FLOWSHEET CONTROLLABILITY ASSESSMENT TOOLS

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Introduction

- ”Is acceptable control possible?”
- ”What makes a plant difficult to control”

So far: Largely been based on engineering experience and intuition.

References

1. Book by S. Skogestad and I. Postlethwaite, ”Multivariable feedback control” (Wiley, 1996)
   - Chapter 5. LIMITATIONS ON PERFORMANCE IN SISO SYSTEMS
   - Chapter 6. LIMITATIONS ON PERFORMANCE IN MIMO SYSTEMS
   - Chapter 10. CONTROL STRUCTURE DESIGN
2. S. Skogestad, ”A procedure for SISO controllability analysis” Comp.Chem.Engng., **Vol. 20**, 373-386, 1996.
3. S. Skogestad, “Design modifications for improved controllability - with application to design of buffer tanks”, Paper 222e, AIChE Annual Meeting, San Francisco, Nov. 13-18, 1994. (available in postscript-file:
   `http://www.chembio.ntnu.no/users/skoge/publications/1994/sisosf.ps`

OUTLINE:

- Why feedback?
- Controllability
  1. Scaling
  2. Time delay, RHP-zero, phase lag
  3. Disturbances
  4. Input constraints
- Application: pH - neutralization process
- Application: Distillation

Notation

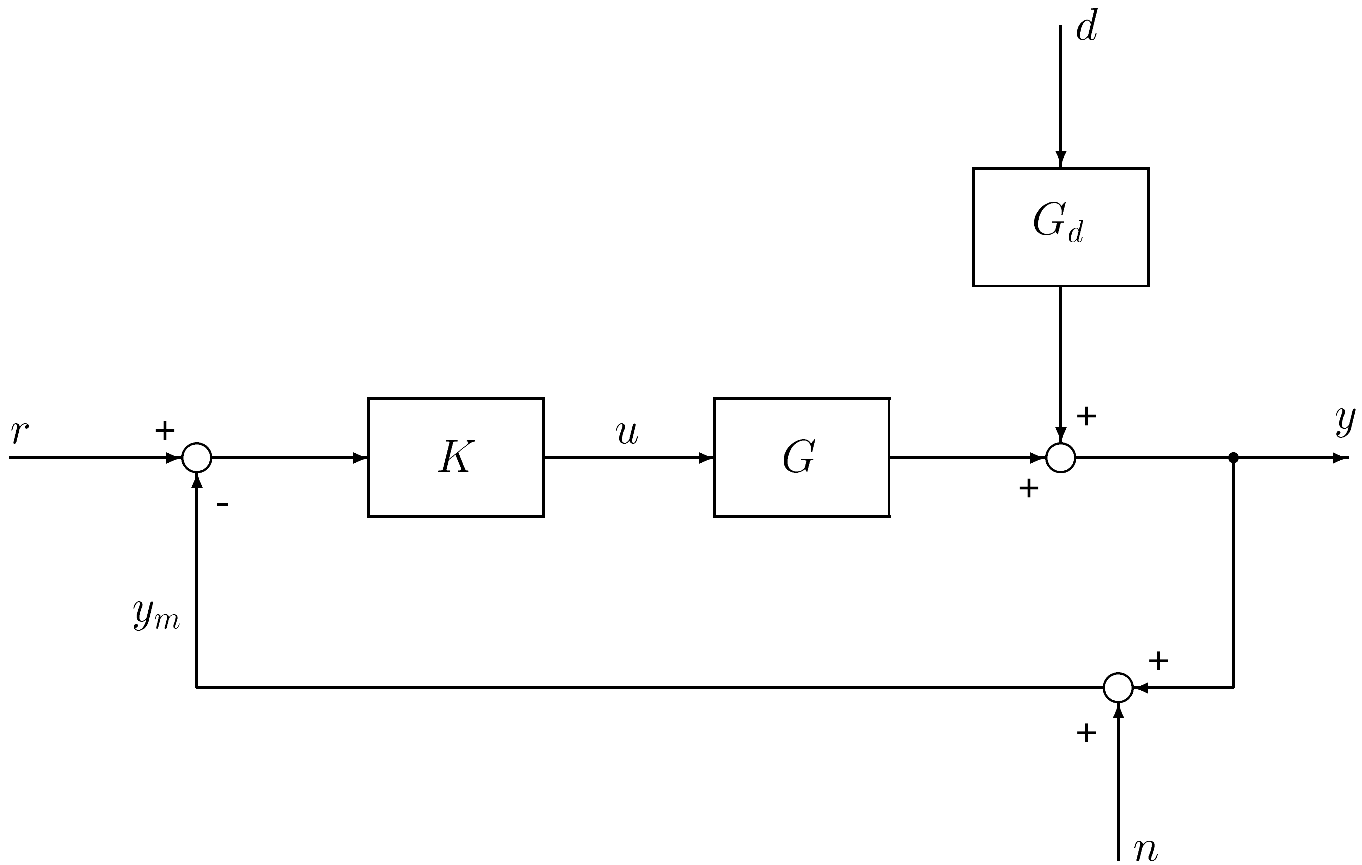


Figure 1: Block diagram of one degree-of-freedom feedback control system

- u - plant inputs (manipulated variables)
- d - disturbance variables
- y - plant outputs (controlled variables)
- r - reference values (setpoint) for plant outputs

