. □

Proposition 3: *For the rational MIMO system $\hat{\mathbf{G}}$ in* (18) *having n_p unstable poles, let $(\bar{\mathbf{P}}, \bar{\mathbf{B}}, \bar{\mathbf{C}})$ be the balanced realization of $\mathcal{U}(\hat{\mathbf{G}})$. Then*

$$I_2^2 = \sum_{i=1}^{n_p} \frac{2|\mathrm{Re}(\bar{\mathbf{P}}_{ii})|}{\sigma_{Hi}^2(\mathcal{U}(\hat{\mathbf{G}})^*)} \qquad (53)$$

$$I_\infty = \underline{\sigma}_H^{-1}(\mathcal{U}(\hat{\mathbf{G}})^*) \qquad (54)$$

Proof:

(1) For (53), based on the expression for I_2^2 (26) ,

$$I_2^2 = \mathrm{tr}(\mathbf{B}^*\mathbf{X}\mathbf{Y}\mathbf{X}\mathbf{B}) = \mathrm{tr}(\bar{\mathbf{B}}^*\bar{\mathbf{X}}\bar{\mathbf{Y}}\bar{\mathbf{X}}\bar{\mathbf{B}}) = \mathrm{tr}(\bar{\mathbf{B}}\bar{\mathbf{B}}^*\bar{\mathbf{X}}\bar{\mathbf{Y}}\bar{\mathbf{X}})$$

Define $\mathbf{\Sigma}_H = \mathrm{diag}(\sigma_{Hi}(\mathcal{U}(\hat{\mathbf{G}})^*))$. Since $(-\bar{\mathbf{P}}^*, \bar{\mathbf{C}}^*, \bar{\mathbf{B}}^*)$ is the balanced realization of $\mathcal{U}(\hat{\mathbf{G}})^*$, using (51) and setting $\bar{\mathbf{X}} = \bar{\mathbf{Y}} = \mathbf{\Sigma}_H^{-1}$,

$$\begin{aligned} I_2^2 &= \mathrm{tr}\big[(\bar{\mathbf{P}}\mathbf{\Sigma}_H + \mathbf{\Sigma}_H\bar{\mathbf{P}}^*)\mathbf{\Sigma}_H^{-3}\big] \\ &= \mathrm{tr}(\bar{\mathbf{P}}\mathbf{\Sigma}_H^{-2}) + \mathrm{tr}(\mathbf{\Sigma}_H^{-2}\bar{\mathbf{P}}^*) = \sum_{i=1}^{n_p} \frac{|\bar{\mathbf{P}}_{ii} + \bar{\mathbf{P}}_{ii}^*|}{\sigma_{Hi}^2(\mathcal{U}(\hat{\mathbf{G}})^*)}, \end{aligned}$$

where $|\bar{\mathbf{P}}_{ii} + \bar{\mathbf{P}}_{ii}^*| = 2|\mathrm{Re}(\bar{\mathbf{P}}_{ii})|$.

(2) For (54), based on (27) and (49)

$$I_\infty = \underline{\lambda}^{-1/2}(\mathbf{X}^{-1}\mathbf{Y}^{-1}) = \underline{\sigma}_H^{-1}(\mathcal{U}(\hat{\mathbf{G}})^*). \qquad \square$$

The expressions (53)–(54) show that I_2 and I_∞ mainly depend on $\sigma_{Hi}(\mathcal{U}(\hat{\mathbf{G}})^*)$, which is a measure of joint controllability and observability of the unstable poles.

Glover (1986) studied the robust stability of systems in the presence of additive unstructured uncertainty. With the additive description of uncertainty, maximizing robust stability is equivalent to minimizing the $\mathcal{H}_\infty$ norm of transfer matrix from disturbances to inputs. Thus, the results of Glover (1986) are also applicable to the present case of minimization of input energy required for stabilization. The expression for I_∞ as derived here is as an alternative proof of the similar result by Glover (1986), but is generalized to the case where $\mathbf{W}_u$ and $\mathbf{G}_w$ can be minimum phase and share common unstable poles with the system.

