Tuning for Smooth PID Control with Acceptable Disturbance Rejection

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The objective of most published tuning rules is “tight” or fast control, subject to avhieving acceptable robustness. This gives a maximum limit on the controller gain. In industrial practice, however, the objective is often “smooth” or slow control, subject to achieving acceptable performance in terms of disturbance rejection. This paper provides the corresponding minimum limit on the controller gain. In terms of appropriately scaled variables, the minimum controller gain is approximately 1, which justifies default settings commonly used in industry.

1. Introduction

Two main approaches to controller tuning are as follows:

(1) Tight control: Fastest possible control subject to achieving acceptable robustness; and

(2) Smooth control: Slowest possible control subject to achieving acceptable disturbance rejection.

Although “smooth” control is probably the more common objective in industrial practice, almost all published proportional−integral−derivative (PID) tuning rules[1−6] aim at tight control. The model-based direct synthesis approaches[4−6] have the closed-loop time constant τ_c (alternatively known as ϵ or λ) as a tuning parameter, but also in these works the emphasis is to obtain a lower bound on τ_c (tight control).

In this paper, we will mainly consider the simple Skogestad internal model control (SIMC) PID rule,[6] which has τ_c as a tuning parameter, but it is stressed that the results can be applied also to other rules that have a free tuning parameter. For a first-order plus delay process with gain k, time constant τ_1, and time delay θ,

$$g(s) = k\frac{e^{-\theta s}}{\tau_1 s + 1} \tag{1}$$

and PI controller,

$$c_{PI}(s) = K_c\left(1 + \frac{1}{\tau_I s}\right) \tag{2}$$

the SIMC PI settings for the controller gain and integral time are[6]

$$K_c = \frac{1}{k}\cdot\frac{\tau_1}{\tau_c + \theta} \tag{3}$$

$$\tau_I = \min(\tau_1, 4(\tau_c + \theta)) \tag{4}$$

A second-order plus delay process model results in a PID controller.[6] The tuning parameter τ_c is free to be chosen, but it is generally bounded by the limits for tight ($\tau_{c,\,min}$) and smooth ($\tau_{c,max}$) control:

$$\tau_{c,min} \le \tau_c \le \tau_{c,max} \tag{5}$$

The limit $\tau_{c,min}$ depends mainly on the robustness requirements with respect to the delay θ, and the SIMC recommendation[6] for “tight” control is to select the closed-loop response time equal to the delay:

$$\tau_{c,min} = \theta \tag{6}$$

For a first-order plus delay process, this choice gives a robust design with a gain margin between 2.96 and 3.14, a sensitivity peak (M_s) between 1.70 and 1.59, and a maximum allowed time delay error (increase) between 159% and 214%. Note that it is possible to use an even lower value for τ_c if we accept less robustness. For example, the classical Ziegler−Nichols tuning rules[1] are quite aggressive and correspond to $\tau_c \approx 0$.

It is clear that the “SIMC-rule for tight control” ($\tau_c = \theta$) is *not* a good choice in many cases. First, with no delay ($\theta = 0$), eq 6 gives $\tau_c = 0$ and eq 3 gives an infinite controller gain, which obviously cannot be used in practice. More generally, one prefers to use a larger value for τ_c than that given in eq 6 in order to obtain smoother control with

(i) less input usage,

(ii) less sensitivity to measurement noise,

(iii) better robustness, and

(iv) less disturbing effect on the rest of the plant.

For example, the “λ tuning approach”, commonly used in the pulp and paper industry, normally gives a much larger value for τ_c; one recommendation is to use a closed-loop time constant which is 3× the process time constant,[7] i.e., $\lambda = \tau_c = 3\tau_1$. However, the problem is that the resulting disturbance response may be unacceptable (too slow). The question, therefore, is as follows: What is the slowest (“smoothest”) control we can allow, or equivalently, what is $\tau_{c,max}$?

The goal of this paper is to derive this limit, when the performance requirement is to achieve a specified level of disturbance rejection. In turns out to be simplest to derive a lower limit on the lower gain, $K_{c,min}$. We can then use eq 3 to obtain

$$\tau_{c,max} = \frac{\tau_1}{k}\cdot\frac{1}{K_{c,min}} - \theta \tag{7}$$

and we can obtain the corresponding value of τ_I from eq 4.

