Selection of variables for stabilizing control using pole vectors

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Abstract

For a linear multivariable unstable plant we consider the problem of finding the best pairing of a single actuator (input) and a single noisy measurement (output) such that the plant is stabilized with minimum input usage. For cases with a single unstable mode, the solution to this problem is to select the input and output corresponding to the largest element in the input and output pole vectors, respectively. In fact, this choice minimizes both the $\mathcal{H}_2$- and $\mathcal{H}_\infty$-norms of the transfer function KS from plant output to plant input. The pole vectors thus provide a powerful tool for independent selection of inputs (actuators) and outputs (sensors) for stabilizing control.

1 Introduction

Stabilization is a key reason for using feedback control. From linear system theory we know that a plant is stabilizable if all of its unstable modes are observable from its output y and state controllable from its input u. In most cases stabilization is performed at the lowest layer in the control hierarchy using single-input single-output (SISO) controllers. A critical issue is then usually to avoid saturation of the input used for stabilization, because otherwise the system effectively becomes open-loop and stability is lost. This motivates the following problem (see Figure 1):

- Which manipulated input (actuator) u_j and which controlled output (measurement) y_k should be selected for stabilizing control in order to minimize input usage?

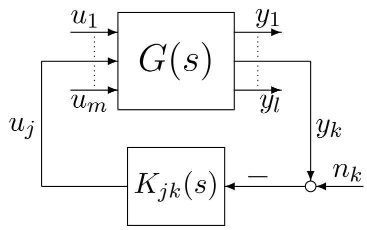

Figure 1: Plant G with stabilizing control loop $u_j \leftrightarrow y_k$

This important problem has attracted little attention in the system theory literature, although there is some related work (Wang and Davison, 1973; Benninger, 1986; Tarokh, 1985; Tarokh, 1992; Hovd and Skogestad, 1992; Lunze, 1992; Li *et al.*, 1994*a*; Li *et al.*, 1994*b*).

More generally, we want to minimize the required magnitude of the stabilized transfer function $[KS]_{jk}$ from the selected output y_k to the selected input u_j. This follows since for a plant $y = Gu + G_d d$ with feedback control $u = -K(y + n - r)$ the closed-loop input signal is

$$u = -KS(\underbrace{n + G_d d}_{\text{unavoidable}} - r)$$

where $S = (I + GK)^{-1}$ (the disturbance d and reference r are not shown in Figure 1). Thus, to minimize the required (unavoidable) input usage (u) due to measurement noise (n) and disturbances (d), we should minimize the norm of KS. However, the presence of an unstable (Right half plane - RHP) pole imposes a minimum value on the norm of KS (Havre and Skogestad, 2001), and this is the basis for the results presented in this paper.

The outline of the paper is as follows: In multivariable system the poles have directions associated with them, and in Section 2 we quantify these by defining the input and output pole vectors. In Section 3 we study the stochastic problem of minimizing the input energy required for stabilization in the presence of white measurement noise, or equivalently the problem of minimizing the $\mathcal{H}_2$-norm of $[KS]_{jk}$. We show that the minimum value is explicitly given in terms of the corresponding elements in the pole vectors. In Section 4 we derive identical results in terms of the $\mathcal{H}_\infty$-norm, and the main result in the paper is summarized in Theorem 3. It shows that the required input usage for stabilization, both in terms of the $\mathcal{H}_2$ and $\mathcal{H}_\infty$-norms, is minimized by selecting the input and output corresponding to the largest element in the input and output pole vectors, respectively. In section 5 we discuss the implications of these results for actuator/measurement selection and give a simple example. The main limitation of the theoretical results, namely that they only hold for a single unstable pole, is discussed in Section 6. We here also justify the usefulness of the pole vectors for a single stable pole. The conclusions are given in Section 7.

The presentation in this paper is brief in places, and for detailed proofs and additional examples we refer to Chapter 6 of the thesis by Havre (1998).

Notation is fairly standard. We consider a linear plant with state-space realization

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t), \quad y = Cx(t) + Du(t)$$

where t is time, $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the input, $y(t) \in \mathbb{R}^l$ is the output, and A, B, C, D are real matrices of appropriate dimensions. The corresponding transfer function matrix from inputs u to

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outputs y is

$$G(s) = C(sI-A)^{-1}B + D \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

We will use the following indexes (subscripts): i for the states x, j for the inputs u, and k for the outputs y. We let $p_i = \lambda_i(A)$ denote the i'th pole of $G(s)$, where $\lambda_i(A)$ is the i'th eigenvalue of A. When we refer to the "mode" p_i we mean the dynamic response associated with p_i. For a system $z = M(s)w$ the $\mathcal{H}_\infty$-norm of M is

$$\|M(s)\|_\infty = \sup_\omega \bar{\sigma} M(j\omega) = \sup_{w(t)\neq 0} \frac{\|z(t)\|_2}{\|w(t)\|_2}$$

where $\|z(t)\|_2$ is the usual Euclidian vector norm. The $\mathcal{H}_2$-norm of M is

$$\|M(s)\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{tr}(M(j\omega)^H M(j\omega))d\omega} = \sup_{w(t)=\text{unit impulses}} \|z(t)\|_2$$

