Controller Design for Serial Processes

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Abstract

In this paper, we aim at obtaining insight into how a multivariable feedback controller works, with special attention to serial processes. Serial processes are important in the process industry, and the structure of this process makes it simple to classify the different elements of the multivariable controller.

In particular, we focus on the difference between the feedforward and feedback parts of the controller. Feedforward control may improve the performance significantly, but is sensitive to uncertainty, especially at low frequencies. Feedback control is very effective at lower frequencies where high feedback gains are allowed.

An example of neutralization of an acid in a series of three tanks is used to illustrate the ideas.
Keywords: Control structure, Serial process, Multivariable control, Feedforward, Feedback

1 Introduction

Before designing and implementing a multivariable controller, there are some questions that are important to answer: What will the multivariable controller really attempt to do? Will a multivariable controller significantly improve the performance as compared to a simpler scheme? How accurate a model is needed?

Conceptually, a multivariable controller uses the two basic principles of *"feedforward"* action, based mainly on the model (for example the off-diagonal decoupling elements of the controllers), and *feedback* correction, based mainly on the measurements. There is a fundamental difference between feedforward and feedback controllers with respect to their sensitivity to uncertainty. Feedforward control is sensitive to *static* uncertainty, whereas feedback is not. On the other hand, aggressively tuned feedback controllers are very sensitive to uncertainty in the *high-frequency (crossover) frequency region*. Similar differences with respect to uncertainty can be found for multivariable controllers.

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A multivariable controller often yields significant *nominal* improvements compared to local single-loop control, and this is largely because of the "feedforward" action. With *model error*, the feedforward effect may in fact lead to worse performance. The use of feedback from downstream measurements depends less on the model, as use of high feedback gains at low frequencies removes the steady-state error. However, at higher frequencies high feedback gains may lead to stability problems, and it is at these higher frequencies one may have the largest benefit of the model-based "feedforward" action of the multivariable controller.

In this paper, we discuss these issues for the important class of *serial processes*, in which the states in one process unit influence the states in the downstream unit, but *not* the other way round. This structure is very common in the process industry, where the outlet flow of one process enters into the next. One example, which will be studied in Section 4, is neutralization performed in several tanks in series. Examples of processes that are not serial are processes with some kind of recycle of material or energy. Even for such processes, however, parts of the process may be modelled as a serial process, if the outlet variations of the last unit are dampened through other process units before it is recycled.

Buckley (1964) discusses control structure design for serial processes and distinguishes between material balance control (control of inventory or pressure by flow rate adjustments) and product quality control (control of quality parameters such as concentration). Shinskey (1973) and McMillan (1984) present methods for the design of pH neutralization processes. Mixing tanks are used to dampen disturbances, and they find that the total volume may be reduced by use of multiple stages with one control loop for each tank. Another advantage with multiple stages is that one may use successively smaller and smaller control valves, leading to a more precise manipulated variable in the last stage. McMillan and Shinskey both recommend different sized tanks to avoid equal resonance frequencies in the tanks, but this has later been questioned (Walsh, 1993; Faanes and Skogestad, 2004*b*).

A discussion on the open-loop response of serial processes is given by Marlin (1995, p. 156). Morud and Skogestad (1996) note that the poles and zeros of the transfer function of a serial process are the poles and zeros of the transfer functions of the individual units. Thus, the overall response may be predicted directly from the individual units, in contrast to for example processes with recycle. Many series connections of processing units are not really serial processes, as the response of each unit also depends on the downstream unit (for example if the outlet flow rate from a unit depends on the pressure in the subsequent unit) (Marlin, 1995), (Morud, 1995, Chapter 4), (Morud and Skogestad, 1995). Morud and Skogestad denote the latter process structure *cascades*, whereas Marlin uses the terms *noninteracting* and *interacting* series, respectively, for the two structures.

In Section 2 we develop the model structure for serial processes and discuss some of their properties. In Section 3 the control of serial processes is discussed, and the division of the controller in feedforward, feedback and resetting blocks is presented. One popular multivariable controller is MPC, and to be able to use the theory for linear systems, we summarize in Appendix A how to express an unconstrained MPC combined with a state estimator on state space and transfer function form, see more details in Faanes and Skogestad (2003). The ideas of the paper are further illustrated through an example with pH neutralization in three stages (section 4). The paper is concluded by a short discussion (section 5) and conclusion (section 6).

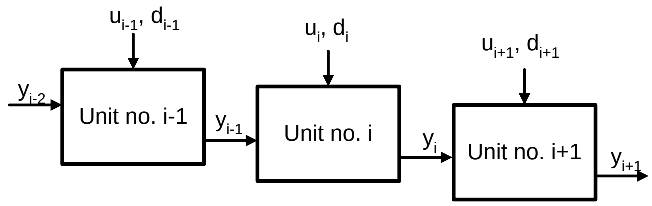


Figure 1: Serial process with exogenous variables u_i (manipulated) and d_i (disturbances) into unit i. The vector y_i represents the outflow of unit i, which continues into unit number $i+1$.

2 Model structure of serial processes

We define a serial process by the following (also see Figure 1):

A serial process can be divided into a series of sub-processes or units, where the states in each unit depend on the states in the unit itself (x_i), the states in the upstream unit (x_{i-1}), and the exogenous variables (u_i, d_i) to the unit.

The model for unit no. i can then be expressed as

$$\frac{d}{dt}x_i = f_i\left(x_i, x_{i-1}, u_i, d_i\right) \tag{1}$$

where x_i and x_{i-1} are the state vectors for unit i and unit $i-1$ respectively, and the external input is divided into a vector of manipulated inputs, u_i, and disturbances, d_i. We further define the outputs from a unit as a function of the states of this unit

$$y_i = g_i(x_i) \tag{2}$$

It is easy to also include direct throughput terms, i.e., define $y_i = g_i(x_i, x_{i-1}, u_i, d_i)$, but it makes the expressions below slightly more complex.

We linearize (1) and (2) around a working point, introduce $A_{i,j} = \partial f_i/\partial x_j; j = i, i-1$, $B_i = \partial f_i/\partial u_i, C_i = \partial g_i/\partial x_i$, and $E_i = \partial f_i/\partial d_i$ and let the variables represent deviations from this working point. Applying Laplace transformation, and recursively inserting for variables from the upstream process unit, we obtain:

$$y(s) = G(s)u(s) + G_d(s)d(s) \tag{3}$$

where we have defined the total output vector, $y(s)$, as all the outputs, $u(s)$ as all the manipulated inputs, and $d(s)$ as all the disturbances. Defining

$$M_i = (sI - A_{i,i})^{-1} \tag{4}$$

we get

$$G(s) = \begin{bmatrix} C_1 M_1 B_1 & 0 & 0 & \cdots & 0 \\ C_2 M_2 A_{2,1} M_1 B_1 & C_2 M_2 B_2 & 0 & \cdots & \vdots \\ \vdots & \vdots & \ddots & & 0 \\ C_n M_n \prod_{r=1}^{n-1} \left[A_{n-r+1,n-r} M_{n-r}\right] B_1 & C_n M_n \prod_{r=1}^{n-2} \left[A_{n-r+1,n-r} M_{n-r}\right] B_2 & \cdots & \cdots & C_n M_n B_n \end{bmatrix}$$
$$= \begin{bmatrix} G_{1,1} & 0 & 0 & \cdots & 0 \\ G_{2,1} & G_{2,2} & 0 & \cdots & \vdots \\ \vdots & \vdots & \ddots & & 0 \\ G_{n,1} & G_{n,2} & \cdots & \cdots & G_{n,n} \end{bmatrix} \quad (5)$$

and

$$G_d(s) = \begin{bmatrix} G_{d,1,1} & 0 & 0 & \cdots & 0 \\ G_{d,2,1} & G_{d,2,2} & 0 & \cdots & \vdots \\ \vdots & \vdots & \ddots & & 0 \\ G_{d,n,1} & G_{d,n,2} & \cdots & \cdots & G_{d,n,n} \end{bmatrix} \quad (6)$$

where n is the number of units. G_d is identical to G except in G_d B_i is replaced by E_i (the disturbances to each unit are assumed independent).