2. Derivation of Lower Limit on Controller Gain

The linear transfer function model in deviation variables is written (Figure 1)

$$y = g(s)u + g_d(s)d \tag{8}$$

Here, u is the manipulated input (controller output), d is the disturbance, y is the controlled output, $g(s)$ is the process transfer

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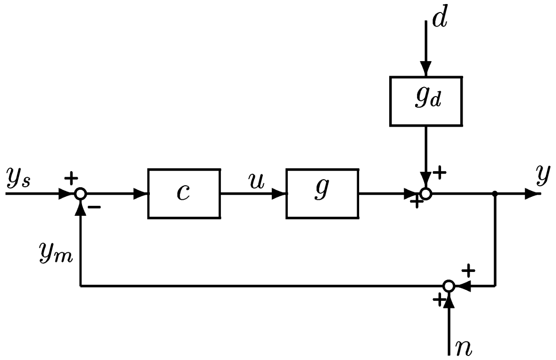

Figure 1. Block diagram of feedback control system.

function, and $g_d(s)$ is the disturbance transfer function. The Laplace variable s is often omitted to simplify notation. With feedback control, we have $u = c(s)(y_s - y)$, where $c(s)$ is the feedback controller. In the following, we do not consider setpoint changes, i.e., $y_s = 0$ in terms of deviation variables. The effect of the disturbance d on the control output y under closed-loop control is then

$$y = \frac{g_d(s)}{1 + g(s)c(s)} d \tag{9}$$

We consider the following performance requirement: For any sinusoidal disturbance d of magnitude $|d_0|$, $d(t) = d_0 \sin \omega t$, the resulting output variation y should be less than $|y_{max}|$, i.e., $y(t) \leq |y_{max}|$.

For simplicity, we mostly assume that $|d_0|$ and $|y_{max}|$ are constant, independent of frequency ω (rad/s), but the derivation also holds if they are frequency-dependent. Using eq 9, the performance requirement $|y| \leq |y_{max}|$ becomes

$$\frac{|g_d(j\omega)|}{|1 + gc(j\omega)|} |d_0| \leq |y_{max}|$$

or equivalently

$$|1 + gc(j\omega)| \geq |g_d(j\omega)| \frac{|d_0|}{|y_{max}|}; \qquad \forall \omega \tag{10}$$

The exact requirement (eq 10), which should be satisfied at all frequencies to achieve acceptable disturbance rejection, can be used to obtain "smooth" controller settings numerically.

To obtain explicit controller settings and obtain insight, we now want to use eq 10 to derive bounds on the controller gain. To this effect, consider the graphical illustration of eq 10 in Figure 2. Define ω_c as the gain crossover frequency where $|gc|$ drops below 1 (see Figure 2), and divide the frequency range in two, as follows:

(1) High Frequencies. For $\omega \geq \omega_c$, we have $|1 + gc| \approx 1$ and we must from eq 10 require that

$$|g_d(j\omega)| \leq \frac{|y_{max}|}{|d_0|}; \qquad \forall \omega \geq \omega_c \tag{11}$$

A necessary requirement for eq 11 is that it is satisfied at frequency ω_c,

$$|g_d(j\omega_c)| \leq \frac{|y_{max}|}{|d_0|} \tag{12}$$

(in process control, this condition is usually also sufficient because eq 11 is usually most easily satisfied at low frequencies).

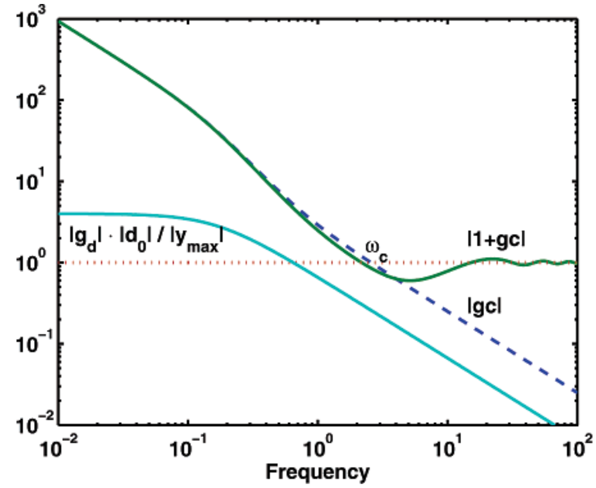

Figure 2. Frequency plots with "tight" SIMC PI settings. Performance requirement $|1 + gc| \geq |g_d|\cdot|d_0|/|y_{max}|$ (eq 10) is easily satisfied at all frequencies. Data: $g_d = g = 4\ (e^{-0.2s})/(6s + 1)$, $|y_{max}| = 1$, $|d_0| = 1$, PI control with $K_c = 3.75$ and $\tau_I = 1.6$.