Remark 2: In general, the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms of a transfer matrix can be arbitrarily far apart. Proposition 3 shows that when input norm is minimized, the ratio I_2/I_∞ is always bounded as

$$2\frac{\underline{\sigma}_H^2(\mathcal{U}(\hat{\mathbf{G}})^*)}{\bar{\sigma}_H^2(\mathcal{U}(\hat{\mathbf{G}})^*)}\sum_{i=1}^{n_p}|\mathrm{Re}(\bar{\mathbf{P}}_{ii})| \le \frac{I_2^2}{I_\infty^2} \le 2\sum_{i=1}^{n_p}|\mathrm{Re}(\bar{\mathbf{P}}_{ii})|, \qquad (55)$$

where $\bar{\mathbf{P}}$ is the state matrix of the balanced realization of $\mathcal{U}(\hat{\mathbf{G}})$. The closeness of I_2 and I_∞ follows from the fact that when input usage is minimized, the corresponding AREs (22)–(23) for the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal controller design problems are the same. The ratio $\kappa_H = \bar{\sigma}_H(\mathcal{U}(\hat{\mathbf{G}})^*)/\underline{\sigma}_H(\mathcal{U}(\hat{\mathbf{G}})^*)$ is the condition number of $\mathcal{U}(\hat{\mathbf{G}})^*$ expressed in terms of Hankel singular values and can be interpreted similar to the Euclidian condition number. A system that has a large Euclidian condition number has strong directionality and may be difficult to control (Skogestad and Postlethwaite 1996). Similarly, κ_H can be large due to small $\underline{\sigma}_H(\mathcal{U}(\hat{\mathbf{G}})^*)$ indicating that the input requirement for stabilization is large. When $\kappa_H = 1$, the upper and lower bounds on I_2^2/I_∞^2 in (55) are the same with $I_2^2/I_\infty^2 = 2\sum_{i=1}^{n_p}|\mathrm{Re}(\bar{\mathbf{P}}_{ii})|$.

In this paper, we assumed that the disturbances enter the closed loop system through the output channels. Proposition 3 can easily be applied to cases where disturbances enter through the input channels by setting $\mathbf{G}_w = \mathbf{G}$ (see figure 4). For minimum phase systems affected by input disturbances, the expressions for achievable input performance are much simplified, as derived by Chen *et al.* (2003) for I_2. The result of Chen *et al.* (2003) is shown to be a special case of Proposition 3 by the next Corollary.

Corollary 3: *With reference to figure* 4, *let* $\mathbf{G}$ *be minimum phase*, *right invertible and have* n_p *unstable poles*. *Then*,

$$I_2^2 = 2\sum_{i=1}^{n_p}\mathrm{R}e(p_i); \quad I_\infty = 1. \tag{56}$$

Proof Let $\mathbf{G} = \mathbf{G}_s\mathcal{B}_{pi}^{-1}$ such that $\mathbf{G}_s$ is stable. With $\mathbf{G}_w = \mathbf{G}$ and using (15),

$$\begin{aligned}\|\mathbf{T}_{uw}\| &= \|(\mathbf{I} + \mathbf{K}\mathbf{G}_s\mathcal{B}_{pi}^{-1})^{-1}\mathbf{K}\mathbf{G}_s\mathcal{B}_{pi}^{-1}\| \\ &= \|(\mathbf{I} + \hat{\mathbf{K}}\mathcal{B}_{pi}^{-1})^{-1}\hat{\mathbf{K}}\|,\end{aligned}$$

where $\hat{\mathbf{K}} = \mathbf{K}\mathbf{G}_s$. Let$(\bar{\mathbf{P}}, \bar{\mathbf{B}}, \bar{\mathbf{C}}, \bar{\mathbf{D}})$ be the balanced realization of $\mathcal{B}_{pi}^{-*}$. Since $\mathcal{B}_{pi}^{-*}$ is all-pass and stable, $\sigma_{Hi}(\mathcal{B}_{pi}^{-*}) = 1$ (Glover 1984). Then, using Proposition 3, $I_\infty = 1$ and $I_2^2 = \sum_{i=1}^{n_p} 2|\mathrm{Re}(\bar{\mathbf{P}}_{ii})|$. The expression for I_2 follows by noting that $\bar{\mathbf{P}}_{ii} = p_i$ (cf. (7)). □

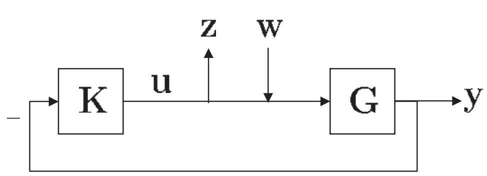

Figure 4. Closed loop system with disturbances entering through input channels.