2 Pole vectors

For a pole p_i the corresponding right eigenvector $\boldsymbol{t}_i$ ("output state direction") and left eigenvector $\boldsymbol{q}_i$ ("input state direction") are defined by

$$A\boldsymbol{t}_i = p_i \boldsymbol{t}_i; \quad \boldsymbol{q}_i^H A = p_i \boldsymbol{q}_i^H$$

We usually normalize the eigenvectors to have unit length, i.e. $\|\boldsymbol{t}_i\|_2 = 1$ and $\|\boldsymbol{q}_i\|_2 = 1$. The *input pole vector* associated with the pole p_i is defined as

$$\boldsymbol{u}_{p,i} = B^H \boldsymbol{q}_i \tag{1}$$

and the *output pole vector* is defined as

$$\boldsymbol{y}_{p,i} = C\boldsymbol{t}_i \tag{2}$$

For a given realization (A, B, C, D) and normalized eigenvectors, the pole vectors corresponding to a distinct pole p_i are unique up to the multiplication of a complex scalar c of length 1 ($|c| = 1$). For a repeated pole p_i (not distinct) there may be more than one linearly independent eigenvector, in which case the eigenvectors and pole vectors associated with p_i are matrices. (These technical issues are not important for this paper, since all theorems are for distinct poles.) To motivate the introduction of pole vectors, consider for the case when all n poles are distinct the following dyadic expansion of the transfer function,

$$G(s) = \sum_{i=1}^n \frac{1}{\boldsymbol{q}_i^H \boldsymbol{t}_i} \cdot \frac{C\boldsymbol{t}_i \boldsymbol{q}_i^H B}{s-\lambda_i} + D = \sum_{i=1}^n \frac{1}{\boldsymbol{q}_i^H \boldsymbol{t}_i} \cdot \frac{\boldsymbol{y}_{p,i}\boldsymbol{u}_{p,i}^H}{s-\lambda_i} + D \tag{3}$$

(It is common to assume that the eigenvectors have been scaled such that $\boldsymbol{q}_i^H \boldsymbol{t}_i = 1$, but we do require this here.) Note here that $\boldsymbol{t}_i \boldsymbol{q}_i^H$ is a rank-one $n \times n$ matrix and $\boldsymbol{y}_{p,i}\boldsymbol{u}_{p,i}^H$ is a rank-one $l \times m$ matrix, whereas the inner product $\boldsymbol{q}_i^H \boldsymbol{t}_i H$ is a scalar. Douglas and Athans (1996) note that $\boldsymbol{u}_{p,i} = B^H \boldsymbol{q}_i$ is "an indication of how much the i'th mode is excited by the inputs", and that $\boldsymbol{y}_{p,i} = C\boldsymbol{t}_i$ is "an indication of how much the i'th mode is observed in the outputs". Indeed, the pole vectors may be used for checking the state controllability and observability of a system, and from linear system theory we have that (Zhou *et al.*, 1996, p.52)).

- The mode p_i is controllable if and only if $\boldsymbol{u}_{p,i} = B^H \boldsymbol{q}_i \neq 0$ (for *all* left eigenvectors $\boldsymbol{q}_i$ associated with p_i).

- The mode p_i is observable if and only if $\boldsymbol{y}_{p,i} = C\boldsymbol{t}_i \neq 0$ (for *all* right eigenvectors $\boldsymbol{t}_i$ associated with p_i).

(the need to consider *all* eigenvectors only applies when p_i is a repeated pole, because otherwise the eigenvectors are unique). It follows that a system is controllable (observable) if and only of every mode p_i is controllable (observable). Furthermore, a mode p_i is controllable from an input u_j if the j'the element in $\boldsymbol{u}_{p,i}$ is nonzero, and observable from an output y_k if the k'the element in $\boldsymbol{y}_{p,i}$ is nonzero.

From the latter results it seems clear that the magnitudes of elements in the input pole vector $\boldsymbol{u}_{p,i}$ give information about from which input the i'th mode is most controllable, and that the magnitude of the elements in the output pole vector $\boldsymbol{y}_{p,i}$ give information about in which output the i'th mode is most observable. The objective of this paper is to confirm this intuition in terms of which input and output to select for stabilizing control.

REMARK 1. The eigenvectors $\boldsymbol{t}_i$ and $\boldsymbol{q}_i$, as well as the length of the pole vectors, depend on the realization (A, B, C, D). However, for distinct poles the corresponding normalized pole vectors or *pole directions*, defined by $\boldsymbol{u}_{p,i}/\|\boldsymbol{u}_{p,i}\|_2$ and $\boldsymbol{y}_{p,i}/\|\boldsymbol{y}_{p,i}\|_2$, are unique (independent of the realization) up to the multiplication of a complex scalar c of length 1 ($|c| = 1$). This implies that the relative magnitude of the elements in the pole vectors are independent of the realization, so a ranking of inputs and outputs based on selecting large elements in the pole vectors is independent of the realization.