Consider the special case, studied in this paper, of a first-order plus delay process with input disturbance, $g_d(s) = g(s) = k\, e^{-\theta s}/(\tau_1 s + 1)$. Since $|g_d(j\omega)|$ drops with frequency, condition 12 is most easily satisfied when ω_c is large (tight control), so we must at least require that eq 12 is satisfied when we use the tight SIMC rule in eq 6, which gives $\omega_c \approx 0.5/\theta$.[6] We assume $\omega_c\tau_1 > 1$ (i.e., $\theta < 0.5\tau_1$), because otherwise feedback control cannot give any performance improvement for this disturbance, and with this assumption we have $|g_d(j\omega_c)| \approx k/(\omega_c\tau_1)$. To satisfy the necessary requirement (eq 12), we must then require the following:

$$\theta < 0.5 \frac{\tau_1}{k} \frac{|y_{max}|}{|d_0|} \tag{13}$$

Note that eq 13, or more generally eq 12, is a process "controllability" condition, which must be satisfied because otherwise the process is not controllable with any controller. In summary, to satisfy eq 10 at high frequencies for the case with an input disturbance to a first-order plus delay process, we must make the following requirements in terms of the time delay:

Requirement R1 (for feedback control to be useful):

$$\theta < 0.5\tau_1$$

$$\textbf{Requirement R2} \text{ (controllability): } \theta < 0.5 \frac{|\tau_1|}{|k|} \frac{|y_{max}|}{|d_0|}$$

(2) Low Frequencies. For $\omega < \omega_c$, we have $|1 + gc| \approx |gc|$ (see Figure 2), and requirement 10 gives the following lower limit on the frequency-dependent controller gain

$$|c(j\omega)| \geq |c_{min}(j\omega)| = \frac{|g_d(j\omega)|}{|g(j\omega)|} \cdot \frac{|d_0|}{|y_{max}|}; \qquad \forall \omega < \omega_c \tag{14}$$

Requirement 14 may be rewritten as

$$|c(j\omega)| \geq |c_{min}(j\omega)| = \frac{|u_0(j\omega)|}{|y_{max}|}; \quad \forall \omega < \omega_c \tag{15}$$

where

$$|u_0(j\omega)| \equiv \frac{|g_d(j\omega)| \cdot |d_0|}{|g(j\omega)|} \quad (16)$$

is the magnitude of the input change needed to reject a sinusoidal disturbance of magnitude $|d_0|$ at low frequencies, $\omega < \omega_c$. This interpretation follows since, at low frequencies $y \approx 0$, and then from eq 8, the required input to reject the disturbance is $u = -(g_d/g)d$. From eq 15, we then derive the following useful rule at lower frequencies, $\omega < \omega_c$, where control is effective: *The minimum controller gain at a given frequency is approximately equal to input change required for disturbance rejection at this frequency divided by the allowed output variation*. As expected, tight control (with $|y_{max}|$ small) requires a large controller gain $|c|$, as do large disturbances (with $|u_0|$ large).

PI and PID control. For a "pure" proportional controller, we have $c(s) = K_c$, and for acceptable disturbance rejection, we must from eq 15 require

$$K_c \geq \max_{\omega < \omega_c} \frac{|u_0(j\omega)|}{|y_{max}|} \quad (17)$$

For PI and PID control, the integral action may take care of some of the disturbance rejection, and we may generally reduce K_c below the value given by eq 17. This is illustrated in Figure 3, where the bound (eq 15) is shown for PI and PID control for the case where $|u_0(j\omega)|$ is assumed to be constant. [Note: The "ideal" form is used for the PID controller, $c_{PIDS}(s) = K_c(1 + 1/\tau_I s + \tau_D s)$.] Several things are worth noting from Figure 3. First, we see that the integral action makes it easier to satisfy the bound (eq 15) at frequencies lower than $\omega_I \equiv 1/\tau_I$ (the derivative action also makes it easier to satisfy it a high frequencies, but this is generally less important because eq 15 only needs to be satisfied up to frequency w_c). Second, we observe that, for both PI and PID control, the minimum value of the controller gain $|c(j\omega)|$ as a function of frequency is equal to K_c, independent of the values of τ_I and τ_D:

$$\min_{\omega} |c_{PI}(j\omega)| = \min_{\omega} |c_{PID}(j\omega)| = K_c \quad (18)$$

(This holds for the ideal PID form, not for the cascade PID form.) A sufficient (conservative) condition for satisfying eq 15 is, therefore, to replace $|c|$ by K_c, but this just rederives eq 17, which we know is conservative for PI and PID control.