5.2. *Time delay systems*

In extending Proposition 2 to MIMO systems, we use a similar method as used for SISO systems, i.e. by using a rational approximation of the time delay system and then letting the order of approximation approach infinity. We consider systems that can be expressed as

$$\hat{\mathbf{G}} = (\mathbf{G}_w)_{sm}^{-1}\mathbf{G}(\mathbf{W}_u)_{sm}^{-1} = \tilde{\mathbf{G}} \circ \boldsymbol{\Theta}; \quad \boldsymbol{\Theta} = \left[e^{-\theta_{ij}s}\right], \tag{57}$$

where $\tilde{\mathbf{G}}$ is the delay-free part of the system. A system as $\hat{\mathbf{G}}$ in (57) with delay associated with individual elements of the transfer matrix, which cannot be separated at inputs or outputs, is sometimes referred to as a multiple delay system in the literature. It is pointed out that (57) does not represent the most general case and in practice is satisfied only when the $\mathbf{W}_u$ and $\mathbf{G}_w$ are diagonal. The remaining discussion in this section is limited to the cases where $n_y \geq n_u$ and similar expressions for $n_y < n_u$ can be obtained with minor modifications.

Lemma 5: Consider $\mathbf{H} \leftrightarrow (\mathbf{P}, \mathbf{B}, \mathbf{C})$ such that $\mathbf{P} = \mathrm{diag}(p_1, \ldots, p_{n_p}), \mathrm{Re}(p_i) > 0,\ p_i \neq p_j$. Let $\mathbf{H}_1 \in \mathcal{RH}_\infty$ with no zeros at p_i. Then

$$\mathcal{U}(\mathbf{H}_1 \circ \mathbf{H}) = \sum_{i=1}^{n_p}\frac{1}{s - p_i}\mathbf{H}_1(p_i) \circ (\mathbf{C}_i\mathbf{B}_i'). \tag{58}$$

The proof of Lemma 5 is similar to the proof of Lemma 3 and is omitted. We make the following additional technical assumption:

Asumption 3: *Let* $\mathcal{U}(\tilde{\mathbf{G}}) \leftrightarrow (\mathbf{P}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}})$. *Then the matrix* $(\tilde{\mathbf{C}}_i\tilde{\mathbf{B}}_i') \circ \boldsymbol{\Theta}(p_i)$ *has full column rank for all* $i = 1, \ldots, n_p$.

Proposition 4: *Consider that the MIMO system expressed by* (57) *has distinct poles and the system satisfies Assumption* 3. *Let* $\mathcal{U}(\tilde{\mathbf{G}}) \leftrightarrow (\mathbf{P}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}})$ *such that* $\mathbf{P} = \mathrm{diag}(p_1, \ldots, p_{n_p})$, $\mathrm{Re}(p_i) > 0$. *If* $\mathbf{G}_p \leftrightarrow (\mathbf{A}_p, \mathbf{B}_p, \mathbf{C}_p)$, *where*

$$\begin{aligned}\mathbf{A}_p &= \mathrm{diag}(p_1\mathbf{I}_{n_u}, \ldots, p_{n_p}\mathbf{I}_{n_u}); \quad \mathbf{B}_p = \left[\mathbf{I}_{n_u}, \ldots, \mathbf{I}_{n_u}\right]' \\ \mathbf{C}_p &= \left[(\tilde{\mathbf{C}}_1\tilde{\mathbf{B}}_1') \circ \boldsymbol{\Theta}(p_1), \ldots, (\tilde{\mathbf{C}}_{n_p}\tilde{\mathbf{B}}_{n_p}') \circ \boldsymbol{\Theta}(p_{n_p})\right]\end{aligned}$$

$I_2(\hat{\mathbf{G}}) = I_2(\mathbf{G}_p)$, $I_\infty(\hat{\mathbf{G}}) = I_\infty(\mathbf{G}_p)$.