REMARK 2. From (3), and even more so from the theorems below, we see that the inner product $\boldsymbol{q}_i^H \boldsymbol{t}_i$ of the eigenvectors influences the magnitude of the transfer function and thus the magnitude input usage (although it does not influence the relative ranking of alternative inputs and outputs). To include more directly this term, it is recommended for practical applications to compute the following *scaled pole vectors*:

$$\tilde{\boldsymbol{u}}_{p,i} = \boldsymbol{u}_{p,i}/\sqrt{\boldsymbol{q}_i^H \boldsymbol{t}_i} \quad \text{and} \quad \tilde{\boldsymbol{y}}_{p,i} = \boldsymbol{y}_{p,i}/\sqrt{\boldsymbol{q}_i^H \boldsymbol{t}_i}$$

REMARK 3. Above the pole directions were defined in terms of the state space matrices A, B and C. The pole directions may alternatively be defined in terms of the transfer matrix, by evaluating $G(s)$ at the pole $p_i = \lambda_i(A)$. The matrix is infinite in the direction of the pole, and we may write

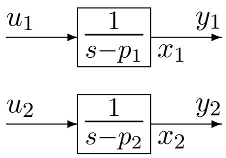
$$G(p_i)\boldsymbol{u}_{p_i} = \infty \cdot \boldsymbol{y}_{p_i} \tag{4}$$

which gives insight into the significance of the pole directions. The pole directions may then in principle be obtained from an SVD of $G(p_i) = U\Sigma V^H$. Then u_{p_i} is the first column in V (corresponding to the infinite singular value), and y_{p_i} the first column in U.

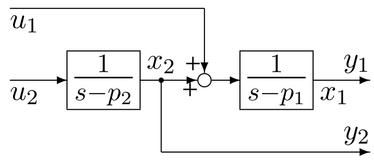
The following simple example illustrates the concept of pole vectors.

EXAMPLE 1 We consider the following parallel and series structures:

(A) Systems in parallel:



(B) Systems in series:



$$G_A(s) = \begin{bmatrix} \frac{1}{s-p_1} & 0 \\ 0 & \frac{1}{s-p_2} \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{cc|cc} p_1 & 0 & 1 & 0 \\ 0 & p_2 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

$$G_B(s) = \begin{bmatrix} \frac{1}{s-p_1} & \frac{1}{(s-p_1)(s-p_2)} \\ 0 & \frac{1}{s-p_2} \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{cc|cc} p_1 & 1 & 1 & 0 \\ 0 & p_2 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

Normalized pole vectors (the first column corresponds to $p_1 = 1$ and the second to $p_2 = 2$):

$$U_p = Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad U_p = Q = \begin{bmatrix} -0.707 & 0 \\ 0.707 & -1 \end{bmatrix}$$

$$Y_p = T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad Y_p = T = \begin{bmatrix} 1 & 0.707 \\ 0 & 0.707 \end{bmatrix}$$

For example, we see that in both cases mode p_2 is not state controllable from u_1 (since $u_{p,12} = 0$), and in both cases mode p_1 is not observable from y_2 (since $y_{p,21} = 0$). This also agrees with the block diagram representation of the systems.

3 Stabilizing control with minimum input energy ($\mathcal{H}_2$-norm)

3.1 SISO control with minimum input energy

In this section we consider the following problem[1]:

PROBLEM 1 **(SISO input energy**, *see Figure 1). Consider a plant G with a single mode $p \in \mathbb{C}_+$ $(\mathrm{Re}\, p > 0)$ and white measurement noise n_k of unit intensity in each output y_k. Find the best pairing $u_j \leftrightarrow y_k$, such that the plant is stabilized with minimum expected input energy*

$$J(j,k) = E\left\{ \lim_{T\to\infty} \frac{1}{T} \int_0^T u_j^2(t)dt \right\} \tag{5}$$

At first sight it is not clear that the output selection problem is included at all, since the outputs do not enter into the objective (5) explicitly. However, the output selection problem is included implicitly through the measurement noise and the expectation operator E.
For this problem an analytical solution can be found in terms of the pole vectors:

[1]We consider a specific pole $p = p_i$ and the subscript i is omitted in the following.

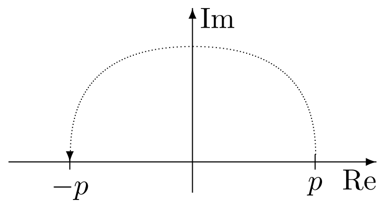


Figure 2: State feedback with minimum input usage mirrors the pole from RHP to LHP

THEOREM 1 **(Solution to Problem 1).** *The minimum input energy J, for a specific input j and output k is*

$$J(j,k)_{min} = \frac{8p^3(\boldsymbol{q}^H\boldsymbol{t})^2}{u_{p,j}^2 y_{p,k}^2} \tag{6}$$

where p is the pole, $u_{p,j}$ is the j'th element in the input pole vector, $y_{p,k}$ is the k'th element in the output pole vector, and $\boldsymbol{q}$ and $\boldsymbol{t}$ are the left and right eigenvectors corresponding to the mode p. The numerator in (6) is independent of the selection of input and output. Hence, to minimize the input energy required for stabilization with SISO control one should

- *Select the input j corresponding to the largest entry $|u_{p,j}|$ in the input pole vector $\boldsymbol{u}_p$.*
- *Select the output k corresponding to the largest entry $|y_{p,k}|$ in the output pole vector $\boldsymbol{y}_p$.*

Because of the separation theorem we may prove (6) by first finding the best input using state feedback (LQR) under the assumption of perfect measurement of all states, and then constructing the optimal state observer (LQE). For our LQR-problem, it is well-known (Kwakernaak and Sivan, 1972) that the minimum input energy for stabilization is obtained when the state feedback $u(t) = -Kx(t)$ mirrors the unstable poles across the imaginary axis, see Figure 2. Similarly, for our dual LQE-problem with zero process noise and unit intensity measurement noise, the unstable observer pole is mirrored across the imaginary axis by the use of the output to state estimate feedback.