To avoid this conservativeness, assume that the integral action ensures that condition 15 is (easily) satisfied at frequencies lower than ω_I, and we consider the frequency range from ω_I to ω_c where proportional action is needed for disturbance rejection. Introducing eq 18, condition 15 then gives the following requirement for acceptable disturbance rejection

$$K_c \geq K_{c,min} = \frac{|u_0|}{|y_{max}|} \quad (19)$$

where $|u_0|$ is the required input magnitude for disturbance rejection at high frequencies (in the frequency range from ω_I and ω_c). More precisely,

$$|u_0| \equiv \max_{\omega \in [\omega_I, \omega_c]} |u_0(j\omega)| \quad (20)$$

where $\omega_I = 1/\tau_I$ is the frequency up to which integral action is effective, $\omega_c \approx 1/\tau_c$ is the closed-loop bandwidth, and $u_0(j\omega)$ is defined in eq 16.

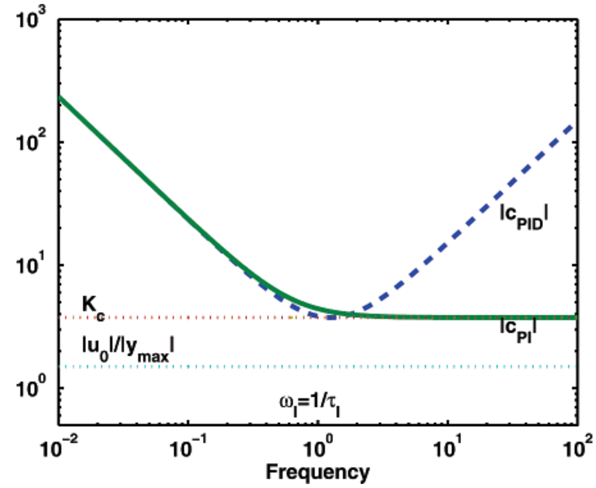

Figure 3. Controller gain $|c|$ as a function of frequency for PI and PID controller, together with the lower bound $|u_0|/|y_{max}|$ for acceptable disturbance rejection. Data PI controller: $K_c = 3.75$, $\tau_I = 1.6$. Data for PID-controller: $K_c = 3.75$, $\tau_I = 1.6$, $\tau_D = 0.2$.

The bound $K_{c,min}$ in eq 19 is tight in most cases. The main exception is when the integral time is small such that $\omega_I > \omega_c$ (i.e., $\tau_I < \tau_c$). There is then no frequency range where proportional action is required for disturbance rejection, and the mimum controller gain will be smaller than the value given in eq 19. Recall that the SIMC tuning rule in eq 4 is $\tau_I = \min(\tau_1, 4(\tau_c + \theta))$, so with the SIMC tunings, this only happens if $\tau_I = \tau_1 < \tau_c$, that is, when the closed-loop response time τ_c is longer than the open-loop time constant τ_1. A typical example may be a flow control loop.

Load Disturbance. For the very common case of an *input (load) disturbance*, we have $g_d = g$, and for acceptable disturbance rejection, we must from eq 14 require $|c(j\omega)| \geq |d_0(j\omega)|/|y_{max}|$, $\omega < \omega_c$. From eq 19, we then get for PI and PID control

$$\text{Load disturbance: } K_c \geq K_{c,min} = \frac{|d_0|}{|y_{max}|} \quad (21)$$

where $|d_0|$ is the maximum expected disturbance magnitude in the frequency range from ω_I to ω_c (where proportional control is required). This bound guarantees that $|y| < |y_{max}|$ in response to sinusoidal load disturbances of magnitude $|d_0|$.

3. PI Example

Consider a first-order with delay process

$$g(s) = g_d(s) = k\frac{e^{-\theta s}}{\tau s + 1} \quad (22)$$

with gain $k = 4$, time constant $\tau_1 = 6$, and time delay $\theta = 0.2$. As a disturbance, we consider an input (load) disturbance of magnitude $|d_0| = 1$, and we assume that the plant has been scaled such that the performance requirement is that the output deviation should stay within $\pm|y_{max}| = \pm 1$. It is also desirable that control is as smooth as possible, which means that we want K_c as small as possible. From eq 21, we must for a sinusoidal disturbance select

$$K_c \geq K_{c,min} = |d_0|/|y_{max}| = 1$$

to achieve acceptable disturbance rejection.

Tight Control ($\tau_{c,min}$). For comparison, let us first consider the response with tight control. According to the SIMC rule

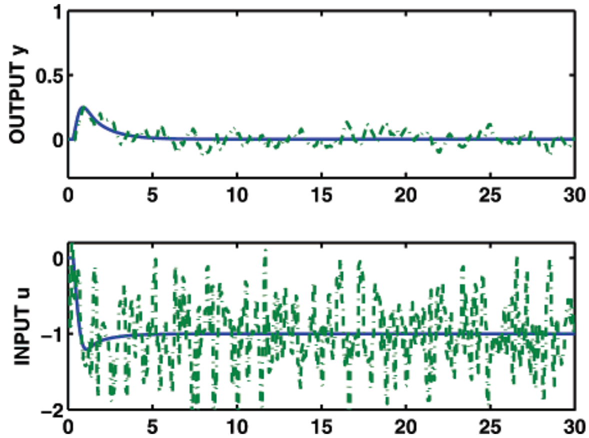

Figure 4. Response to step load disturbance for process (eq 22) with "tight" SIMC PI settings (eq 23) with $\tau_c = 0.2$: dashed line = with measurement noise n ($\pm$0.1 standard deviation); solid line = no noise.