We see that $G(s)$ and $G_d(s)$ are both lower block triangular. From (5) and (6), we can deduce the following properties:

- The state vector of a process unit is not influenced by control inputs and disturbances to downstream units.
- The influence from a control input or a disturbance which enters an upstream unit, q, is dampened by the transfer function
  $C_i \left(sI - A_{i,i}\right)^{-1} \prod_{r=1}^{i-q} \left[A_{i-r+1,i-r} \left(sI - A_{i-r,i-r}\right)^{-1}\right]$
  before it reaches the output of unit i.
- The open loop stability of the total process is given by the stability of each unit since the elements in G and G_d consists of products of M_i's.
- $G(s)$ and $G_d(s)$ are block diagonal at infinite frequency ($s \to \infty$).

Note that the nominal model of unit i can be expressed as

$$y_i = G_{i,i} u_i + \tilde{G}_{d,i} y_{i-1} + G_{d,i,i} d_i \quad (7)$$

where $\tilde{G}_{d,i}$ is the transfer function from "disturbances" due to variations in the upstream unit, $i-1$ to output y_i:

$$\tilde{G}_{d,i}\left(s\right) \stackrel{\text{def}}{=} G_{i,i-1} G_{i-i,i-1}^{-1} \quad (8)$$

This is illustrated in Figure 2.

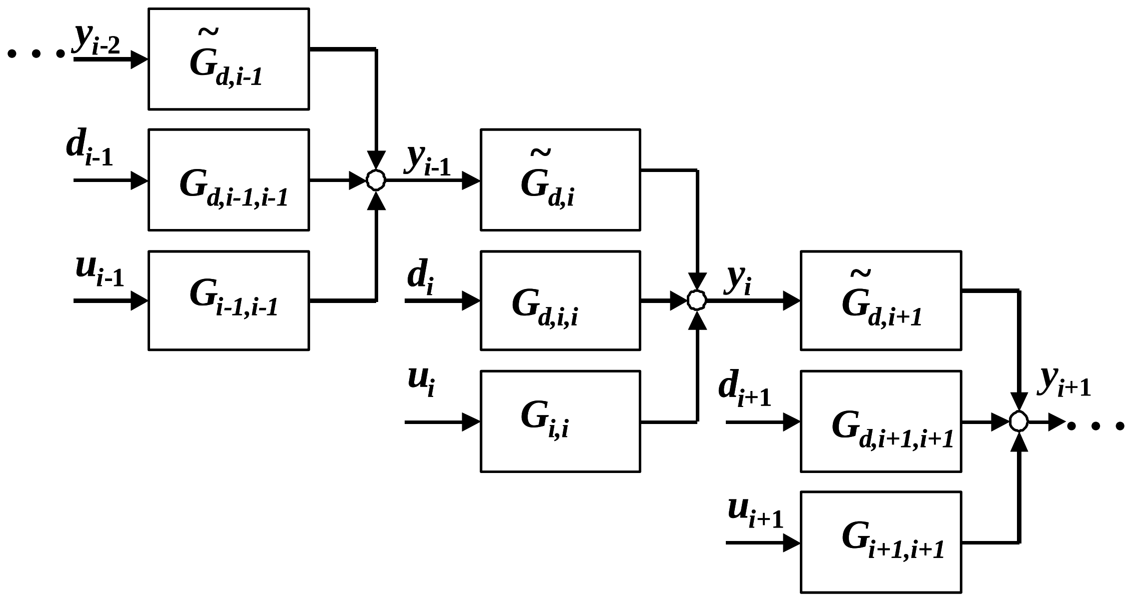


Figure 2: Model structure for a serial process

3 Control structures for serial processes

In the previous section, we introduced the concept of serial processes and Equations (3)-(6) summarize the linearized model. If we for simplicity assume that the set-points are constant ($y_r = 0$), and we want to control all the outputs, the control inputs are given by:

$$u(s) = K(s)y(s) \tag{9}$$

where $K(s)$ is the controller. We divide the controller $K(s)$ into $n \times n$ blocks of the same size as the blocks in $G(s)$:

$$K(s) = \begin{bmatrix} K_{11} & K_{12} & \dots & K_{1n} \\ K_{21} & K_{22} & \dots & K_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ K_{n1} & K_{n2} & \dots & K_{nn} \end{bmatrix} \tag{10}$$

These controller blocks can be divided into three groups:

Blocks on the diagonal ($K_{i,i}$) These blocks use local control, where inputs to a unit are used to control outputs of the same unit.

Blocks below the diagonal ($K_{i,j}$, $i > j$) Through these blocks an output from an upstream unit directly affects the input in a downstream unit. Since upstream units act as disturbances to downstream units (see (7)), these controller blocks may be viewed as *"feed-forward" elements*.

Blocks above the diagonal ($K_{i,j}$, $i < j$) These blocks represents feedback from the outputs of downstream units. Intuitively, when the effective delay through the units is large, these blocks seem ineffective since the local feedback always will be quicker. There are, however, several cases when it may prove useful:

1. We have no relevant control inputs downstream so local control is impossible.
2. The downstream actuators are slow, so that it actually is more efficient to manipulate the upstream control inputs.
3. There are not enough degrees of freedom in the downstream units.
4. The control inputs downstream are constrained, and insufficient to compensate for the disturbances.
5. The downstream actuators are expensive to use.

In the latter two cases, the upstream manipulated variable can be used to (slowly) drive the downstream ones to zero or to some other ideal resting value. This is called input resetting and is normally used for systems where we have more control variables than outputs (e.g., (Skogestad and Postlethwaite, 1996, page 418)).

In analyzing the controller, it is useful to plot the gain of the controller elements as a function of frequency. A key point is to find out whether there is integral action in the feedback part of the controller. Integral action requires high gain at low frequencies, but it is not always straightforward to interpret the gain plot of the controller elements as seen later in the example. Instead, it is proposed to consider the individual gains of the sensitivity function, $S(j\omega) = (I + L(j\omega))^{-1}$ where $L(j\omega) = G(j\omega)K(j\omega)$ is the loop transfer function. The usefulness of S is seen from the following expression for the control error

$$e = -Sy_r + SG_d d \tag{11}$$

where $e = y - y_r$, y_r is the reference, d is the disturbance and G_d is the (open loop) transfer function matrix from the disturbance to the output. To have no steady-state offset in an output we need that *all* elements in the corresponding *row* of S to be zero at steady state ($\omega = 0$).

3.1 Single loop controllers

Only diagonal blocks (local control)

Local control is by far the most common control element,

$$\text{Local control:} \qquad u_i = K_{i,i}(s)y_i \tag{12}$$

With only local control and three units ($n = 3$), the loop transfer function becomes

$$\begin{aligned} L &= \begin{bmatrix} G_{11} & 0 & 0 \\ G_{21} & G_{22} & 0 \\ G_{31} & G_{32} & G_{33} \end{bmatrix} \begin{bmatrix} K_{11} & 0 & 0 \\ 0 & K_{22} & 0 \\ 0 & 0 & K_{33} \end{bmatrix} \\ &= \begin{bmatrix} G_{11}K_{11} & 0 & 0 \\ 0 & G_{22}K_{22} & 0 \\ 0 & 0 & G_{33}K_{33} \end{bmatrix} \end{aligned} \tag{13}$$

From this it follows that the stability of the closed-loop system (S) is determined by the blocks on the diagonal only. That is, we have closed-loop stability if and only if each of the individual loops $(I + G_{i,i}K_{i,i})^{-1}$ are stable.