Process models in deviation variables

$$y = G(s)u + G_d(s)d$$

- G - effect of change in plant inputs on outputs
- G_d - effect of disturbances on outputs

Feedback control

$$u = K(s)(r - y)$$

Closed-loop response

$$y = \underbrace{(I + GK)^{-1}}_{S} G_d d + \underbrace{GK(I + GK)^{-1}}_{T} r$$

- S - sensitivity function
- $T = I - S$ - complementary sensitivity function

THE SENSITIVITY FUNCTION

Control error with no control ("open-loop", $u = 0$)

$$e_o = y_o - r = G_d d - r$$

Control error with **feedback control** ("closed-loop", $u = K(r-y)$)

$$e_c = y_c - r = SG_d d - Sr = Se_o$$

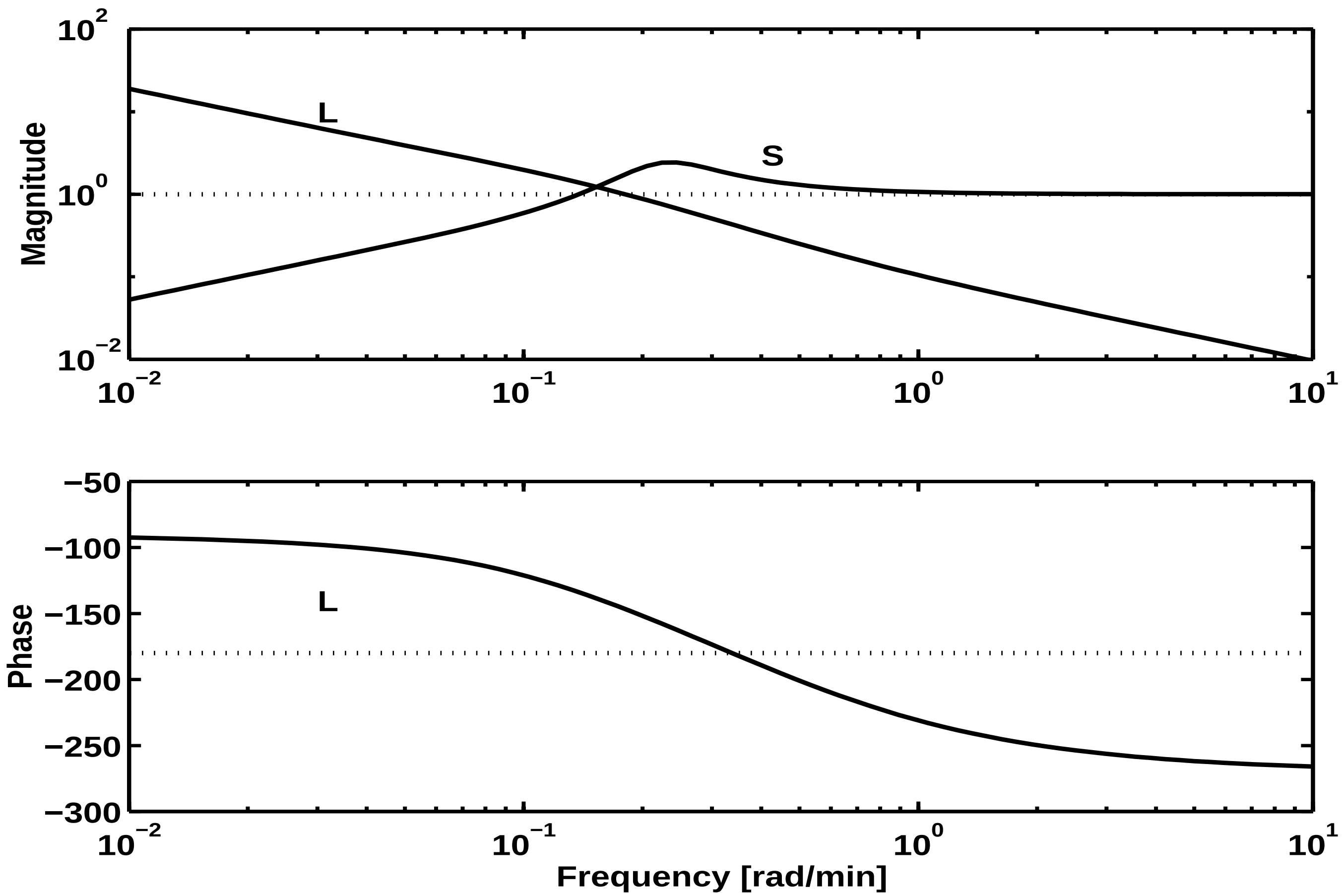
where the **sensitivity function** is

$$S = \frac{1}{1+L}; \quad L = GK$$

$\Rightarrow$ The effect of feedback is given by S.

- Sensitivity S is small (and control performance is good) at frequencies where the loop gain L is much larger than 1.
- Integral action yields $S = 0$ at steady-state
- Problem: Must have $|L| < 1$ at frequencies where phase shift through L exceeds -180^o (Bode's stability condition).

Plot of typical L and S



Bandwidth ω_B: Frequency up to which control is effective

Closed loop response time $\tau_c \quad \approx 1/w_B$

- Low frequencies ($\omega < \omega_B$): $|S| < 1$. Feedback improves performance ($|S| < 1$)
- Intermediate frequencies (around ω_B): Peak with $|S| > 1$. Feedback degrades performance
- High frequencies ($w \gtrsim 5w_B$): $S \approx 1$. Feedback has no effect
- Generally: "Resonance" peak in $|S|$ around the bandwidth. For example, $|S| = \sqrt{2} = 1.4$ at the frequency where $|L| = 1$ if the phase margin is 45^o.
- At high frequencies: Process lags make $|L| \to 0$ so $|S| \to 1$

WHY FEEDBACK CONTROL?