Proof of (6).

LQR: Optimal state feedback to input u_j. In this case, the problem is to minimize the input usage due to non-zero initial states x_0, i.e. minimize the deterministic cost

$$J_{\mathrm{LQR}}(j) = \int_0^\infty u_j^2(t)dt$$

The corresponding Riccati equation with zero weight on the states and unity weight on the input becomes

$$A^TX + XA - XBe_je_j^TB^TX = 0$$

where e_j is a unit vector with 1 in position j and 0 in the other elements. With a single real pole p the solution is

$$X = \frac{2p}{u_{p,j}^2}\boldsymbol{q}\boldsymbol{q}^T \geq 0$$

and the optimal state feedback gain becomes

$$K_j = e_j^TB^TX = \frac{2p}{u_{p,j}}\boldsymbol{q}^T \tag{7}$$

LQE: Kalman filter (state observer) based on y_k. There is no process noise and the Riccati equation becomes

$$YA^T + AY - YC^T e_i e_i^T CY = 0$$

The solution is $Y = \frac{2p}{y_{p,k}^2}\boldsymbol{t}\boldsymbol{t}^T \geq 0$ so the optimal feedback gain from output y_k to the state estimate becomes

$$K_{f,k} = YC^T e_k = \frac{2p}{y_{p,k}}\boldsymbol{t} \tag{8}$$

Finally, to obtain the value of the expected input energy J, we use (Kwakernaak and Sivan, 1972, Theorem 5.4 part (d) page 394–395).

$$J(j,k) = \mathrm{tr}\left\{XK_{f,k}K_{f,k}^T\right\} = \mathrm{tr}\left\{\frac{2p}{u_{p,j}^2}\boldsymbol{q}\boldsymbol{q}^T\frac{2p}{y_{p,k}}\boldsymbol{t}\frac{2p}{y_{p,k}}\boldsymbol{t}^T\right\} = \frac{8p^3}{u_{p,j}^2 y_{p,k}^2}(\boldsymbol{q}^T\boldsymbol{t})^2$$

$\square$

3.2 MIMO control with minimum input energy

We here consider the same problem as above, but with multivariable (MIMO) control.

THEOREM 2 **(MIMO input energy)**. *Consider a plant G with a single unstable mode $p \in \mathbb{C}_+$ and with white measurement noise n_k of unit intensity in each output y_k. The minimal achievable input energy required for stabilization,*

$$J = E\left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T u^T(t)u(t)dt\right\} \tag{9}$$

is given in terms of the pole vectors:

$$J_{min} = \frac{8p^3\cdot(\boldsymbol{q}^T\boldsymbol{t})^2}{\|\boldsymbol{u}_p\|_2^2\cdot\|\boldsymbol{y}_p\|_2^2} \tag{10}$$

By comparing the minimum value of $J(j,k)$ (SISO control) with the minimum value of J (MIMO control) we can quantify the extra input energy needed to stabilize the plant using SISO control compared to full multivariable control. As expected, this is directly given by the relative magnitudes of the elements in the pole vectors:

$$\frac{\sqrt{J(j,k)_{min}}}{\sqrt{J_{min}}} = \frac{\|\boldsymbol{u}_p\|_2\cdot\|\boldsymbol{y}_p\|_2}{|u_{p,j}|\cdot|y_{p,k}|} \geq 1$$

3.3 Interpretation in terms of the $\mathcal{H}_2$-norm

The above theorems may alternatively be interpreted in terms of the $\mathcal{H}_2$-norm of the closed-loop transfer function KS from plant inputs to plant outputs. This follows since (e.g. (Zhou *et al.*, 1996)):

$$\min_{K_{jk}}\|K_{jk}S_{kk}(s)\|_2 = \sqrt{J(j,k)_{min}} \quad \text{where} \quad S_{kk}(s) = (1+G_{kj}K_{jk}(s))^{-1} \tag{11}$$

$$\min_{K}\|KS(s)\|_2 = \sqrt{J_{min}} \quad \text{where} \quad S(s) = (I+GK)^{-1} \tag{12}$$

4 Stabilizing control with minimum input usage ($\mathcal{H}_\infty$-norm)

Interestingly, almost identical results can be derived in terms of the $\mathcal{H}_\infty$-norm. Thus, the $\mathcal{H}_2$- and $\mathcal{H}_\infty$-norms give the same the best input-output pairing for stabilizing a plant G with a single unstable mode.