Only blocks below the diagonal (pure feedforward)

The use of measurements in upstream units in the control of a unit is denoted feedforward control:

$$\text{Feedforward } (i > j): \qquad u_i = K^{\mathrm{FF}}_{i,j}(s) y_j \tag{14}$$

With "pure" feedforward control (only feedforward elements), the controller does not influence the stability of the closed-loop system, S.

From (7) and (8) we find that perfect nominal control is obtained by selecting

$$K^{\mathrm{FF}}_{i,i-1} = -G^{-1}_{i,i}\tilde{G}_{d,i} \tag{15}$$

$$K^{\mathrm{FF}}_{i,i-2} = \cdots = K^{\mathrm{FF}}_{i,1} = 0 \tag{16}$$

The reason for the zero in (16) is that the disturbance is already eliminated by (15). If (15) cannot be realised, for example if it is not causal, (15) must be modified:

$$K^{\mathrm{FF}}_{i,i-1,-} = -G^{-1}_{i,i,-}\tilde{G}_{d,i} \tag{17}$$

where subscript minus indicates that negative delays and other non-causal elements of the (total) controller has been removed (this is a simplification of the $\mathcal{H}_2$ optimal feedforward controller given by Lewin and Scali (1988) and Scali *et al.* (1989)). As an example, let

$$G_{i,i} = \frac{ke^{-\theta s}}{\tau s + 1}; \quad \tilde{G}_{d,i} = \frac{k_d e^{-\theta_d s}}{\tau s + 1} \tag{18}$$

Then

$$K^{\mathrm{FF}}_{i,i-1,-} = \begin{cases} -\left(k_d/k\right) e^{-(\theta_d - \theta)s}; & \theta_d > \theta \\ -\left(k_d/k\right); & \theta_d \le \theta \end{cases} \tag{19}$$

When (15) cannot be realised, feedforward from units $i-2, i-3, \ldots$ can be useful. For example, if it is causal, the following feedforward controller from unit $i-2$ eliminates the control error that remains after $K^{\mathrm{FF}}_{i,i-1,-}$:

$$K^{\mathrm{FF}}_{i,i-2} = -G^{-1}_{i,i,-}\left(I - G_{i,i}G^{-1}_{i,i,-}\right)\tilde{G}_{d,i}\left(I - G_{i-1,i-1}G^{-1}_{i-1,i-1,-}\right)\tilde{G}_{d,i-1} \tag{20}$$

See Appendix B for a derivation of (20).

Feedforward control is generally sensitive to uncertainty, and we will now consider its effect. The nominal model is given by (7), and the actual model (with uncertainty) is

$$y'_i = G'_{i,i}u_i + \tilde{G}'_{d,i}y'_{i-1} + G'_{d,i,i}d_i \tag{21}$$

A pure feedforward controller from upstream units then yields the following actual control error:

$$e'_i \stackrel{\mathrm{def}}{=} y'_i - y_{r_i} = \tilde{G}'_{d,i}y'_{i-1} + \sum_{j=1}^{i-1} G'_{i,i}K^{\mathrm{FF}}_{i,i-j}y'_{i-j} + G'_{d,i,i}d_i - y_{r_i} \tag{22}$$

With "ideal" feedforward control based on the nominal model, as given by (15) and (16), the actual control error becomes

$$\begin{aligned} e'_i &= \left(\tilde{G}'_{d,i} - G'_{i,i}G^{-1}_{i,i}\tilde{G}_{d,i}\right)y'_{i-1} + G'_{d,i,i}d_i - y_{r_i} \\ &= \underbrace{\left(I - G'_{i,i}G^{-1}_{i,i}\tilde{G}_{d,i}\tilde{G}'^{\dagger}_{d,i}\right)}_{E_{d,i}}\tilde{G}'_{d,i}y'_{i-1} + G'_{d,i,i}d_i - y_{r_i} \end{aligned} \tag{23}$$

where † denotes generalized inverse (e.g., Zhou *et al*. (1996, page 76)), and $E_{d,i}$ is a relative model error in $G_{i,i}\tilde{G}^{\dagger}_{d,i}$. In particular, for scalar blocks

$$E_{d,i} = 1 - \frac{G'_{i,i}/\tilde{G}'_{d,i}}{G_{i,i}/\tilde{G}_{d,i}} \tag{24}$$

Thus, model errors at any frequency directly influence the actual control error. Upon comparing the response with control in (23) with the response without control ($u_i = 0$ in (21)) we see that "feedforward" (decoupling) control has a positive (dampening) effect on disturbances from upstream units at frequencies ω where

$$\|E_{d,i}(\omega)\| < 1 \tag{25}$$

or in words, as long as the relative error in $G_{i,i}\tilde{G}^{\dagger}_{d,i}$ is less than 1 in magnitude. Here, we use an appropriate norm dependent on the definition of performance.

External disturbances entering directly into the process at unit i, d_i, are (of course) not dampened by feedforward control from upstream units, but if d_i is measured, then separate feedforward controllers may be designed for d_i. Feedforward control from the reference, y_{r_i}, is also necessary to avoid control error if $y_{r_i} \neq 0$ and no feedback is applied.

Lower block-triangular control

A lower (block) triangular controller will result if we combine local feedback and feedforward from upstream units,

$$\begin{array}{ll} \text{Local control } (i = j): & u_i = K_{i,i}(s)y_i \\ \text{Feedforward } (i > j): & u_i = K^{\mathrm{FF}}_{i,j}(s)y_j \end{array}$$

The loop transfer function now becomes ($n = 3$):

$$\begin{aligned} L &= \begin{bmatrix} G_{11} & 0 & 0 \\ G_{21} & G_{22} & 0 \\ G_{31} & G_{32} & G_{33} \end{bmatrix} \begin{bmatrix} K_{11} & 0 & 0 \\ K^{\mathrm{FF}}_{21} & K_{22} & 0 \\ K^{\mathrm{FF}}_{31} & K^{\mathrm{FF}}_{32} & K_{33} \end{bmatrix} \\ &= \begin{bmatrix} G_{11}K_{11} & 0 & 0 \\ G_{21}K_{11} + G_{22}K^{\mathrm{FF}}_{21} & G_{22}K_{22} & 0 \\ G_{31}K_{11} + G_{32}K^{\mathrm{FF}}_{21} + G_{33}K^{\mathrm{FF}}_{31} & G_{32}K_{22} + G_{33}K^{\mathrm{FF}}_{32} & G_{33}K_{33} \end{bmatrix} \end{aligned} \tag{26}$$

The diagonal elements are feedback elements, where most of the control benefits are achieved simply by using sufficiently high gains, and an accurate process model is not needed. The main problem is that too high gain may give closed-loop instability.

As for the local feedback (diagonal) control structure, the stability of the closed-loop system (S) is determined only by the blocks on the diagonal, that is, we have closed-loop stability if and only if each of the local loops $(I + G_{i,i}K_{i,i})^{-1}$ are stable (e.g., Skogestad and Postlethwaite (1996)).