- Why use feedback rather than simply feedforward control?

Three fundamental reasons:

1. Stabilization. **Only** possible with feedback
2. Unmeasured disturbances
3. Model uncertainty (e.g. change in operating point)

- Feedback is most effective when used locally (because then response can be fast without inducing instability)

CONTROLLABILITY ANALYSIS

Before attempting controller design one should analyze the plant:

- Is it a difficult control problem?
- Does there exist a controller that meets the specs?
- How should the process be changed to improve control?

QUALITATIVE RULES from Seborg et al. (1989)
(chapter on “The art of process control”):

1. *Control outputs that are not self-regulating*
2. *Control outputs that have favorable dynamic and static characteristics, i.e., there should exist an input with a significant, direct and rapid effect.*
3. *Select inputs that have large effects on the outputs.*
4. *Select inputs that rapidly effect the controlled variables*

- Seems reasonable, but what is “self-regulating”, “large”, “rapid” and “direct” ?
- Objective: quantify !

DEFINITION

(INPUT-OUTPUT) CONTROLLABILITY =
The ability to achieve acceptable control performance.

More precicely: To keep the outputs (y) within specified bounds or displacements from their setpoints (r), in spite of unknown changes (e.g., disturbances (d) and plant changes) using available inputs (u) and available measurements (e.g.,y_m or d_m).

- A plant is controllable if there **exists** a controller that yields acceptable performance.
- Thus, controllability is independent of the controller, and is a property of the plant (process) only.
- It can only be affected by changing the plant itself, that is, by **design modifications.**
  - measurement selection
  - actuator placement
  - control objectives
  - design changes, e.g., add buffer tank
- Surprisingly, methods for controllability analysis have been mostly qualitative.
- Most common: The "simulation approach" which requires a specific controller design and specific values of disturbances and setpoint changes.
  BUT: Is result a fundamental property of the plant or does it depends on these specific choices?
- Here: Present quantitative controllability measures to replace this **ad hoc** procedure.

“PERFECT CONTROL” and plant inversion. (Morari, 1983)

$$y = G(s)\,u + G_d(s)\,d$$

Ideal feedforward control, $y = r$:

$$u = G^{-1}\,r - G^{-1}G_d\,d \qquad (1)$$

Feedback control:

$$u = G^{-1}T\,r - g^{-1}TG_d\,d \qquad (2)$$

For frequencies below the bandwidth $(\omega < \omega_B) : T \approx I$: Then (2) =(1).

Controllability is limited if G^{-1} cannot be realized:

- Delay (Inverse yields prediction)
- Inverse response = RHP-zero (Inverse yields instability)
- Input constraints (Inverse yields saturation)
- Uncertainty (Inverse not correct)

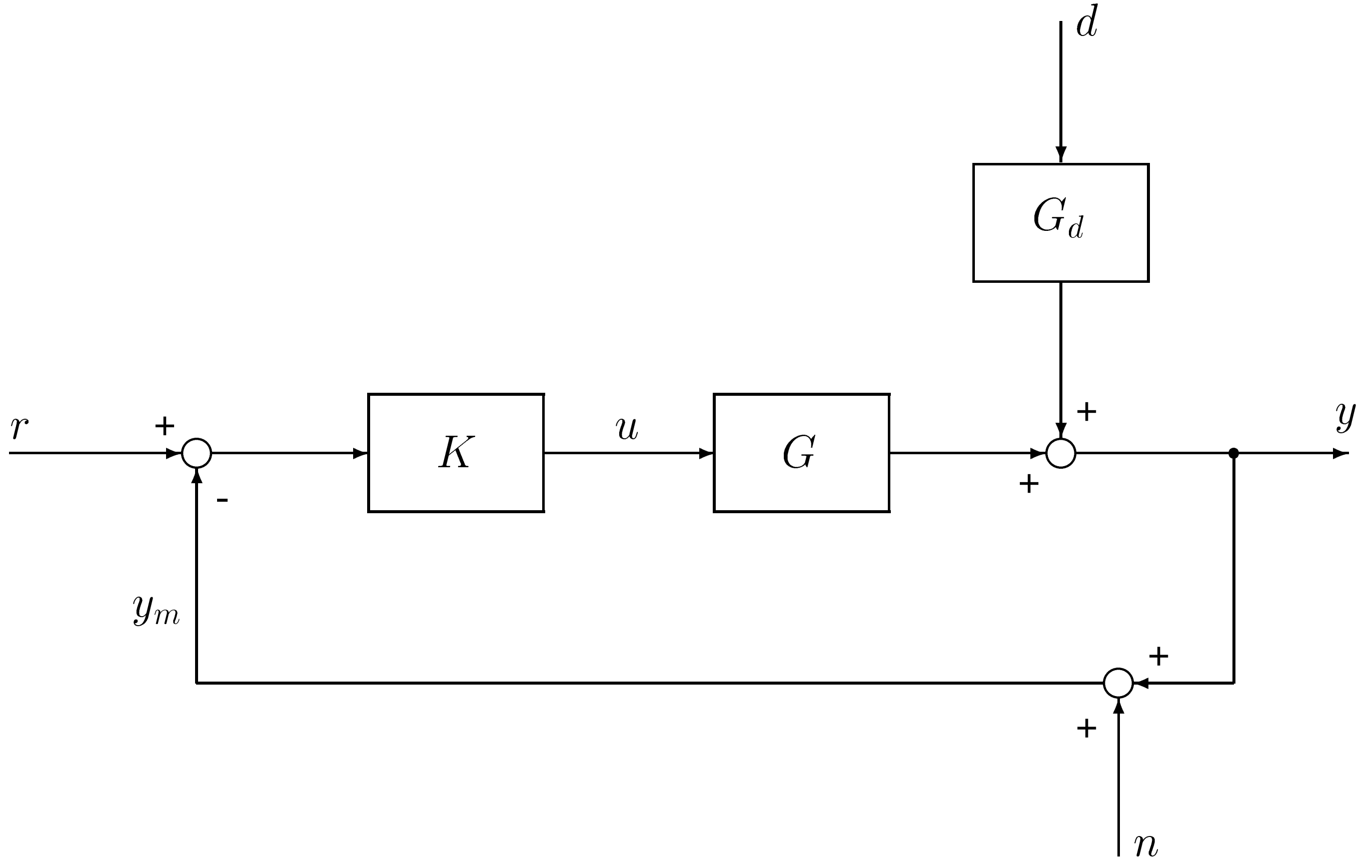
POOR CONTROLLABILITY CAN BE CAUSED BY:

1. Delay or inverse response in $G(s)$
2. or $G(s)$ is of "high order" (tanks-in-series) so that we have an "apparent delay"
3. Constraints in the plant inputs (a potential problem if the plant gain is small)
4. Large disturbance effects (which require "fast control" and/or large plant inputs to counteract)
5. Instability: Feedback with the active use of plant inputs is required. May be unable to react sufficiently fast if there is an effective delay in the loop. And: May have problems with input saturation if there is measurement noise or disturbances
6. With feedback: Delay/inverse response or infrequent or lacking measurement of y. May try
   (a) Local feedback (cascade) based on another measurement, e.g. temperature
   (b) Estimation of y from other measurements

7. Nonlinearity or large variations in the operating point which make linear control difficult. May try
   (a) Local feedback (inner cascades)
   (b) Nonlinear transformations of the inputs or outputs, e.g. $\ln y$
   (c) Gain scheduling controllers (e.g. batch process)
   (d) Nonlinear controller
8. MIMO RHP-zeros: May have internal couplings resulting in multivariable RHP-zeros $\Rightarrow$ Fundamental problem in controlling some combination of outputs.
9. MIMO plant gain: May not be able to control all outputs independently (if the "worst case" plant gain $\underline{\sigma}(G)$ is small).
10. MIMO interactions: May have large RGA-elements (caused by strong two-way interactions between the outputs) which makes multivariable control difficult.
11. Feedforward control: Should be considered if feedback control is difficult (e.g. due to delays in the feedback loop or MIMO interactions) and an "early" measurement of the disturbance is possible.

Would like to quantify this!

SCALING IS CRITICAL



Consider persistent sinusoids.
Assume that G and G_d are scaled such that all signals have magnitude less than 1 at each frequency:

$d = \pm 1$: Largest expected disturbance

$u = \pm 1$: Largest allowed input (e.g., constraint)

$e = \pm 1$: Largest allowed control error

$r = \pm R_{max}$: Largest expected reference change

Scaling procedure

Model in unscaled variables

$$y' = G'(s)u' + G'_d(s)d'$$

$$e' = y' - r'$$

Scaled variables: Normalize each variable by maximum allowed value

$$d = \frac{d'}{d_{max}}, u = \frac{u'}{u_{max}},\ e = \frac{e'}{e_{max}},\ y = \frac{y'}{e_{max}},\ r = \frac{r'}{e_{max}}$$

where

- u_{max} - largest allowed change in u (saturation constraints)
- d_{max} - largest expected disturbance
- e_{max} - largest allowed control error for output
- r_{max} - largest expected change in setpoint

Note: e, y and r are in the same units: Must be normalized with the same factor (e_{max}). Let

$$R_{max} = \frac{r_{max}}{e_{max}}$$

R_{max}: Largest setpoint change relative to largest allowed control error. Most cases: $R_{max} \geq 1$.

With these scalings we have at all frequencies

$$|d(j\omega)| \leq 1,\ |u(j\omega)| \leq 1,\ |e(j\omega)| \leq 1,\ |r(j\omega)| \leq R_{max}$$

Scaled transfer functions

$$G(s) = G'(s)\frac{u_{max}}{e_{max}}; \quad G_d(s) = G'_d(s)\frac{d_{max}}{e_{max}}$$

Scaled model

$$y = G(s)u + G_d(s)d$$

$$e = y - r$$

Example: First-order with delay process

$$g(s) = k\frac{e^{-\theta s}}{1+\tau s}; \quad G_d(s) = k_d\frac{e^{-\theta_d s}}{1+\tau_d s}$$

+ Measurement delays: θ_m, θ_{md}.

Problem: What values are desired for good controllability?