(eq 6), we select $\tau_c = \theta = 0.2$, and from eqs 3 and 4, the PI settings are

$$K_c = 3.75, \quad \tau_I = 1.6 \tag{23}$$

We note that K_c is 3.75× the minimum required value, so we expect that the output response is much better than the requirement of staying within ±1. This is also seen from the frequency plots in Figure 2, where we find that the value of $|g_d|\cdot|d_0|/|y_{max}|$ is only 0.31 of $|1 + g_c|$, even at the "worst" frequency (with the smallest margin).

In Figure 4 we also show the simulated time response to a unit *step* input disturbance (a step disturbance is not covered by the theory in this paper, which is for sinusoidal disturbances, but it is included because of the popularity of step responses in process control). The output deviation $y(t)$ in Figure 4 is <0.25, well-below $|y_{max}| = 1$. However, because of the high controller gain, the input usage and also the output response is sensitive to measurement noise n (dashed line in Figure 4). Simulations show that it does not help to filter the measurement y_m in this case.

Smooth control ($\tau_{c,max}$). The above response is unnecessarily fast, and from the derivation in the beginning of the example, the controller gain may be reduced to $K_c = K_{c,min} = 1$. From eq 7, this corresponds to $\tau_{c,max} = 1.3$ and eq 4 gives the following "smooth" SIMC PI settings:

$$K_c = 1; \quad \tau_I = \min\{6, \underbrace{4 \cdot (1.3 + 0.2)}_{6}\} = 6 \tag{24}$$

The frequency plots in Figure 5 show that the performance requirement (eq 10) is satisfied at all frequencies, although the margin is much smaller than with the tight settings in Figure 2. Figure 5 gives that the value of $|g_d|\cdot|d_0|/|y_{max}|$ is 0.82 of $|1 + gc|$ at the "worst" frequency (with the smallest margin). The value 0.82 is less than 1 because of the approximation $|1 + gc| \approx |gc|$ used in the derivation of $K_{c,min} = 1$. Note in Figure 5 that the curve for $|gc|$ is difficult to identify because it closely follows $|1 + gc|$ at low frequencies and $|g_d|\cdot|d_0|/|\,y_{max}|$ at high frequencies.

The corresponding simulated response to a step disturbance in Figure 6 has a maximum deviation for y of 0.66. The deviation is below $|y_{max}| = 1$, partly because of the approximation $|1 + gc| \approx |gc|$ and partly because a unit step disturbance in this case is "better" than the worst-case unit sinusoidal disturbance. Also note from the simulation that the input usage

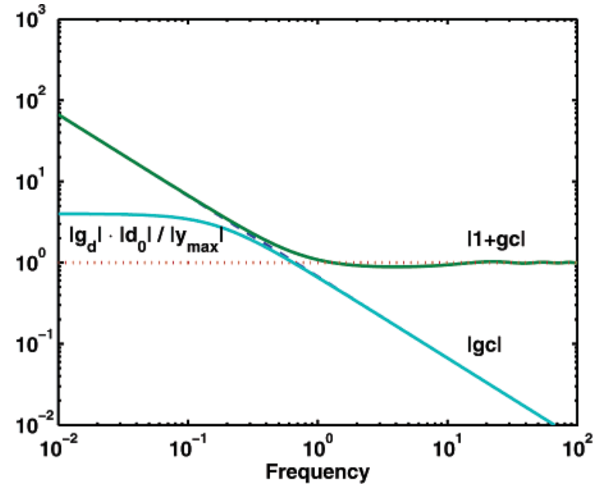

Figure 5. Frequency plots with "smooth" SIMC PI settings (eq 24). Performance requirement $|1 + gc| \geq |g_d|\cdot|d_0|/|y_{max}|$ (eq 10) is satisfied at all frequencies. Data: $g_d = g = 4\,(e^{-0.2s})/(6s + 1)$, $|y_{max}| = 1$, $|d_0| = 1$, PI control with $K_c = 1$ and $\tau_I = 6$.