Note that we also obtain this control structure if an inverse-based (decoupling) design method ($K(s) = k(s)G^{-1}(s)$) is used. An example of an inverse based controller is IMC decoupling (Morari and Zafiriou, 1989), $K_{\mathrm{IMC}} = W_1 G^{-1} W_2$ where W_1 and W_2 are (block) diagonal matrices (with blocks corresponding to the blocks in G). For this controller we obtain the following diagonal and sub-diagonal blocks:

$$K_{\mathrm{IMC},i,i} = W_{1_{i,i}} G^{-1}_{i,i} W_{2_{i,i}} \tag{27}$$

$$K^{\mathrm{FF}}_{\mathrm{IMC},i,i-1} = -W_{1_{i,i}} G^{-1}_{i,i} G_{i,i-1} G^{-1}_{i-1,i-1} W_{2_{i-1,i-1}} \tag{28}$$

where $W_{j_{i,i}}$ denotes block (i,i) of weight matrix W_j (this is the integrator). (27) and (28) can be verified by calculating that $GG^{-1} = I$. Since the stability is determined by the diagonal blocks, and these are the scaled inverse of the blocks of G, the weights can be selected independently for each unit, e.g. using the method of Rivera *et al.* (1986) (for scalar blocks). If G is not invertible, e.g., due to right half plane zeros and delays, the not invertible part of G is essentially factored out before the inversion (for details, see Morari and Zafiriou (1989)).

Using (8), we note that the sub-diagonal part of the IMC controller, (28), is identical to the ideal feedforward controller (15), except for the weights. Integral action in the feedback part of the controller ($K_{IMC,i,i}$) requires an integrator in either $W_{1_{i,i}}$ or $W_{2_{i,i}}$. For example, we may choose $W_{2_{i,i}} = \frac{1}{\tau_{\mathrm{CL}} s} I$ where τ_{CL} is the desired closed loop time constant (Rivera *et al.*, 1986). Thus, we see from (28) that also the "feedforward" gain will be amplified at low frequencies.

Let us now consider the effect of model uncertainty for this case. The nominal model is given by (7) and the actual model by (21). A lower triangular controller yields the following actual control error:

$$e_i' \stackrel{\text{def}}{=} y_i' - y_{r_i} = S_i' \left(\tilde{G}_{d,i}' y_{i-1}' + \sum_{j=1}^{i-1} G_{i,i}' K_{i,i-j} y_{i-j}' + G_{d,i,i}' d_i - y_{r_i} \right) \tag{29}$$

where (e.g., Skogestad and Postlethwaite (1996))

$$S_i' = \left(1 + G_{i,i}' K_{i,i}\right)^{-1} = S_i \left(1 + E_i T_i\right)^{-1} \tag{30}$$

Here S_i and T_i are the nominal sensitivity and complementary sensitivity functions, respectively, and E_i is the relative error in G (note that E_i in Section 2 denoted something else).

Upon comparing the closed-loop response in (29) with the open loop response in (21) we see the following:

1. Effective local feedback control ($\|S_i(j\omega)\| \ll 1$) dampens disturbances from the preceding unit (y_{i-1}), external disturbances entering the process at unit i, and also the effect of the model error (E_i) and errors in the feedforward control.

2. For frequencies where feedback control is not effective, i.e., $\|S_i(j\omega)\| \geq 1$ we may benefit from feedforward control. We can apply the results from Section 3.1, (15)-(25), except that (20) must be modified due to the feedback control in unit $i-1$:

$$\begin{aligned} K_{i,i-2}^{\mathrm{FF}} = -G_{i,i}^{-1} \left(I - G_{i,i} G_{i,i,-}^{-1}\right) \tilde{G}_{d,i} \left(I - G_{i-1,i-1} K_{i-1,i-1}\right)^{-1} \\ \left(I - G_{i-1,i-1} G_{i-1,i-1,-}^{-1}\right) \tilde{G}_{d,i-1} \end{aligned} \tag{31}$$

3. As for the pure feedforward case, external disturbances entering directly at unit i, d_i, are not dampened by the feedforward control from upstream units, so they must be handled by the feedback control. If this is not sufficient, and provided d_i is measured, a separate feedforward controller may be designed for d_i.

For serial processes with a lower block-triangular controller, it is particularly simple to identify feedforward and feedback controller elements, but similar differences between the elements occur for most multivariable controllers. Such insights are important for example when evaluating how the controller is affected by model error.

A more general analysis of feedforward control under the presence of uncertainty is given elsewhere Faanes and Skogestad (2004*a*) (or Faanes (2003, Chapter 6)).

3.2 Full controller

With a full controller, as in (10), and three units ($n = 3$), the loop transfer function becomes

$$L = G(s)K(s) = \begin{bmatrix} G_{11}K_{11} & G_{11}K_{12} & G_{11}K_{13} \\ G_{21}K_{11} + G_{22}K_{21} & G_{21}K_{12} + G_{22}K_{22} & G_{21}K_{13} + G_{22}K_{23} \\ G_{31}K_{11} + G_{32}K_{21} + G_{33}K_{31} & G_{31}K_{12} + G_{32}K_{22} + G_{33}K_{32} & G_{31}K_{13} + G_{32}K_{23} + G_{33}K_{33} \end{bmatrix} \quad (32)$$

In this case, the stability of the closed-loop system is affected by all elements in the controller K (and in G).

As illustrated in the case study in Section 4, also in this case the controller blocks below the diagonal have properties similar to feedforward control.

3.3 Final control only in last unit (input resetting)

In many serial processes, the output from the last unit is the most important for the overall plant economics, and the inputs in the upstream units are extra degrees of freedom. These are normally used for local disturbance rejection by controlling the outputs in upstream units. The inputs towards the end of the process can then be reset to some ideal resting value by adjusting the upstream unit set-points.

We may then use the following control elements:

Local control ($i = j$)	$u_i = K_{i,i}(s)\left[y_{r_i} - y_i\right]$
Feedforward ($i > j$)	$u_i = K^{\mathrm{FF}}_{i,j}(s) y_j$
Input resetting ($i < j$)	$y_{r_i} = K^{\mathrm{IR}}_{i,j}(s)\left[u_{r_j} - u_j\right]$

The controller can usually be tuned in a rather simple sequential manner. The feedforward elements are normally the fastest acting and should normally be designed first. The local feedback controllers can be tuned almost independently. Finally, the slow input resetting is added, which will not affect the closed-loop stability if it is sufficiently slow.

4 Case study: pH neutralization

4.1 Introduction

Neutralization of strong acids or bases is often performed in several steps (tanks). The reason for this is mainly that with a single tank, the pH control is not quick enough to compensate for disturbances (Skogestad, 1996). McMillan (1984) uses an analogy from golf: the difficulty of controlling the pH in one tank is compared to getting a hole in one. Using several tanks, and smaller valves for addition of reagent for each tank, is similar to reaching the hole with a series of shorter and shorter strokes.

In the present example, we want to compare different control structures for neutralization of a strong acid in three tanks (see Figure 3). This is clearly a serial process. The aim of the control is to keep the outlet pH from the last tank constant despite changes in inlet pH and inlet flow rate. For each tank, the pH can be measured, and the reagent (here base) can be added. Figure 3 shows the process with only local control in each tank (K diagonal).