Qualitative results:

	Feedback control	Feedforward control
k	Large	Large
τ	Small	Small
θ	Small	Small
k_d	Small	Small
τ_d	Large	Large
θ_d	No effect	Large
θ_m	Small	No effect
θ_{md}	No effect	Small

Step response controllability analysis

- Disturbance response with maximum disturbance ($d = 1$):

Figure 2: Response for step disturbance

- Response to maximum plant input ($u = 1$) is similar (but with k, τ and θ)

- Steady-state: Need $k > k_d$ to reject disturbance (otherwise inputs will saturate)
- Slopes of initial responses: Would like $k/\tau > k_d/\tau_d$ (to avoid input saturation)
- **Maximum response time with feedback**: Time from disturbance is detected on output until output exceeds allowed value of $1 \approx \tau_d/k_d$
- **Minimum response time with feedback**: Sum delays around the loop = $\theta + \theta_m$
- **To counteract disturbance (**$|y| < 1$**) with feedback need:** $\theta + \theta_m < \tau_d/k_d$
- **Feedforward control**.

  Time when $y = 1$: ("minimum reaction time") $\approx \tau_d/k_d + \theta_d$ To counteract disturbance with feedforward control need:

$$\theta + \theta_{md} < \tau_d/k_d + \theta_d$$

  Delay in disturbance model helps with feedforward.

CONTROLLABILITY RESULTS IN FREQUENCY DOMAIN

TIME DOMAIN τ [min] $\leftrightarrow$ FREQUENCY DOMAIN ω [rad/min]

$$\tau \approx \frac{1}{\omega}$$

Frequency domain more general than step response!

1. Disturbances (speed of response)
2. Time delay, RHP zero, Phase lag
3. Input constraints
4. Instability
5. Summary

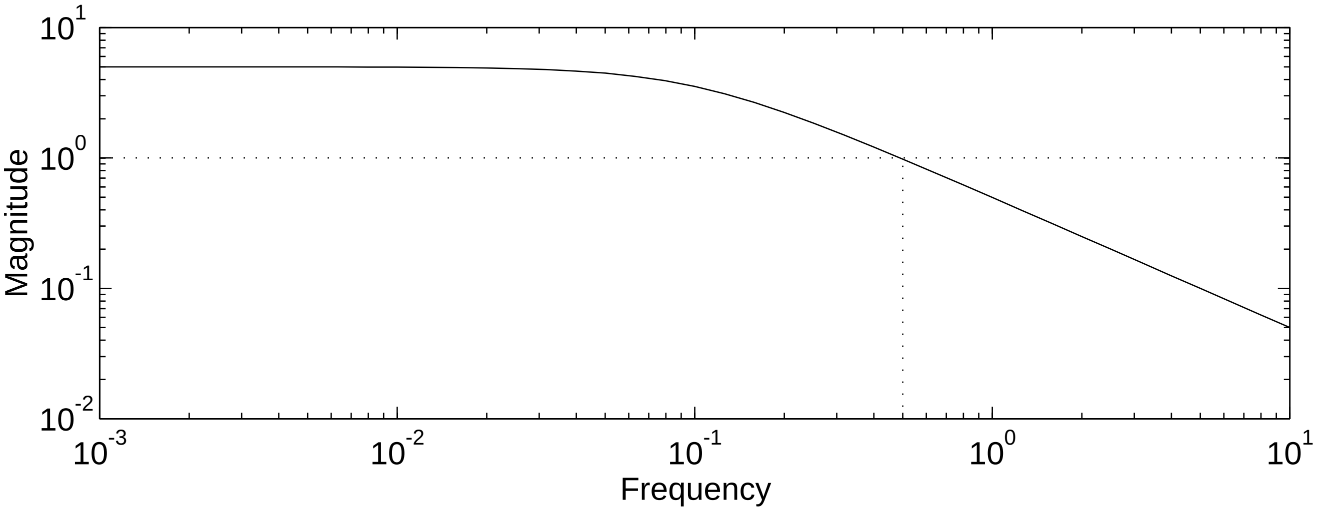
1. DISTURBANCES (speed of response)

Without control: $y = G_d d$

Worst-case disturbance: $|d| = 1$. Want $|y| < 1$

$\Rightarrow$ Need control at frequencies $\omega < \omega_d$ where $|G_d| > 1$.

$\Rightarrow$ Bandwidth requirement: $\omega_B > \omega_d$



More specifically: With feedback control $y = SG_d\, d$ we must require $|SG_d(j\omega)| < 1$, or

$$|1 + L| > |G_d|$$

Thus, at frequencies where feedback is needed for disturbance rejection ($|G_d| > 1$), we want the loop gain $|L|$ to be larger than the disturbance transfer function, $|G_d|$ (appropriately scaled).

Example.

$$G_d(s) = \frac{k_d e^{-\theta_d s}}{1+\tau_d s}; \quad k_d = 5, \tau_d = 10 \text{ [min]}$$

Get $\omega_d \approx k_d/\tau_d = 0.5$ rad/min. Bandwidth requirement

$$\omega_B > \omega_d = k_d/\tau_d$$

or equivalently in terms of the closed-loop response time

$$\tau_c < \tau_d/k_d$$

$\Rightarrow$ Min. response time 2 min.

REMARKS

1. Note: $\omega_B \approx w_{cg} > w_{180}$.
2. "Large disturbances (k_d large) with fast effect (τ_d small) requires fast control".
3. Recall the following rule from Seborg *et al*:

   - "Control outputs that are not self-regulating"

   This rule can be quantified as follows:

   - Control outputs y for which $|G_d(j\omega)| > 1$ at some frequency.