Proof: Let $\boldsymbol{\Theta}$ be approximated by an nth order rational function as before. As $n \to \infty$, using Lemma 5 and the same arguments as used in the proof of Proposition 5,

$$\mathcal{U}(\hat{\mathbf{G}}) = \sum_{i=1}^{n_p}\frac{1}{s - p_i}(\tilde{\mathbf{C}}_i\tilde{\mathbf{B}}_i') \circ \boldsymbol{\Theta}(p_i). \tag{59}$$

Due to Assumption 3, $(1/(s - p_i))\boldsymbol{\Theta}(p_i) \circ (\mathbf{C}_i\mathbf{B}_i') \leftrightarrow (p_i\mathbf{I}_{n_u}, \mathbf{I}_{n_u}, \boldsymbol{\Theta}(p_i) \circ (\mathbf{C}_i\mathbf{B}_i'))$. Then the result follows by considering the aggregation of these subsystems. □

It is interesting to note that when $\mathbf{\Theta}$ is unstructured (delays cannot be separated at inputs or outputs), stabilization of the irrational system with n_p unstable poles is equivalent to stabilizing a rational system with $n_p \times n_u$ unstable poles. For systems not satisfying Assumption 3, the triplet $(\mathbf{A}_p, \mathbf{B}_p, \mathbf{C}_p)$ is not necessarily a minimal realization. This assumption can be relaxed for generalization purposes, but this makes the expressions difficult and complex. A practical case, where Assumption 3 is always violated, occurs when the delays are associated with the sensors or actuators of the system. Systems with delay associated with sensors are handled next and the expressions for systems with delay associated with actuators can be obtained analogously.

Corollary 4: *Let* $\hat{\mathbf{G}} = \text{diag}(e^{-\theta_i s})\tilde{\mathbf{G}}$ *and* $\mathcal{U}(\tilde{\mathbf{G}}) \leftrightarrow (\mathbf{P}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}})$ *such that* $\mathbf{P} = \text{diag}(p_1, \ldots, p_{n_p})$, $\text{Re}(p_i) > 0$. *Let* $\mathbf{G}_p \leftrightarrow (\mathbf{P}, \tilde{\mathbf{B}}, \mathbf{C}_p)$, *where*

$$\mathbf{C}_p = \left[\text{diag}(e^{-\theta_i p_1})\tilde{\mathbf{C}}_1, \ldots, \text{diag}(e^{-\theta_i p_{n_p}})\tilde{\mathbf{C}}_{n_p}\right]$$

Then, $I_2(\hat{\mathbf{G}}) = I_2(\mathbf{G}_p)$ *and* $I_\infty(\hat{\mathbf{G}}) = I_\infty(\mathbf{G}_p)$.

The proof of Corollary 4 follows by considering the dyadic expansion of $\hat{\mathbf{G}}$ in (59) and noting that $(\tilde{\mathbf{C}}_i\tilde{\mathbf{B}}_i') \circ \mathbf{\Theta}(p_i) = \text{diag}(e^{-\theta_i p_i})\tilde{\mathbf{C}}_i\tilde{\mathbf{B}}_i'$. It was shown earlier that for SISO systems, I_2 and I_∞ are non-increasing functions of θ, but this does not hold for MIMO systems.

Example 3: Consider the system $\mathbf{G} = \tilde{\mathbf{G}} \circ \mathbf{\Theta}$, where

$$\tilde{\mathbf{G}} = \left[\begin{array}{cc|cc} 0.2 & 0 & 2 & 3 \\ 0 & 0.5 & 1 & 4 \\ \hline 3 & 2 & 0 & 0 \\ 5 & 3 & 0 & 0 \end{array}\right]; \quad \mathbf{\Theta} = \begin{bmatrix} e^{-\alpha_1 s} & e^{-\alpha_2 s} \\ e^{-\alpha_2 s} & e^{-\alpha_1 s} \end{bmatrix}.$$

The variation of I_∞ with α_1, α_2 is shown in figure 5, which leads to the counter intuitive conclusion that the input requirement for stabilization of MIMO systems can decrease when the delay in some of the elements of the system increases. When $\alpha_1 \neq \alpha_2$, by virtue of Proposition 4, the unstable projection of the irrational system has 4 unstable poles (2 poles each at 0.2 and 0.5). When $\alpha_1 = \alpha_2 = \alpha$, $\mathbf{G}$ can be expressed as $\mathbf{G} = \tilde{\mathbf{G}}e^{\alpha s}$. Then, using Corollary 4, the unstable projection of the irrational system has only 2 unstable poles.