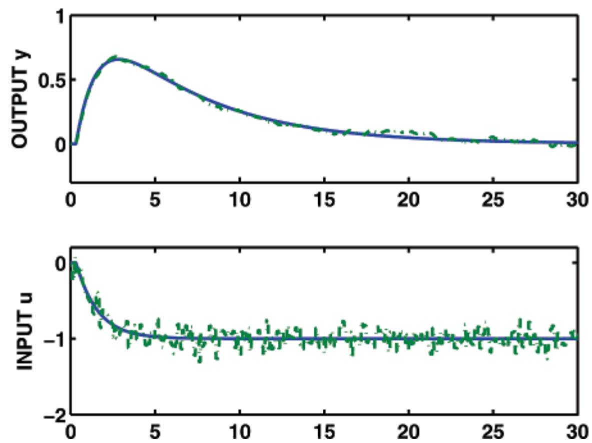

Figure 6. Response to step load disturbance for process (eq 22) with "smooth" SIMC PI settings (eq 24) with $\tau_c = 1.3$: dashed line = with measurement noise n ($\pm$0.1 standard deviation); solid line = no noise.

is smooth and that the sensitivity to noise is significantly reduced. Thus, this tuning is preferred in practice.

4. Discussion

Default Industrial PI Settings. The bound in eq 19 confirms the common default factory setting with a controller gain of approximately 1. This follows since, in industrial practice, the variables, or more precisely the instrument ranges, are often scaled such that

$$|u_0| \approx |y_{max}|$$

For example, we may have $|u_0| \approx |y_{max}| = 50\%$ (of full signal span). Here, $|u_0|$ is the expected input variation for disturbance rejection and y_{max} is the acceptable output deviation. From eq 19, it then follows that we need $K_c \geq 1$ for acceptable disturbance rejection.

There are, of course, exceptions to the rule $K_{c,\,min} = 1$. First, if the variables have been scaled differently (or not at all), then we should use $K_{c,min} = |u_0|/|y_{max}|$ from eq 19. Second, even when the scaling $|u_0| \approx |y_{max}|$ has been used, one may for some PI controlllers use a lower value than 1 for the controller gain. This holds for cases where the disturbance is handled by the integral action, as illustrated in Figure 3. This may happen either if (i) the disturbance occurs at frequencies lower than $1/\tau_I$ or if (ii) the integral time is small ($\tau_I < \tau_c$) such that the proportional action is only active outside the closed-loop bandwidth. This is likely to happen for processes with a small time constant τ_1, like a flow loop, because such loops often have a small integral time ($\tau_I \approx \tau_1$).

Averaging Level Control. A well-known case where smooth control is desired is for "averaging level control", where we use a tank in order to smoothen flow disturbances. Here, the main control objective is not to control the level tightly but rather to have smooth input usage (smooth flow variations), subject to the requirement of stabilizing the system and keeping the level h within bounds when there are flow disturbances. That is, the requirement is to keep $|y| = |\Delta h| \leq |\Delta h_{max}|$ (m) for a change in the flow of magnitude $|d_0| = |u_0| = |\Delta q_0|$ (m^3/s).

From eq 21, the minimum controller gain for averaging level control is then

$$K_c \geq K_{c,min} = \frac{|\Delta q_0|}{|\Delta h_{max}|} \tag{25}$$

which agrees with the value normally recommended (e.g., ref 8). Next, consider the integral time. For level control, the process transfer function $g(s)$ from u (flowrate q) to y (level h) is close to integrating (with τ_1 in eq 1 being very large) and can be written

$$g(s) = \frac{k'}{s}\, e^{-\theta s}$$

where $k' = k/\tau_1$ is the initial slope of the response. Combining eqs 3 and 4 gives the SIMC setting for the integral time of an integrating process

$$\tau_I = \frac{4}{K_c k'} \tag{26}$$

This agrees with the industrially recommended value in ref 9, and keeping τ_I above this value avoids the "slow" oscillations that may otherwise result from having two integrators in the control loop.[6] To derive an alternative expression for τ_I, let τ_{tank} denote the time it would take for the tank level to exceed its allowed bound (i.e., for the level to change by $|\Delta h_{max}|$) in response to a maximum flow increase of magnitude Δq_0 for the case with no control. If the tank is nominally half full, then τ_{tank} is equal to the tank residence time V/q. We then have $k' = |\Delta h_{max}|/(|\Delta q_0| t_{tank})$, and with the minumum controller gain $K_c = |\Delta q_0|/|\Delta h_{max}|$, the integral time from eq 26 becomes

$$\tau_I = 4\tau_{tank}$$

Thus, the integral time for smooth level control should be $\sim 4\times$ the residence time.