4. NOTE: Delay in disturbance model has no effect on required bandwidth.
5. BUT with feedforward control (measure disturbance): Delay makes control easier.
6. Scaling critical for evaluating the effect of disturbances.

2. TIME DELAY etc.

- For stability: Need $|L| < 1$ for $w > w_{180}$ where w_{180} is where phase lag in L is -180^o.
- (Unavoidable) phase lag is caused by time delay, RHP-zero, lags etc. Collect these effects in "effective delay". Example:

$$G(s) = \frac{ke^{-\theta s}(-\theta_z s + 1)}{(\tau_1 s + 1)(\tau_2 s + 1)(\tau_3 s + 1)\cdots}$$

  Use ("half rule")

$$\mathrm{PI-control:} \quad \theta_{eff} = \theta + \theta_z + \frac{\tau_2}{2} + \sum_{i\geq 3} \tau_i$$

$$\mathrm{PID-control:} \quad \theta_{eff} = \theta + \theta_z + \frac{\tau_3}{2} + \sum_{i\geq 4} \tau_i$$

- For acceptable control: Need $|L| < 1$ at frequencies larger than $1/\theta_{eff}$ (approximately), i.e. there is an upper bound on the bandwidth

$$\omega_B < \frac{1}{\theta_{eff}}$$

  or equivalently a lower bound on the closed-loop response time

$$\tau_c > \theta_{eff}$$

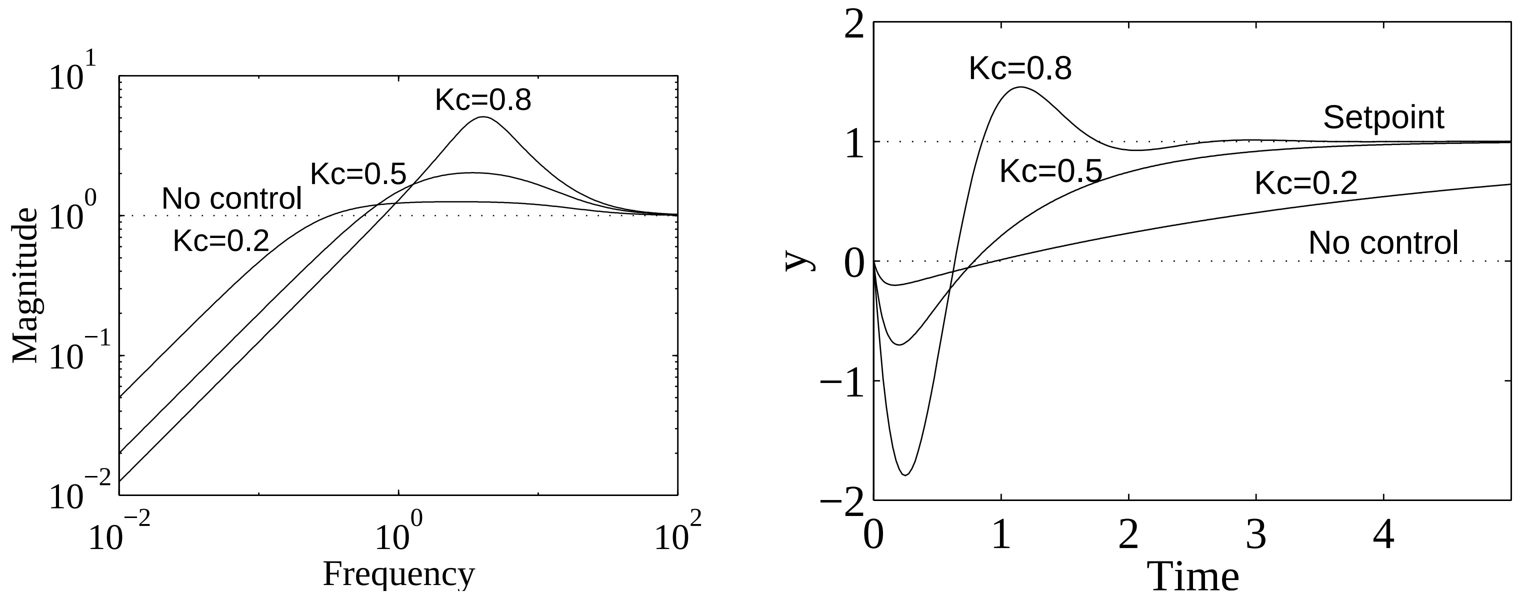
- BUT: For disturbance rejection there is a lower bound on the bandwith $\omega_B > \omega_d$
- To satisfy both we MUST require

$$\boxed{\omega_d < \frac{1}{\theta_{eff}}} \tag{3}$$

IF THIS IS NOT OK, THEN NO FEEDBACK CONTROLLER WILL GIVE ACCEPTABLE PERFORMANCE.

- NOTE: scaling is critical for using this experession!

This example shows that we need $w_B < 1/\theta_{eff}$ (in this example $\theta_{eff} = 1$).



(a) Sensitivity function

(b) Response to step in reference

Figure 3: Control of plant with RHP-zero at $z = 1$ using negative feedback

$$G(s) = \frac{-s+1}{s+1}$$

$$K(s) = K_c \frac{s+1}{s} \frac{1}{0.05s+1}$$

$K_c = 0.5$ corresponds to “ideal” response in terms of minimum ISE

3. INPUT CONSTRAINTS

Process model

$$y = Gu + G_d d$$

1. Worst-case disturbance: $|d| = 1$. To achieve *perfect control* ($e = 0$) with $|u| < 1$ we must require

$$\boxed{|G| > |G_d|} \quad \text{at frequencies where } \ |G_d| > 1 \tag{4}$$

2. Worst-case reference: $|r| = R_{max}$. To achieve perfect control ($y = r$) with $|u| < 1$ we must require

$$\boxed{|G| > |R_{max}|} \quad \forall \omega \le \omega_r \tag{5}$$

Remarks.