With slight abuse of terminology, the case of $\alpha_1 = \alpha_2 = \alpha$ can be interpreted as the system having 4 unstable poles and 2 unstable zeros at 0.2 and 0.5. Thus, when $\alpha_1 \neq \alpha_2$, these RHP zeros differ from their nominal values of 0.2 and 0.5 and effectively reduce the joint controllability and observability of the unstable poles. Keeping α_1 (or α_2) constant and increasing α_2 (or α_1), these RHP zeros recede away from the unstable poles reducing the input requirement for stabilization. It is also worth pointing out that similar to input performance, an increase in time delay can also improve the output performance (Skogestad and Postlethwaite 1996, p. 220).

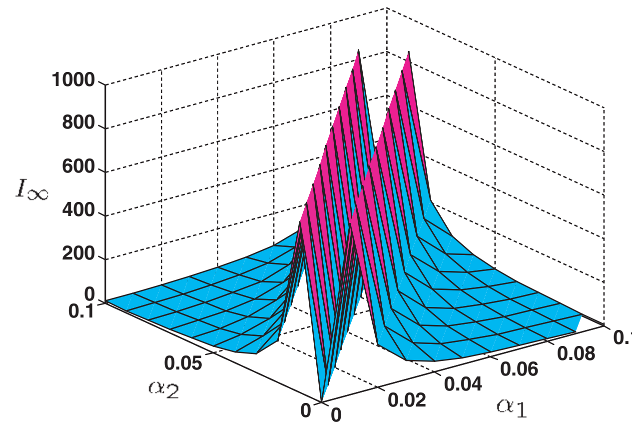

Figure 5. Variation of I_∞ with α_1 and α_2. This counter-example shows that the input requirement for stabilization can decrease with increase in time delay for MIMO systems.

When the system has a single unstable pole, the expressions for I_2 and I_∞ simplify considerably, as shown next.

Corollary 5: *Consider a MIMO system* $\hat{\mathbf{G}}$ *that is expressed by* (57) *and satisfies Assumption* 3. *If* $\hat{\mathbf{G}}$ *has a single real unstable pole* p,

$$I_2^2 = \frac{8p^3}{\sum_{i=1}^{n_u} \sigma_i^2((\tilde{\mathbf{C}}\tilde{\mathbf{B}}') \circ \mathbf{\Theta}(p))} \qquad I_\infty = \frac{2p}{\underline{\sigma}((\tilde{\mathbf{C}}\tilde{\mathbf{B}}') \circ \mathbf{\Theta}(p))}, \tag{60}$$

where $\mathcal{U}(\tilde{\mathbf{G}}) \leftrightarrow (p, \tilde{\mathbf{B}}, \tilde{\mathbf{C}})$.

Proof: Define $\mathbf{G}_p \leftrightarrow (p\mathbf{I}_{n_u}, \mathbf{I}_{n_u}, (\tilde{\mathbf{C}}\tilde{\mathbf{B}}') \circ \mathbf{\Theta}(p))$. Now, similar to the proof of Proposition 4, it can be shown that $I_2(\hat{\mathbf{G}}) = I_2(\mathbf{G}_p)$, $I_\infty(\hat{\mathbf{G}}) = I_\infty(\mathbf{G}_p)$. Since $\mathbf{G}_p$ has a single pole repeated n_u times, $\mathbf{M} = (1/2p)[\mathbf{1}_{n_u}, \ldots, \mathbf{1}_{n_u}]$. Using the alternate expression for the Hankel singular values (50),

$$\sigma_{Hi}(\mathbf{G}_p^*) = \left(\frac{1}{2p}\right)\lambda_i^{1/2}\left[\left((\tilde{\mathbf{C}}\tilde{\mathbf{B}}') \circ \mathbf{\Theta}(p)\right)^*\left((\tilde{\mathbf{C}}\tilde{\mathbf{B}}') \circ \mathbf{\Theta}(p)\right)\right]$$
$$= \left(\frac{1}{2p}\right)\sigma_i\left[(\tilde{\mathbf{C}}\tilde{\mathbf{B}}') \circ \mathbf{\Theta}(p)\right]. \tag{61}$$