Controllability Implications. Consider an input (load) disturbance of magnitude $|d_0|$. In eq 21, we derived the minimum controller gain for acceptable disturbance rejection, $K_c \geq K_{c,min} = |d_0|/|y_{max}|$. From eq 7, the corresponding value for the SIMC tuning parameter for a first-order plus delay process is

$$\tau_{c,max} = \frac{\tau_1}{k} \cdot \frac{|y_{max}|}{|d_0|} - \theta \tag{27}$$

However, there is also a minimum value $\tau_{c,\,min} = \theta$ (SIMC rule (eq 6)) to have acceptable robustness for a time delay θ, and we must require $\tau_{c,\,max} > \tau_{c,min} = \theta$. This results in the controllability requirement

$$\theta \leq 0.5\, \frac{\tau_1}{k} \cdot \frac{|y_{max}|}{|d_0|} \tag{28}$$

which is identical to eq 13 (requirement R2). We note from eq 28 that a small delay θ is required if we have a tight performance requirement ($|y_{max}|$ small), a large disturbance ($|d_0|$ large), or a "fast-acting" disturbance ($k' = k/\tau_1$ large).

For the averaging level control problem, we have $k' = |\Delta h_{max}|/(|\Delta q_0|\tau_{tank})$, and the controllability requirement becomes $\theta \leq 0.5\tau_{tank}$. This is reasonable, as the time delay θ must be less than the time τ_{tank} it takes for the level to exceed its bound.

Sinusoidal Responses. The frequency plots (e.g., Figures 2 and 5) give the open-loop and closed-loop responses for $y(t)/|y_{max}|$ to a sinusoidal disturbance $d(t) = |d_0| \sin \omega t$ as a function of frequency ω. Specifically, the open-loop value of $|y(t)|/|y_{max}|$ (as $t \to \infty$) is equal to $|g_d||d_0|/|y_{max}|$, which is given by the "gap" between the lines for 1 (dotted) and $|g_d||d_0|/|y_{max}|$ (for the specific example, there is a "negative gap" at low frequencies, which signals that feedback control is required to achieve acceptable performance in response to disturbances). The closed-loop value is equal to $|g_d||d_0|/|y_{max}|\cdot(1/|1+L|)$, which is given by the "gap" (which should be large) between the lines for $|1 + L|$ and $|g_d||d_0|/|y_{max}|$. For the specific example, the gap is large when we use tight settings (Figure 2) but small when we use smooth settings (Figure 5).

Assumption of Sinusoidal Disturbances. The derivation of the results in this paper is based on the assumption of sinusoidal disturbances. This may seem to be a restrictive assumption but actually is not. First, the set of sinusoidal signals is much richer and more general than, for example, a step change. Second, sinusoidal disturbances are common in real systems, as a visit to any industrial plant would show. Actually, step changes, which are frequently used in academic simulation studies including this paper, almost never occur in practice.

There are no general relationships between a sinusoidal time response (the frequency domain) and a step response, but as illustrated by the example, they are quite closely related for simple processes of the kind commonly encountered in process control.

Generalization to Decentralized Control of Multivariable Systems. The results in this paper can be directly generalized to decentralized control of multivariable systems by introducing the closed-loop disturbance gain.[10,11] Consider a square multivariable plant, $y = G(s)u + G_d(s)d$, controlled by a diagonal (decentralized) controller, $C(s) = \mathrm{diag}\{c_i\}$. As before, consider a sinusoidal disturbance d of magnitude $|d_0|$ and assume that the maximum allowed variation in each output is $|y_i| \leq |y_{i,max}|$ (performance requirement). We want to derive an expression for the minimum controller gain $|c_{i,min}|$.

Let the diagonal matrix $\tilde{G} = \mathrm{diag}\{g_{ii}\}$ contain the diagonal elements of G (corresponding to the selected pairings for decentralized control). The closed-loop response becomes $y = (I + GC)^{-1}G_d(s)d$ where, for decentralized control, the sensitivity matrix $S = (I + GC)^{-1}$ may be rewritten as[10]

$$S = (I + \tilde{S}(\Lambda - I))^{-1}\, \tilde{S}\Lambda \tag{29}$$

Here $\tilde{S} = (I + \tilde{G}C)^{-1} = \mathrm{diag}\{1/(1 + g_{ii}c_i)\}$ is a diagonal matrix containing the sensitivities of the individual loops (which is not equal to the diagonal elements of S). $\Lambda = \tilde{G}G^{-1}$ is the "performance relative gain array" (PRGA), which has the same diagonal elements as the "conventional" relative gain array (RGA). At low frequencies, $\omega < \omega_c$, where feedback is effective, we have $\tilde{S} \approx 0$ and eq 29 becomes $S \approx \tilde{S}\Lambda$. The closed-loop response then becomes