1. Recall the following rule from the introduction:

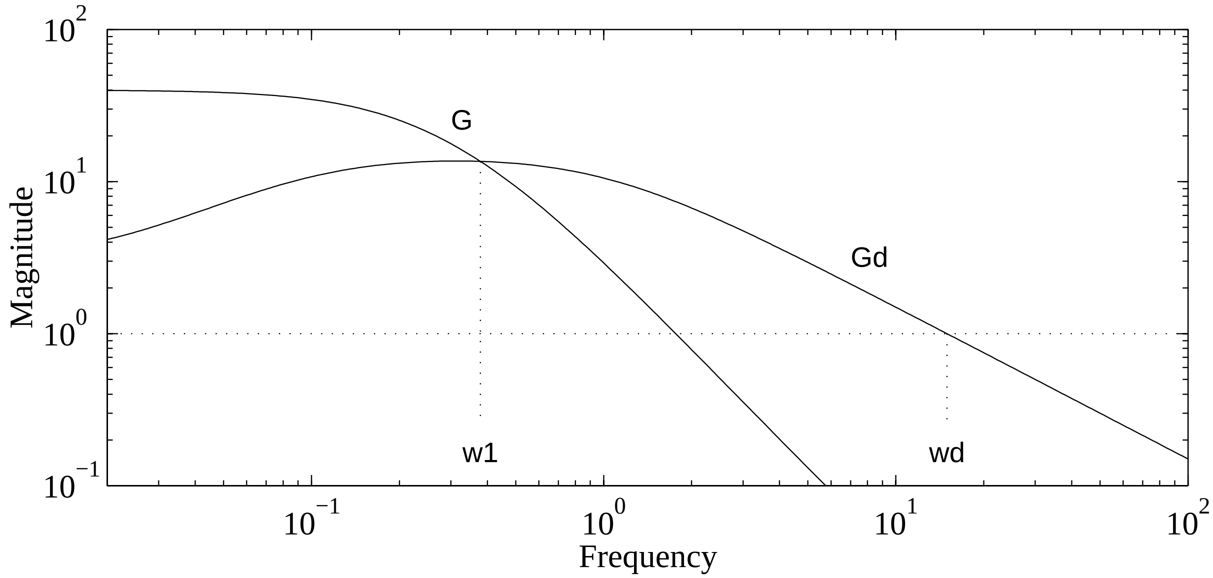
   - "Select inputs that have large effects on the outputs."

   This rule may be quantified as follows:

   - In terms of scaled variables: Need $|G| > |G_d|$ at frequencies where $|G_d| > 1$, and $|G| > R_{max}$ at frequencies where command following is desired.

2. Bounds (4) and (5) apply also to feedforward control.

3. For "acceptable" control ($|e| < 1$ we may relax the requirements to $|G| > |G_d| - 1$ and $|G| > |R_{max}| - 1$ but this has little practical significance.



Input saturation is expected for disturbances at intermediate frequencies from ω_1 to ω_d
A buffer tank may be added to reduce the effect of a disturbance.
It reduces the distrurbance effect at frequencies above $1/t_{buffer}$.

1. Reduces w_d and thus the requirement for speed of response
2. Lowers $|G_d|$ and thus the requirement for input usage.

4. INSTABILITY

$$g(s) = \frac{1}{s-p}$$

One “limitation” : **Feedback control** is required.

Use P-controller $c(s) = K_c$. Get

$$L(s) = \frac{K_c}{s-p}$$

$|L|$ crosses 1 at $\omega_B = K_c$. Furthermore

$$S(s) = \frac{1}{1+L(s)} = \frac{s-p}{s-p+K_c}$$

$\Rightarrow$ need $K_c > p$ to stabilize plant.
$K_c = 2p$ gives minimum input usage for stabilization

Conclusion: Bandwidth needed for unstable plant

$$\omega_B > 2p \quad \text{or} \quad \tau_c < 0.5/p$$

- “Must respond quicker than time constant of instability ($1/p$)”.

SUMMARY OF SISO CONTROLLABILITY RULES

Now we can quantify!

1. AVOID INPUT SATURATION (constraints). Must require (scaled model!)

$$|G| > |G_d|$$

   at frequencies where $|G_d| > 1$.

2. REJECT DISTURBANCES. Must require fast control:

$$\tau_c < \omega_d^{-1}$$

   where ω_d is the frequency where $|G_d| = 1$ (scaled model!)