THEOREM 3 **(Stabilizing SISO Control with minimum $\mathcal{H}_2$ and $\mathcal{H}_\infty$input usage).** *Consider a plant G with a single unstable mode $p \in \mathbb{C}_+$. The minimum achievable $\mathcal{H}_2$- and $\mathcal{H}_\infty$-norm of the closed-loop transfer function $K_{jk}S_{kk}$ from output y_k to the input u_j is then*

$$\min_{K_{jk}(s)} \|K_{jk}S_{kk}(s)\|_\infty = \frac{1}{\sqrt{|2p|}} \min_{K_{jk}(s)} \|K_{jk}S_{kk}(s)\|_2 = |(G_{kj})_s^{-1}(p)| = \frac{|2p| \cdot |\boldsymbol{q}^H \boldsymbol{t}|}{|u_{p,j}| \cdot |y_{p,k}|} \tag{13}$$

where $u_{p,j}$ is the j'th element in the input pole vector, $y_{p,k}$ is the k'th element in the output pole vector, $\boldsymbol{q}$ and $\boldsymbol{t}$ are the left and right eigenvectors of A corresponding to the pole p, $S_{kk}(s) = (1 + G_{kj}K_{jk}(s))^{-1}$, and the notation $(G_{kj})_s^{-1}(p)$ means: Find the stable version of G_{kj} with the RHP*-pole at $s = p$ mirrored across the imaginary axis, i.e., $(G_{kj}(s))_s = \frac{s-p}{s+p}G_{kj}(s)$, take its inverse, i.e. $(G_{kj}(s))_s^{-1} = ((G_{kj}(s))_s)^{-1}$, and evaluate $(G_{kj}(s))_s^{-1}$ at $s = p$.*

REMARK 1. Note that the scalar $|2p| \cdot |\boldsymbol{q}^H \boldsymbol{t}|$ in (13) is independent of j and k.
REMARK 2. From (13) we see that the best input j and the best output k correspond to minimizing $|(G_{kj})_s^{-1}(p)|$, or equivalently maximizing $|(G_{kj})_s(p)|$. Thus, an alternative to using the pole vectors, is to select the input-output pair (j, k) corresponding to the element in $G_s(p)$ with the largest magnitude. Nevertheless, we recommend using the pole vectors, because this allows for an individual evaluation of inputs and outputs, and also requires fewer evaluations (a plant with m candidate inputs and l candidate outputs, has $m \cdot l$ elements in $G_s(p)$, but only $m + l$ pole vector elements).
REMARK 3. When minimizing the input usage, both in terms of the $\mathcal{H}_2$-norm and the $\mathcal{H}_\infty$-norm, the unstable open-loop pole p is mirrored into the left half plane.
REMARK 4. In general, the values of the $\mathcal{H}_2$- and $\mathcal{H}_\infty$-norms of KS for a given system (with a given controller) may be arbitrary far apart. It is then somewhat surprising that the minimum of $\mathcal{H}_2$-norm and $\mathcal{H}_\infty$-norms differ by a constant factor of $\sqrt{2p}$ (although the two controllers achieving these two minimum values are of course different).
REMARK 5. The $\mathcal{H}_\infty$-controller that achieves the bound in (13) is in general improper.

Proof of Theorem 3.
The identity $\min_{K_{jk}(s)} \|K_{jk}S_{kk}(s)\|_\infty = |(G_{kj})_s^{-1}(p)|$ follows from Havre and Skogestad (2001, Theorem 4 and eq.(26)) for the case with a single unstable mode. The last identity is proved as follows: Since p is the only unstable mode, it follows from (3) that a partial fraction expansion of G contains the following two terms

$$G(s) = \frac{1}{\boldsymbol{q}^H \boldsymbol{t}} \cdot \frac{\boldsymbol{y}_p \boldsymbol{u}_p^H}{s-p} + N(s)$$

where $N(s)$ is stable. Also, $\left(G_{kj}(s)\right)_s = e_k^T \frac{s-p}{s+p} G(s) e_j$ and since $y_{p,k} = e_k^T \boldsymbol{y}_p$ and $u_{p,j} = \boldsymbol{u}_p^H e_j$ we have

$$|(G_{kj})_s(p)| = \left| \frac{1}{\boldsymbol{q}^H \boldsymbol{t}} \frac{y_{p,k} u_{p,j}}{s+p} + \frac{s-p}{s+p} N_{kj}(s) \right|_{s=p} = \frac{|y_{p,k}| \cdot |u_{p,j}|}{|2p| \cdot |\boldsymbol{q}^H \boldsymbol{t}|}$$

The relationship to the $\mathcal{H}_2$-norm follows from Theorem 1 and (11). □

In the following example we design, for a simple SISO plant, $\mathcal{H}_2$- and $\mathcal{H}_\infty$-optimal controllers that achieve the lower bounds on the input usage.