Now, (60) is obtained by substituting (61) in the expressions for I_2 and I_∞ (53)–(54). □

For a system that is delay free and has a single unstable pole, $\mathbf{M} = 1/2p$, $\mathbf{BB}^* = \|\mathbf{B}\|_2^2$ and $\mathbf{C}^*\mathbf{C} = \|\mathbf{C}\|_2^2$. Then, using the alternate expression for Hankel singular values (50),

$$I_2^2 = \frac{8p^3}{\|\mathbf{B}\|_2^2\|\mathbf{C}\|_2^2} \quad I_\infty = \frac{2p}{\|\mathbf{B}\|_2\|\mathbf{C}\|_2}. \tag{62}$$

The expression for I_∞ in (62) was earlier obtained by Havre and Skogestad (2003). Propositions 3 and 4 can be seen as the generalization of the results of Havre and Skogestad (2003) to systems with multiple unstable poles and time delay.

Further, note that for a system with distinct unstable poles, the rows of **B** and columns of **C** matrices of the state-space realization with diagonal state matrix are the same as the input and output pole vectors, respectively (see e.g. Skogestad and Postlethwaite (1996) for definition of pole vectors). Then, it follows from (62) that for a rational system with single unstable pole, the input requirement for stabilization is minimized by selecting the input and output variables corresponding to largest entries in input and output pole vectors, respectively (Havre and Skogestad 2003). This simple "pole-vector" approach avoids the problem of combinatorial complexity, but cannot provide the optimal solution for systems with multiple unstable poles. In the general case, the optimal subset can be found by evaluating the expressions for achievable input performance presented in this paper for different subsets of input and output variables, but such an approach can be computationally intractable. Sequential approaches that provide sub-optimal solutions in reasonable time are discussed in Havre and Skogestad (2003) and Kariwala (2004).

6. Conclusions

We used a state-space framework to obtain analytic expressions for achievable input performance for SISO and MIMO systems with and without time delay. Regarding the factors affecting achievable input performance, the following general conclusions are drawn:

(1) The input performance primarily depends on the joint controllability and observability of unstable poles.
(2) Plant's unstable zeros do not limit the achievable input performance, except when located close to plant's unstable poles resulting in near pole-zero cancellation.
(3) Time delay poses no serious limitation on the achievable input performance for a system with slow instabilities and *vice versa*.
(4) The input performance of a MIMO system, where the delays cannot be separated at the inputs or outputs, can be much worse as compared to a system with delays that can be factored at the inputs or outputs.
(5) In contrast to the SISO systems, the achievable input performance may decrease for MIMO systems with an increase in time delay in some elements of the transfer matrix relating controlled and manipulated variables.

In this paper, we focussed on characterizing the achievable value of the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms of the transfer matrix from disturbances to input, $\mathbf{W}_u\mathbf{KSG}_w$. In turn, these results provide the minimal control effort required for system stabilization. For system stabilization, input saturation is one of the primary concerns. When the achievable bound on $\|\mathbf{W}_u\mathbf{KSG}_w\|_\infty$ exceeds the allowable bounds on the inputs, system stabilization without input saturation is not possible, but the converse is not necessarily true. The issue of input saturation is best handled in the $\mathcal{L}_1$-optimal control framework and this problem can be numerically solved using the linear programming approach of Dahleh and Diaz-Bobillo (1995). Explicit expressions for the achievable input performance in the $\mathcal{L}_1$-optimal control framework can provide additional insights regarding the limiting factors and is an open area for research.

In conjunction with the achievable bounds on the (weighted) sensitivity and complementary functions (Chen 2000), the proposed results on input performance are useful for input–output controllability analysis. The available results, however, consider only one closed-loop transfer matrix of interest at a time and thus may be misleading, for example, for minimum phase stable systems, the achievable bounds on individual closed-loop transfer matrices is zero indicating no limitations. A better approach is to consider the input and output performances together or establish bounds on the achievable output performance with bounded inputs. This problem is somewhat more involved and the results of Pérez *et al.* (2002) and Chen *et al.* (2003) can be seen as good starting points for further research.

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