$$y \approx (I + \tilde{G}C)^{-1}\tilde{G}_d(s)d; \qquad \omega < \omega_c \tag{30}$$

where

$$\tilde{G}_d \equiv \Lambda G_d = \tilde{G}G^{-1}G_d \tag{31}$$

is known as the "closed-loop disturbance gain" (CLDG) under decentralized control. Consider a single disturbance d, in which case G_d is a vector. Using eq 30, the response in output y_i is

$$y_i \approx (1 + g_{ii}c_i)^{-1}\tilde{g}_{d,i}(s)d; \qquad \omega < \omega_c \tag{32}$$

where $\tilde{g}_{di}$ is the ith element in the CLDG vector $\tilde{G}_d$. Using eq 32, the performance requirement $|y_i| \le |y_{i,max}|$ becomes

$$|1 + g_{ii}c_i(j\omega)| \ge |\tilde{g}_{d,i}(j\omega)| \cdot |d_0|/|y_{i,max}|; \qquad \forall\omega < \omega_c$$

and assuming $|1 + g_{ii}c_i| \approx |g_{ii}c_i|$ gives

$$|c_i(j\omega)| \ge |c_{i,min}(j\omega)| = \frac{|\tilde{g}_{d,i}(j\omega)|}{|g_{ii}(j\omega)|} \cdot \frac{|d_0|}{|y_{i,max}|}; \qquad \forall\omega < \omega_c \tag{33}$$

The requirement (eq 33) provides a direct generalization of eq 14. The only change is to replace g_d by the closed-loop disturbance gain $\tilde{g}_{d,i}$ defined in eq 31. Equation 33 has proved to be very useful in applications for tuning decentralized controllers.[11]

Note from eq 31 that G_d is multiplied by the PRGA to get $\tilde{G}_d$. The PRGA corrects for the interactions between the loops. It may seem from eq 31 that plants with large elements in the RGA (and PRGA) need a high controller gain to achieve acceptable performance. This is generally true for setpoint changes (where G_d may be viewed as the setpoint direction, which may take on any value) but not necessarily for disturbances, as illustrated by the following example.

Example. Consider the steady-state model of a distillation column with two outputs y (top and product purity), two inputs u (reflux and boilup), and a single disturbance d (feed composition). For "column A" with 40 stages, a relative volatility of 1.5, and 99% product purities, the steady-state gain matrices are

$$G = \begin{pmatrix} 87.8 & -86.4 \\ 108.2 & -109.6 \end{pmatrix}; \; G_d = \begin{pmatrix} 8.81 \\ 11.19 \end{pmatrix}$$

From G_d, it seems the disturbance affects both products equally, but actually, under decentralized control, there is a large difference. The RGA and PRGA matrices are

$$\text{RGA} = \begin{pmatrix} 35.1 & -34.1 \\ -34.1 & 35.1 \end{pmatrix};$$

$$\text{PRGA} = \Lambda = \tilde{G}G^{-1} = \begin{pmatrix} 35.1 & -27.6 \\ -43.2 & 35.1 \end{pmatrix}$$

and we get

$$\tilde{G}_d = \Lambda G_d = \begin{pmatrix} -0.4 \\ 11.7 \end{pmatrix}$$

Thus, in this case, the interactions actually reduce the steady-state effect of the disturbance on output 1 (top composition) from 8.8 to −0.4, even though the plant has large RGA elements. The effect on output 2 (bottom composition) is almost unchanged at 11. From eq 33, the implication for controller design is that the bottom loop needs to be ~20× faster than the top loop in order to get acceptable performance for a feed composition disturbance.

5. Conclusion

The requirement of acceptable disturbance rejection (output deviation less than $|y_{max}|$ in response to a sinusoidal disturbance of magnitude $|d_0|$) at low frequencies results in a lower limit on the controller gain,

$$|c(j\omega)| \ge |c_{min}(j\omega)| = \frac{|g_d(j\omega)|}{|g(j\omega)|} \cdot \frac{|d_0|}{|y_{max}|}; \qquad \forall\omega < \omega_c \tag{14}$$

(additional controllability requirements R1 and R2, which are independent of the controller, are derived by considering high frequencies). For a ~~load~~ disturbance and PI or PID control, eq 14 gives $K_c \ge K_{c,min} = |u_0|/|y_{max}|$; see eq 19. In words, the minimum controller gain is approximately equal to the input change required for disturbance rejection divided by the allowed output variation. Thus, if the inputs and outputs are scaled with respect to their expected variations (such that $|u_0| \approx |y_{max}|$), the minimum controller gain is approximately 1. A lower value than 1 may be used for cases where the disturbance is handled by the integral action in the controller.

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