EXAMPLE 2 Consider the SISO plant

$$G(s) = \frac{s-2}{(0.1s+1)(s-1)} \stackrel{s}{=} \left[\begin{array}{cc|c} -10 & 0 & \sqrt{120/11} \\ 0 & 1 & \sqrt{10/11} \\ \hline \sqrt{120/11} & -\sqrt{10/11} & 0 \end{array}\right]$$

with an unstable (RHP) pole at $p = 1$ and a RHP-zero at $z = 2$. With the above realization, the eigenvectors and pole "vectors" corresponding to the unstable pole are

$$\boldsymbol{t} = \boldsymbol{q} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{u}_p = 0.9535 \quad \text{and} \quad \boldsymbol{y}_p = -0.9535$$

The $\mathcal{H}_2$-norm of KS is minimized with the following LQG controller:

$$K_{\mathrm{LQG}}(s) = -44\frac{0.1s+1}{s^2+13s+78}$$

The controller is strictly proper with LHP-poles at $-6.5 \pm 5.98j$ and a LHP-zero at -10 which cancels the open-loop stable pole at -10 in the plant. With this controller the closed-loop poles of the minimal realization are located at $\{-1, -1\}$, and we achieve:

$$\|K_{\mathrm{LQG}}S_{\mathrm{LQG}}(s)\|_2 = \frac{\sqrt{8p} \cdot |\boldsymbol{q}^H\boldsymbol{t}|}{|\boldsymbol{u}_p| \cdot |\boldsymbol{y}_p|} = \frac{\sqrt{8 \cdot 1} \cdot 1}{0.9535 \cdot 0.9535} = 3.11$$

The $\mathcal{H}_\infty$-norm of KS is minimized with the following controller

$$K_\infty(s) = -2.2\,\frac{0.1s+1}{0.1s+3.4}$$

The controller is semi-proper, with a LHP-pole at -34 and a LHP-zero at -10 which cancels the corresponding stable pole in G. With this controller the closed-loop pole of the minimal realization of KS is located at -1, and we achieve:

$$\|K_\infty S_\infty(s)\|_\infty = 2.2 = \frac{3.11}{\sqrt{2p}}$$

which as expected is equal to

$$|G_s^{-1}(p)| = \left|\frac{(0.1s+1)(s+1)}{s-2}\right|_{s=1} = \left|\frac{1.1 \cdot 2}{-1}\right| = 2.2$$

Note that $K_\infty S_\infty(s) = -2.2\frac{s-1}{s+1}$ is semi-proper (it remains flat at magnitude 2.2 at all frequencies) so its $\mathcal{H}_2$-norm is infinite.

We have the following generalization of Theorem 3 for multivariable control.

THEOREM 4 **(Stabilizing MIMO Control with minimum $\mathcal{H}_\infty$-norm input usage).** *Consider a plant G with a single unstable pole $p \in \mathbb{C}_+$. The minimum achievable $\mathcal{H}_\infty$-norm of the closed-loop transfer function KS from output y to input u is then*

$$\min_{K(s)} \|KS(s)\|_\infty = \|\boldsymbol{u}_p^H(G_{so}(p))^{-1}\|_2 = \|(G_{si}(p))^{-1}\boldsymbol{y}_p\|_2 \tag{14}$$

where $S(s) = (I + GK(s))^{-1}$, and G_{so} and G_{si} are the stable versions of G with the RHP*-poles mirrored across the imaginary axis and factorized at the output and input, respectively (see Havre and Skogestad (2001) for details), and $\|\cdot\|_2$ denotes the usual Eucledian vector norm.*

Note that this only generalizes part of (13), as it does not relate the minimum $\mathcal{H}_\infty$-norm directly to the pole vectors only or to the minimum $\mathcal{H}_2$-norm of KS.

5 Actuator/measurement selection for stabilizing control

Theorem 3 has the following implication for actuator/measurement selection for a plant with a single unstable mode:

> The required input usage for stabilization, both in terms of the $\mathcal{H}_2$- and $\mathcal{H}_\infty$-norms, is minimized by selecting the output (measurement) y_k corresponding to the largest element in the output pole vector $\boldsymbol{y}_p$, and the input (actuator) u_j corresponding to the largest element in the input pole vector $\boldsymbol{u}_p$.

More precisely, we propose the following procedure for designing a SISO stabilizing controller, assuming that input usage is a concern:

1. Scale the plant inputs and outputs such that a unit change in each input u_j is of equal importance, and a unit change in each output y_k is of equal importance. Specifically , we have

   $$G = D_y^{-1} \hat{G} D_u$$

   where $\hat{G}$ denotes the original (unscaled) model, and the diagonal scaling matrices are

   $$D_y = \mathrm{diag}\{\hat{y}_{k,max}\}, \quad D_u = \mathrm{diag}\{\hat{u}_{j,max}\}$$

   Typically, $\hat{u}_{j,max}$ denotes the maximum allowed input deviation, for example, the distance from the nominal input value to its saturation limit. Typically, $\hat{y}_{k,max}$ denotes the magnitude of the measurement noise (n) plus the expected output deviation due to disturbances (process noise) ($G_d d$).

2. Compute the pole vectors (or pole directions).

3. Select an input u_j corresponding to a large element in the input pole vector $\boldsymbol{u}_p$.

4. Select an output y_k corresponding to a large element in the output pole vector $\boldsymbol{y}_p$.

5. Design a controller for this input/output pairing.

Obviously, the input magnitude is not the only concern when it comes to selecting an input/output-pairing for stabilizing control, and this is the reason for using the term “large” rather than “largest” in step 3 and 4.