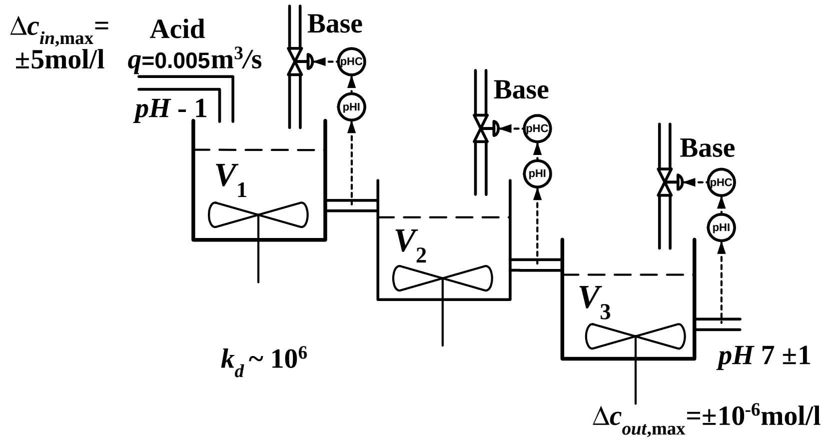


Figure 3: Neutralization of an acid in three tanks in series with local control in each tank.
Data: Outlet requirement: $pH = 7 \pm 1$, set-points tank 2 and 3: $pH = 1.65$ and $pH = 3.8$. Inlet acid flow $pH = -1$ ($= 10\,\mathrm{mol}\,/\,\mathrm{l}$) and flow rate $0.005\,\mathrm{m}^3\,/\,\mathrm{s}$. Reactant (base): $pH = 15$ ($= 10\,\mathrm{mol}\,/\,\mathrm{l}$), nominal flow: $0.005\,\mathrm{m}^3\,/\,\mathrm{s}$. $V_1 = V_2 = V_3 = 13.6/\,\mathrm{m}^3$.

4.2 Model

To study this process we use the models derived in a previous paper Faanes and Skogestad (2004*b*). In each tank we consider the excess H^+ concentration, defined as $c = c_{H^+} - c_{OH^-}$. This gives a bilinear model, which is linearized around a steady-state working point, so that the methods from linear control theory can be used. We get two states in each process unit (tank), namely the excess concentration, c, and the level. The disturbances (mainly feed changes) enter in tank 1. We here assume that there is a delay of $5\,\mathrm{s}$ for the effect of a change in inlet acid or base flow rate or inlet acid concentration to reach the outflow concentration of the tank, e.g. due to incomplete mixing, and a further delay of $5\,\mathrm{s}$ until the change can be measured. In the discrete linear state space model, these delays are represented as extra states (poles in the origin). We assume no further delay in the pipes between the tanks. The levels are assumed to be controlled by the outflows using P controllers such that the time constant for the level is about 1/10 of the residence time (i.e., $q = 0.01\,(V - V_s)$, where V_s is the volume set-point).

The volumes of the tanks are chosen as $13.6\,\mathrm{m}^3$, which are the smallest possible volumes according to the discussion in Skogestad (1996). The concentrations are scaled such that a variation of $c = \pm 1\,pH$ corresponds to a scaled value of ± 1. The control inputs and the disturbances are also scaled appropriately. The linear model is used for multivariable controller design, whereas the simulations are performed on the nonlinear model.

4.3 Model uncertainty

The model presented in the previous section was the *nominal* model, which will be used in the controller design. If the model gives an exact representation of the actual process, we say it is *perfect*. Due to simplifications in the modelling or process variations, there is often a discrepancy between the model and the actual process. Often the model is idealized, i.e., it is simplified, to ease the modelling work, the identification of parameters, and the controller design.

In this example, we use linearized models in the MPC design. In the design of (SISO) feedforward controllers, a further simplification is that outlet flow variations are neglected.

The latter gives a steady-state model error, but dynamically the error is small due to the slow level control. What we here consider as the "actual plant", is the full nonlinear model, possibly with the following errors:

- Offset of 0.2 (in scaled value) in control input u_3 (last tank).
- pH measurement error of -1 in second tank.

4.4 Single loop controllers

The conventional way of controlling this process is to use *local PID-control of the pH in each tank*. Figure 4(a) shows the pH-response in each tank when the acid concentration in the inflow is decreased from $10\,\mathrm{mol}\,/\,\mathrm{l}$ to $5\,\mathrm{mol}\,/\,\mathrm{l}$. As expected since the tank volumes are selected at their minimum (Skogestad, 1996), this control system is barely able to give acceptable control, $pH = 7 \pm 1$ in last tank, even though the PID-controllers are tightly tuned.

We now want to study the use of *feedforward control from upstream units*. As before, we let the pH in the first tank be controlled with local PID control (the same tuning as before), since we do not measure inlet disturbances to tank 1, and feedback is therefore the only possibility. We let the pH in the second and third tanks be controlled with feedforward control only, namely with feedforward from y_1 to u_2 and from y_2 to u_3. With "ideal" feedforward control based on the nominal model, we then find K_{21}^{FF} and K_{32}^{FF} from (17). The net delay is increased to obtain a causal controller with zero or positive delay in the controller. The two feedforward controllers will react $5\,\mathrm{s}$ too late due to the measurement delays in y_1 and y_2, and thereby introduce a transient output error. To avoid this, the last feedforward controller, K_{31}^{FF}, from y_1 to u_3, can be used to eliminate this error as given in (20).

Figure 4(b) shows the nominal response (thick lines), and we can see that perfect control is achieved in tank 3. However, when applied to a more realistic nonlinear model, which incorporates flow rate changes (dotted lines), we find that the feedforward controller fails.

We now *combine the local PID-control* in all the tanks with *feedforward control of tanks 2 and 3* (controllers K_{21}^{FF}, K_{31}^{FF} and K_{32}^{FF}). In K_{31}^{FF} it is now necessary to take into account the feedback loop of tank 2 and use Equation (31). Again, with a perfect model the effect of the disturbance is eliminated. Simulation on the more realistic model reveals as expected an improvement compared to the pure feedback and pure feedforward structures. The feedforward controllers reduce the transient errors, whereas the PID controllers remove the steady-state errors, as illustrated in Figure 4(c).

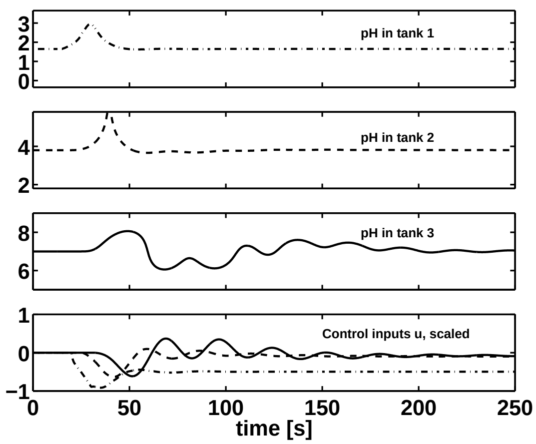
In Figure 5(a) we plot the magnitude of the controller gains (dotted lines). The presence of integral action is recognized from the high gains at low frequencies in the diagonal elements. The sub-diagonal control elements K_{21}^{FF} and K_{32}^{FF} are constant, whereas K_{31}^{FF} only has an effect at high frequencies where K_{32}^{FF} is no longer effective.

Note that with a larger model error, the positive effect of the feedforward controller may be reduced, and the feedforward action may even amplify the disturbances.