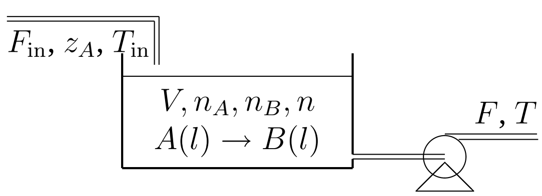


Figure 3: Chemical reactor (CSTR)

EXAMPLE 3 **Stabilization of chemical reactor.** The objective is to design a stabilizing SISO controller for the exotermic CSTR in Figure 3. The candidate inputs and outputs are

$$u = \begin{bmatrix} F \\ T_{in} \end{bmatrix}, \quad y = \begin{bmatrix} V \\ T \end{bmatrix}$$

where F is the outflow from the reactor, T_{in} is the reactor inlet temperature, V is the reactor volume (level), and T is the reactor temperature. The appropriately scaled linear model is

$$G(s) = \begin{bmatrix} \frac{-20}{s} & 0 \\ \frac{-70}{s(s-3.5)} & \frac{20}{s-3.5} \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{cc|cc} 0 & 0 & -1 & 0 \\ 70 & 3.5 & 0 & 20 \\ \hline 20 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

The pole at the origin ($p_1 = 0$) is due to the integrating level, and the unstable pole at $p_2 = 3.5$ is due to the exothermic reaction. The corresponding pole vectors are

$$\boldsymbol{u}_{p,2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \boldsymbol{y}_{p,2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{u}_{p,1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \boldsymbol{y}_{p,1} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

and the inner products of the corresponding eigenvectors are $\boldsymbol{q}_1^H \boldsymbol{t}_1 = 0.05$ and $\boldsymbol{q}_2^H \boldsymbol{t}_2 = 0.05$. From $\boldsymbol{y}_{p,2}$ we see that the unstable mode at $p_2 = 3.5$ is only observable in output 2 (this is also seen easily from $G(s)$), and from $\boldsymbol{u}_{p,2}$ we see that the unstable mode is equally controllable in both inputs. Thus, to minimize the input usage required for stabilization we should use output 2 and any of the two inputs.

Comment: We note from $\boldsymbol{u}_{p,1}$ that the pole at the origin ($p_1 = 0$) is only controllable from input 1, but observable in both outputs . This suggest that we may be able to move both the poles into the LHP if we design a controller using input 1 and output 2. This is indeed confirmed, for example by designing a LQG-controller for element $g_{21}(s)$.

REMARK. For this simple example, we reach the same conclusion easily by looking at the elements of $G(s)$, and indeed, an evaluation of the poles and zeros of the transfer function elements yields invalueable insight. However, for more complicated cases the use of pole vectors is simpler and more reliable numerically.

The theorems, and thus the above procedure for using pole vectors as a tool for selecting stabilizing pairings, applies to one unstable pole at the time. For plants with more than one unstable poles (including plants with a pair of complex unstable poles) it is not possible from the pole vectors to make any conclusive recommendations on which stabilizing loop to close first. For example, as discussed in some more detail below, if we have two RHP-poles close to each other (e.g. a pair of complex RHP-poles) with a real RHP-zero nearby, then stabilization is very difficult, but this will not show up when we compute the pole vectors.

Nevertheless, the pole vectors have proven themselves useful in several applications with more than one unstable mode, including the stabilizing control of the Teneessee-Eastman process (Havre, 1998) (Havre and Skogestad, 1998) with 6 unstable modes, and the selection of pressure sensor location for stabilization of desired two-phase flow regimes in pipelines (Havre *et al.*, 2000) (Storkaas *et al.*, 2001) which has a pair of complex RHP-poles. For such applications the pole vectors need to be interpreted with care and the results need to be checked, for example, by designing controllers. It is recommended to start by using the pole vectors of $G(s)$ to design a controller for the most unstable mode (furthest into the right half plane). Next, obtain the transfer function for the "new" partially stabilized plant, and repeat steps 2-5 until the plant is completely stabilized. In some cases, as illustrated in the reactor example, closing a single loop can stabilize more than one unstable mode.

6 Discussion

6.1 Multiple unstable poles

As just noted, the main limitation with the theoretical results presented in this paper is that they only apply for cases with a single RHP-pole. For cases with multiple RHP-poles, the pole vectors associated with a specific RHP-pole give the input usage required to move this RHP-pole assuming that the other RHP-poles are unchanged. This is of course unrealistic and may lead to misleading results, as is illustrated in the following simple SISO example.

EXAMPLE 4 **Complex RHP-poles with nearby RHP-zero**. Consider the SISO plant

$$G(s) = \frac{s-p}{s^2 - 2ps + p^2 + \varepsilon^2} \stackrel{s}{=} \left[\begin{array}{cc|c} p & -\varepsilon & 1 \\ \varepsilon & p & 0 \\ \hline 1 & 0 & 0 \end{array}\right]$$

For $p > 0$ the plant has two unstable (RHP) complex poles at $p_{1,2} = p \pm \varepsilon j$ and a RHP-zero at p. Independent of the value of $\varepsilon \neq 0$, the left and the right eigenvector matrices for this realization are

$$Q = T = \begin{bmatrix} 0.707 & 0.707 \\ -0.707j & 0.707j \end{bmatrix}$$

(which give $Q^H T = I$) and the matrices consisting of the pole "vectors" are

$$U_p = B^H Q = \begin{bmatrix} \underbrace{0.707}_{u_{p,1}} & \underbrace{0.707}_{u_{p,2}} \end{bmatrix} \quad \text{and} \quad Y_p = CT = \begin{bmatrix} \underbrace{0.707}_{y_{p,1}} & \underbrace{0.707}_{y_{p,2}} \end{bmatrix}$$

The pole vectors thus indicate that stabilization requires only moderate input usage. However, because of the nearby RHP-zero we expect in practice that stabilization of both RHP-poles becomes ingreasingly difficult for small values of ε. This is confirmed by designing LQG-controllers that minimize the input energy J for different values of ε. The closed-loop poles become $p_{1,2} = -p \pm \varepsilon j$, and the following table gives for $p = 2$ the value of J as a function of ε:

ε	1.5	1	0.5	0.1	0.05	0.01
J	2838	14848	$2.54 \cdot 10^5$	$1.64 \cdot 10^8$	$2.62 \cdot 10^9$	$1.64 \cdot 10^{12}$

As expected, the required input energy goes to infinity as ε goes to zero. The pole vectors fail to identify this.

Similar problems occur if we have two real RHP-poles with a real RHP-zero close by.

In summary, the pole vectors are reliable indicators of input usage only for plants with a single real RHP-pole (in this case they also correctly identify the problem with a close-by RHP-zero). For applications with multiple RHP-poles, including cases with complex RHP-poles, the computation of pole vectors may provide valueable insight, but they need to be interpreted with care. In particular, also the zeros and associated directions need to be considered.

6.2 Stable poles: Pole placement with minimum feedback gains

The pole vector results in this paper in terms of minimum input usage apply only to an unstable (RHP) pole, because for a stable plant the minimum input usage is zero. However, from (7) and (8) we note that an alternative interpretation is that pairing on large elements in the pole vectors minimizes the required state feedback gain K_j and observer gain $K_{f,k}$, and this result also generalizes to moving a stable (LHP) pole.

6.2.1 State feedback to input u_j.

We want to move the distinct real open-loop pole p to the closed-loop location μ by the use of state feedback from input u_j. The required state feedback gain vector is

$$K_j = \frac{p-\mu}{u_{p,j}} \boldsymbol{q}^T \tag{15}$$

where $u_{p,j}$ is the j'th element in the input pole vector corresponding to the pole p and $\boldsymbol{q}$ is the corresponding left eigenvector. Here only the scalar $u_{p,j}$ depends on the choice of input j, so it follows that any matrix norm of K_j is minimized by selecting the input j corresponding largest element magnitude in the input pole vector $\boldsymbol{u}_p$.

6.2.2 State observer based on y_k.

Similarly, we want to move the observer pole p to the desired location ν by feedback from output y_k. The required observer feedback gain vector is

$$K_{f,k} = \frac{p-\nu}{y_{p,k}} \boldsymbol{t} \tag{16}$$

where $y_{p,k}$ is the k'th element in the output pole vector corresponding to the pole p and $\boldsymbol{t}$ is the corresponding right eigenvector. Thus, the norm of $K_{f,k}$ is minimized by selecting the output k corresponding largest element magnitude in the output pole vector $\boldsymbol{y}_p$.

The above results provide some theoretical basis for using the pole vectors as a tool selecting an input/output pair for moving a stable pole, including a pole located at the origin.

7 Conclusion

The input and output pole vectors for a pole p are defined as $\boldsymbol{u}_p = B^H \boldsymbol{q}$ (where $\boldsymbol{q}$ is the left eigenvector of A corresponding to the pole p) and $\boldsymbol{y}_p = C\boldsymbol{t}$ (where $\boldsymbol{t}$ is the right eigenvector), or alternatively $G(p)\boldsymbol{u}_p = \infty \cdot \boldsymbol{y}_p$. The element magnitudes of the pole vectors are inversely related to the minimum input usage needed to stabilize one unstable mode using a SISO controller. This holds both in terms of minimum input energy with white noise and for the $\mathcal{H}_2$- and $\mathcal{H}_\infty$-norms of the closed-loop transfer function KS from plant outputs to plant inputs as given in Theorem 3:

$$\min_{K_{jk}(s)} \|K_{jk}S_{kk}(s)\|_\infty = \frac{1}{\sqrt{|2p|}} \min_{K_{jk}(s)} \|K_{jk}S_{kk}(s)\|_2 = |(G_{kj})_s^{-1}(p)| = \frac{|2p| \cdot |\boldsymbol{q}^H \boldsymbol{t}|}{|u_{p,j}| \cdot |y_{p,k}|}$$

where $u_{p,j}$ is the j'th element in the input pole vector, and $y_{p,k}$ is the k'th element in the output pole vector. The pole vectors thus provide a powerful tool for selecting actuators and sensors for stabilizing control. An alternative interpretation of pole vectors, which also hold for a stable mode, is that large elements minimize the required feedback gains for pole placement. The main limitation is that the theoretical results only hold for moving a single mode.

Theorem 3 also provides, for a SISO plant G with a single unstable pole p, a lower bound on the $\mathcal{H}_2$- and $\mathcal{H}_\infty$-norms of KS that needs to be satisfied for any stabilizing controller K:

$$\min_{K(s)} \|KS(s)\|_\infty = \frac{1}{\sqrt{|2p|}} \min_{K(s)} \|KS(s)\|_2 = |G_s^{-1}(p)| \tag{17}$$

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