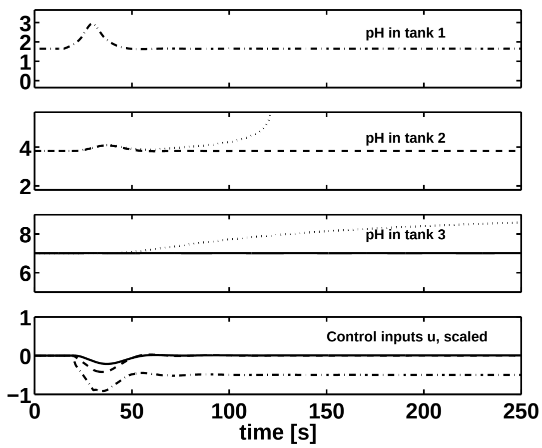
4.5 Multivariable control

Original 3×3 MPC controller

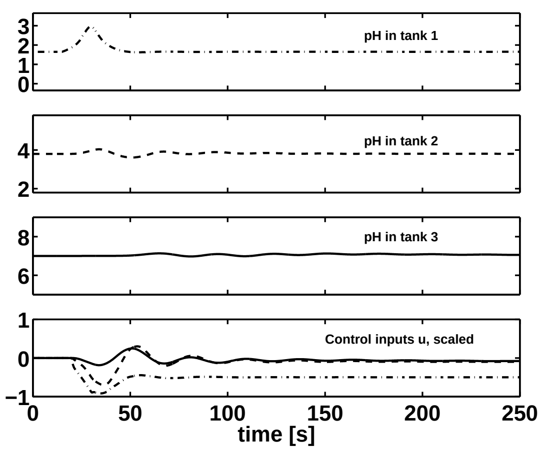
Figure 4(d) shows the response with a 3×3 MPC controller (Muske and Rawlings (1993); see also Appendix A). To obtain the current state at each time step for the controller, a state estimator is used. The estimated states in this "original" MPC-controller also include the two (unmeasured) disturbances, inlet flow rate and inlet excess concentration, modelled as integrated white noise (we will discuss this choice later). The controller design is based on a discretized model, whereas in the simulation only the controller is discrete. Even if this is a feedback controller, we see that the disturbance response is similar to that of combined local feedback and feedforward control in Figure 4(c).



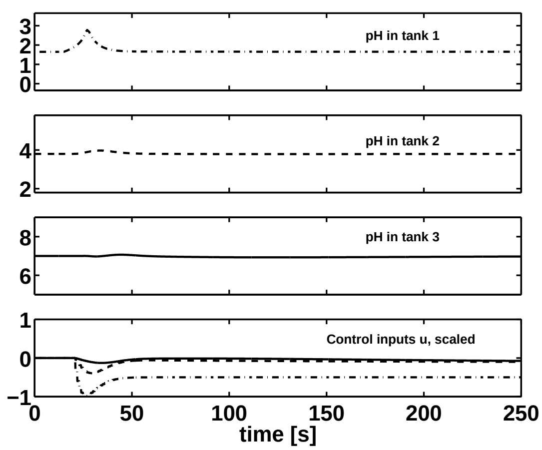
(a) Local feedback PID-control in all three tanks.



(b) Local feedback PID-control in tank 1, and feedforward control of tanks 2 and 3. Dotted lines for tanks 2 and 3: Response with model error.



(c) Local feedback PID-control in all three tanks combined with feedforward control of tanks 2 and 3.



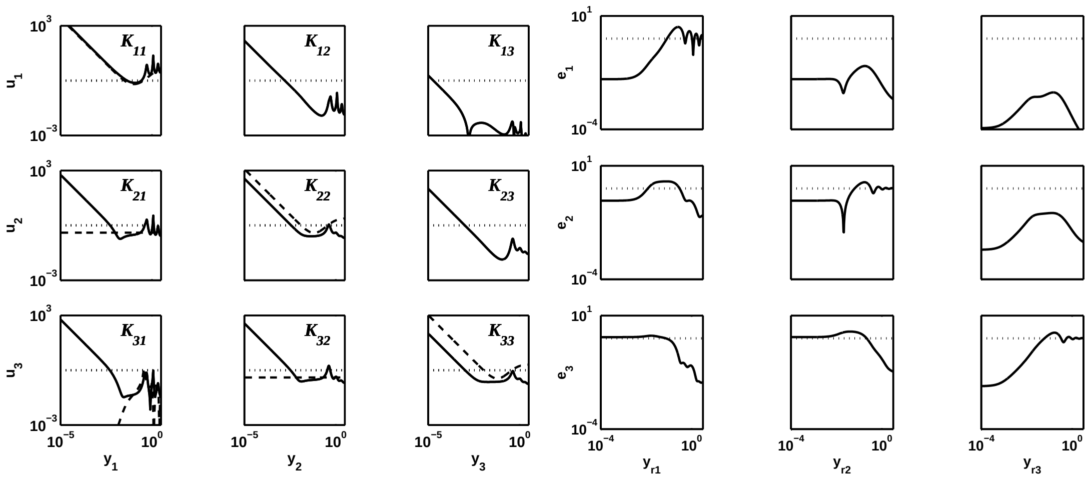
(d) 3×3 multivariable MPC control

Figure 4: Different control structures applied to the neutralization process in Figure 3. Disturbance in inlet concentration occurs at $t = 10\,\mathrm{s}$.

“Feedforward” part of MPC-controller

From the lower plots in Figure 4(a) and Figure 4(d), we can see clearly that MPC has a “feedforward” effect. To study this “feedforward” effect separately, we design a MPC-controller that uses the pH measurement in the first tank only, but adjust the reactant flow rates to all three tanks. The response becomes for the nominal case similar to the simulation with the full MPC-controller in Figure 4(d). If, however, a model error is introduced, e.g. by simulation with the nonlinear model, a steady-state error occurs for outlet pH. The reason for this is the lack of feedback control in the last two tanks.

The individual gains of the 3×3 MPC-controller are shown as a function of frequency in Figure 5(a) (solid lines). Recall that the diagonal control elements are the local controllers in each tank, whereas the elements below the diagonal represent the "feedforward" elements. From these plots, we get an idea of how the multivariable controller works. For example, we see that the control input to tank 1 (row 1) is primarily determined by local feedback, whereas in tanks 2 and 3 (rows 2 and 3) it seems that "feedforward" from the previous tank is more important. In tanks 2 and 3 the control actions are smaller, which is also confirmed in the simulation (Figure 4(d)). The local feedback control elements on the diagonal compare well with the PID controllers (dashed lines), except that the gain is reduced for tanks 2 and 3, but this depends on the tuning of the MPC. At high frequencies, the MPC "feedforward" elements are similar to the manually designed feedforward controllers.



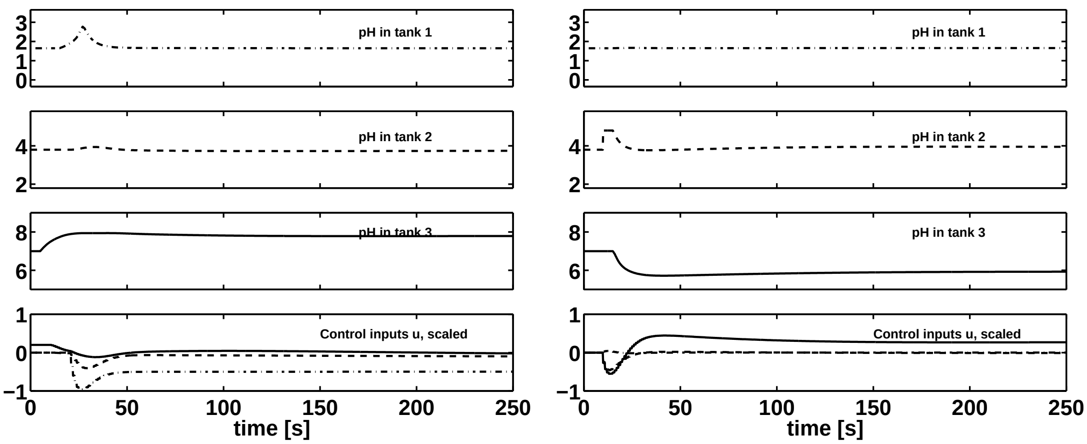
(a) Controller gains for 3×3 MPC (solid). Also shown: Local PID controllers and manually designed feedforward elements (dashed).

(b) Gains of the sensitivity function S for 3×3 MPC

Figure 5: Original 3×3 MPC controller, frequency domain analysis.

Integral action in original 3×3 MPC controller

As mentioned in Section 3, it is not straight-forward to interpret the steady state behaviour from the gain plots of the controller elements when all the elements have large gains at low frequencies. This is shown by considering the controller gain elements for the MPC controller in Figure 5(a). All elements have large gains at low frequencies, so it seems we have integral action in all outputs. However, from Figure 5(b) we see that only the first row in S has all elements small at low frequencies. Thus, only output 1 has integral action. We should therefore expect steady-state offset in tank 3. However, the nominal simulations in Figure 4(d) show no offset. The reason is that the integral effect in the first tank removes the concentration effect, and the “feedforward” control gives the correct compensation for the flow rate disturbance. However, if some unmodelled disturbance or model error is introduced (e.g. a constant offset in u_3 or a measurement error in tank 2), then we get steady-state offset. This indeed is shown in Figure 6. The local PID controllers give no such steady-state offset.



(a) Unmodelled disturbance: Control input u_3 has offset of 0.2 (at time 0).

(b) Measurement error: At time 10 s a pH measurement error of -1 is introduced in tank 2.

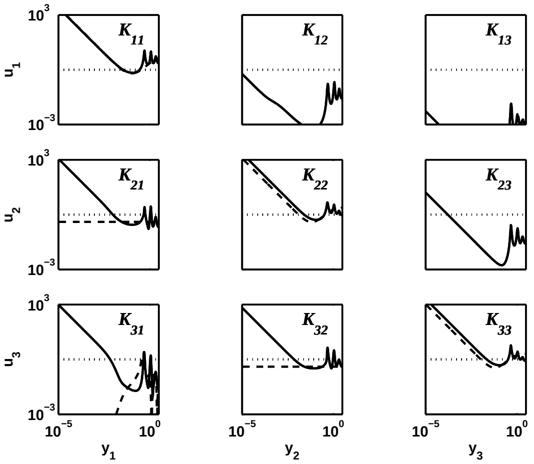
Figure 6: The original 3×3 MPC has insufficient integral action and the pH in tank 3 drifts away when we have model error.

Modified 3×3 MPC-controller with integral action

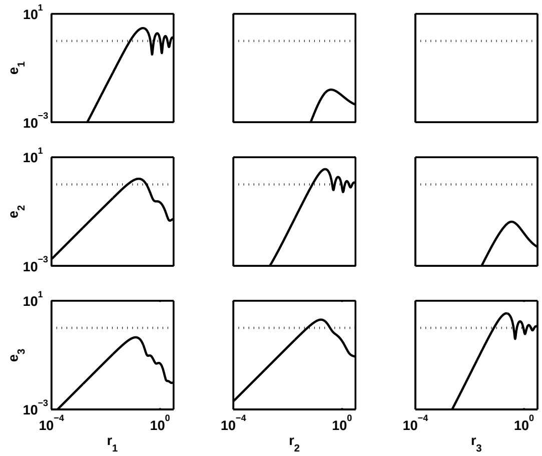
In the “original” estimator used above, we only estimated the inlet disturbances to tank 1. We now redesign the controller by estimating *one disturbance in each tank*: The concentration disturbance to the first tank and disturbances in the manipulated variables in tanks 2 and 3 (u_2 and u_3). The resulting controller gains are shown in Figure 7(a). With this design the gain in $|S(j\omega)|$ is low at low frequencies for all tanks (Figure 7(b)), and the simulations in this case give no steady-state offset, also when errors like in Figure 6 are present. This agrees with the result (Pannocchia and Rawlings, 2003; Faanes and Skogestad, 2003) that the number of disturbance estimates in the controller must equal the number of measurements.

4.6 MPC with input resetting

In the simulations above, we fixed the pH-setpoints in all three tanks. Actually, we are only interested in the pH in the last tank, so that giving set-points for tanks 1 and 2 is not necessary. Since we have three control inputs, this leaves two extra degrees of freedom, which, as described in section 3.3, may be used for input resetting. The MPC controller is easily modified to accommodate this. Figure 8 illustrates how it performs for a unit step in the disturbance (Figure 8(a)) and the controller gains (Figure 8(b)). At steady state, all the required change in base addition is done in the first tank. Since we do not measure the actual base addition, there is no compensation for offset in the control input.

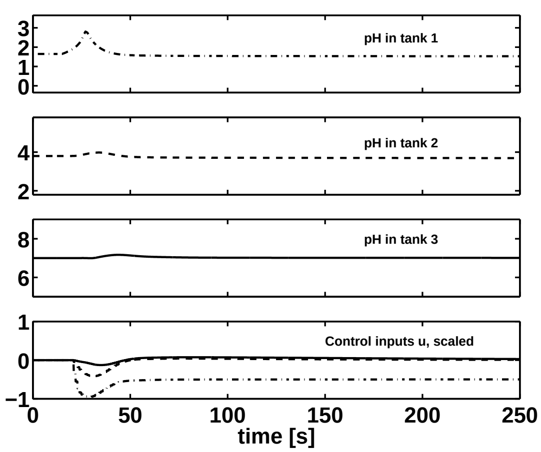


(a) Controller gains for the modified 3×3 MPC (solid). Also shown: Local PID controllers and manually designed feedforward elements (dashed).

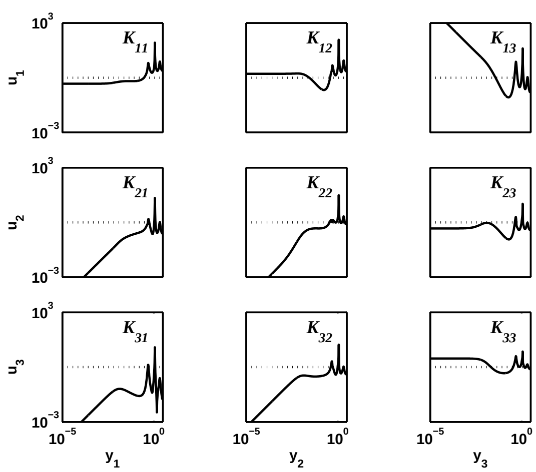


(b) Gains of the sensitivity function.

Figure 7: Modified 3×3 MPC with integral action



(a) Simulation. Step disturbance at time 10 s.



(b) Controller gains

Figure 8: MPC with integral action and input resetting.

5 Discussion

There are several ways to avoid steady-state offset with MPC controllers. The most common method is to estimate the bias in the outputs, i.e. the difference between the predicted and the measured outputs, and compensate for this bias. However, performance is often improved by estimating *input* biases, or disturbances (Muske and Rawlings, 1993; Lee *et al.*, 1994; Lundström *et al.*, 1995; Muske and Badgwell, 2002; Pannocchia and Rawlings, 2003). In this paper, we have followed this approach. We ended up with estimating the concentration disturbance into first tank and input biases for tanks 2 and 3 (three input biases gives similar results). Our controller handles well both input disturbances (see Figure 4(d)) and output disturbances or measurement errors.

We have also tried to estimate output biases, but this gave a very slow settling in response to inlet disturbances. The reason is the long time constants in our process, which give the output bias estimates a ramp form (Lundström *et al.*, 1995). The controller then faces a problem similar to following a ramp trajectory.

In a previous paper, (Faanes and Skogestad, 2004*b*), we found that the minimum volume in each tank is limited by the delays in each tank. In the present paper, we found that with a full multivariable controller, these limitations are theoretically no longer valid provided we have a sufficiently accurate process model. The reason for this is that the multivariable controller does not have to wait for the measurement in last tank before it takes action (due to the "feedforward" effect). To be able to achieve a nominally perfect "feedforward" control effect, the delay from at least one control input to the output must be shorter or equal to the delay from a measurement in the disturbance to the output. Equally important, the effect of model uncertainty must not be too large. If this is satisfied, then one may design smaller tanks compared to the sizes given in Faanes and Skogestad (2004*b*) or reduce the instrumentation.

6 Conclusions

An analysis of the control of a serial process has revealed some interesting properties of multivariable controllers

1. The multivariable controller may rely strongly on feedforward control and thus be sensitive to model errors, including at steady state if there is no integral action in the feedback part of the controller. The lack of integral feedback action may be difficult to see from nominal simulations, but it may be identified by plotting the gains of the sensitivity function as a function of frequency.

2. Note that high controller gains at low frequencies does not guaranty integral action in the presence of uncertainty (see Figure 5).

3. To avoid the above problem of lacking integral action, one may use an observer-based feedback correction with disturbances at as many plant inputs as there are measurements.

4. If some feedforward controller blocks make the performance poorer with model error present, these blocks may be removed by modifying the process model (e.g., by removing parts of the model).

In this study, we considered model predictive control (MPC), but very similar results have also been obtained for a multivariable $\mathcal{H}_\infty$-controller (Faanes and Skogestad, 1999).

7 Acknowledgements

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Appendix A State space MPC used in case study

Here, we briefly describe the MPC controller of Muske and Rawlings (1993) under the assumption that the constraints are not active. For details see Faanes and Skogestad (2003).

The MPC controller uses an estimate of the current states of the process and a state space model to predict future responses to control input movements. By letting the control input change each time step over a certain horizon, and thereafter be held constant, the optimal sequence of control inputs is calculated. The criterion for the optimization is

$$\min_{u_k^N} \sum_{j=0}^{\infty} \left(y_{k+j}^T Q y_{k+j} + u_{k+j}^T R u_{k+j} + \Delta u_{k+j}^T S \Delta u_{k+j} \right) \tag{33}$$

where u_k^N is the vector of N future control inputs, the first at sample number k, y_k is the output vector at sample k, u_k is the control input at sample k, Δu_k is the change in u_k since last time step and Q, R and S are weight matrices. Note that in the criterion we assume that the set-point for the output, $y_r = 0$. Non-zero set-points are handled by a steady-state solver. Only the first control input is applied, since at next time step the whole sequence is recalculated, starting from the states actually obtained at that moment.

Without active constraints the MPC can be represented as state feedback control, i.e. the control input u_k at time step no. k can be expressed by

$$u_k = Kx_k + K_u u_{k-1} \tag{34}$$

where x_k is the state vector at step k and K and K_u are constant matrices, independent of time provided the model is time invariant. The dependence of the control input at the previous step, u_{k-1}, comes from the weight on change in u in the optimization criterion.

Since all the states are not measured, we estimate them for example with a Kalman filter. For the MPC algorithm we use a discretized model with time step 1 second and use a zero order hold method for the discretization since the inputs are held constant between the time steps. In the discretized model, time delays are represented exactly, as long as they are multiples of the time step.

In Faanes and Skogestad (2003) we derive a state space formulation for the controller and the estimator:

$$x_{k+1}^K = Ax_k^K + By_k^m + Ey_r \tag{35}$$

$$u_k = Cx_k^K + Dy_k^m + Fy_r \tag{36}$$

where u_k is the control input at sample number k, x_k^K is the controller/estimator state vector, y_k^m is the measurement vector and y_r the reference, which may be seen as a disturbance to the controller. A, B, C, D, E and F are constant matrices.

For frequency analysis of the controller, we may convert this discrete controller into a continuous one using d2c in Matlab (Tustin method), and Laplace transform yields:

$$u(s) = K(s)\, y^m(s) + K_r(s)\, y_r(s) \tag{37}$$

We have chosen weights in the MPC optimization criterion (33) as $Q = diag\,(100,\ 1, 1)$, $R = I$ and $S = 0$. For the estimator, the co-variance matrices are $Q_w = I$ (process noise) and $R_v = I$ (measurement noise).

Appendix B Derivation of equations (20) and (31)

With pure feedforward control, we get the following control error

$$\begin{aligned} e_i &= \tilde{G}_{d,i} y_{i-1} + G_{i,i} K_{i,i-1}^{\mathrm{FF}} y_{i-1} + G_{i,i} K_{i,i-2}^{\mathrm{FF}} y_{i-2} \\ &= \left(I - G_{i,i} G_{i,i,-}^{-1}\right) \tilde{G}_{d,i} y_{i-1} + G_{i,i} K_{i,i-2}^{\mathrm{FF}} y_{i-2} \end{aligned} \tag{38}$$

where we have inserted feedforward from unit $i-1$ from (17). With a combination of feedback and feedforward control, we get (with (17))

$$e_i = (I - G_{i-1,i-1} K_{i-1,i-1})^{-1} \left(I - G_{i,i} G_{i,i,-}^{-1}\right) \tilde{G}_{d,i} y_{i-1} + G_{i,i} K_{i,i-2}^{\mathrm{FF}} y_{i-2} \tag{39}$$

In both cases, "ideal" feedforward requires $e_i = 0$ for all y_{i-1} and y_{i-2}:

$$\left(I - G_{i,i} G_{i,i,-}^{-1}\right) \tilde{G}_{d,i} y_{i-1} + G_{i,i} K_{i,i-2}^{\mathrm{FF}} y_{i-2} = 0 \tag{40}$$

We consider first pure feedforward, $K_{i,i} = K_{i-1,i-1} = 0$, and find the transfer function from y_{i-2} to y_{i-1}:

$$y_{i-1} = \left(\tilde{G}_{d,i-1} + G_{i-1,i-1} K_{i-1,i-2}^{\mathrm{FF}}\right) y_{i-2} \tag{41}$$

$K^{\mathrm{FF}}_{i-1,i-2} = -G^{-1}_{i-1,i-1,-}\tilde{G}_{d,i-1}$ yields

$$y_{i-1} = \left(I - G_{i-1,i-1}G^{-1}_{i-1,i-1,-}\right)\tilde{G}_{d,i-1}y_{i-2} \tag{42}$$

and upon inserting (42) into (40) we obtain

$$G_{i,i}K^{\mathrm{FF}}_{i,i-2} + \left(I - G_{i,i}G^{-1}_{i,i,-}\right)\tilde{G}_{d,i}\left(I - G_{i-1,i-1}G^{-1}_{i-1,i-1,-}\right)\tilde{G}_{d,i-1} = 0$$

leading to (20).

Second, we find the transfer function from y_{i-2} to y_{i-1} for a combination of local feedback and feedforward,

$$y_{i-1} = G_{i-1,i-1}K_{i-1,i-1}y_{i-1} + \left(\tilde{G}_{d,i-1} + G_{i-1,i-1}K^{\mathrm{FF}}_{i-1,i-2}\right)y_{i-2} \tag{43}$$

where $K^{\mathrm{FF}}_{i-1,i-2} = -G^{-1}_{i-1,i-1,-}\tilde{G}_{d,i-1}$. Then

$$y_{i-1} = \left(I - G_{i-1,i-1}K_{i-1,i-1}\right)^{-1}\left(I - G_{i-1,i-1}G^{-1}_{i-1,i-1,-}\right)\tilde{G}_{d,i-1}y_{i-2} \tag{44}$$

and by inserting this into (40) it follows

$$\begin{aligned} G_{i,i}K^{\mathrm{FF}}_{i,i-2} + \left(I - G_{i,i}G^{-1}_{i,i,-}\right)\tilde{G}_{d,i}\left(I - G_{i-1,i-1}K_{i-1,i-1}\right)^{-1} & \\ \left(I - G_{i-1,i-1}G^{-1}_{i-1,i-1,-}\right)\tilde{G}_{d,i-1} &= 0 \end{aligned} \tag{45}$$

which gives (31).