3. EFFECTIVE DELAY. For stability must require slow control:

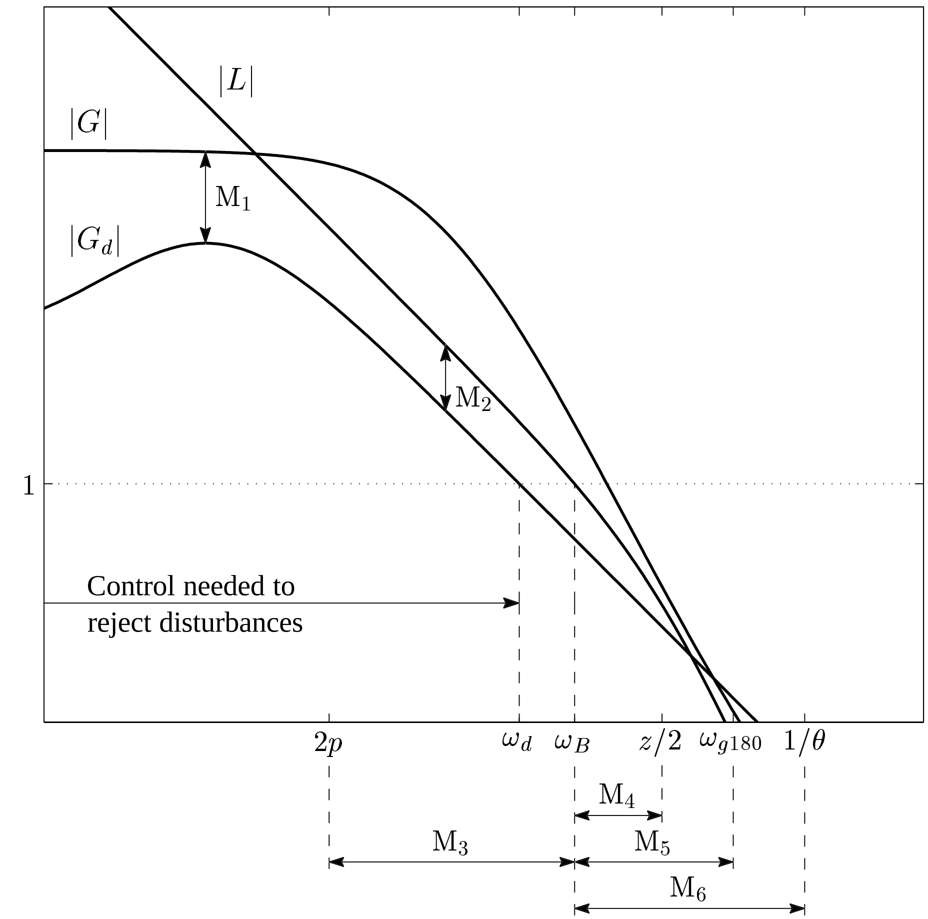
$$\tau_c > \theta_{eff}$$

4. INSTABILITY. Must require fast control

$$\tau_c > p^{-1}$$

THE PLANT IS NOT CONTROLLABLE IF THESE REQUIREMENTS ARE IN CONFLICT

SUMMARY OF CONTROLLABILITY RESULTS



Margins for stability and performance:

M_1 : Margin to stay within constraints, $|u| < 1$.
M_2 : Margin for performance, $|e| < 1$.
M_3 : Margin because of RHP-pole, p.
M_4 : Margin because of RHP-zero, z.
M_5 : Margin because of phase lag, $\angle G(j\omega_{g180}) = -180°$.
M_6 : Margin because of delay, θ.

Margins M_4-M_6 can be combined into $\omega_B < 1/\theta_{eff}$

EXERCISES

Problem 1

$$G(s) = \frac{2}{s+1} \qquad G_d(s) = \frac{3}{5s+1}$$

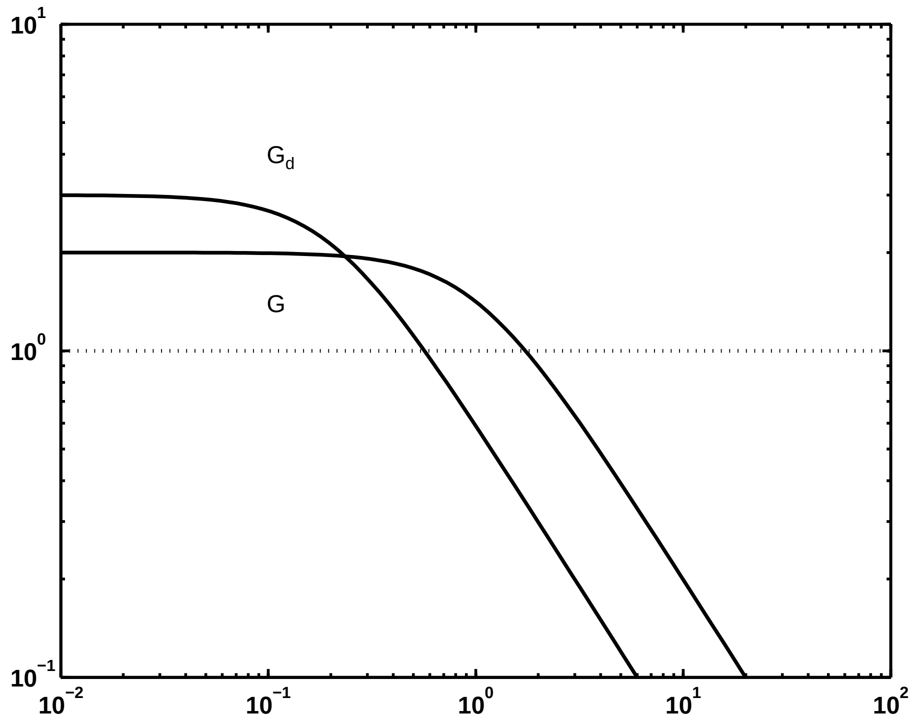
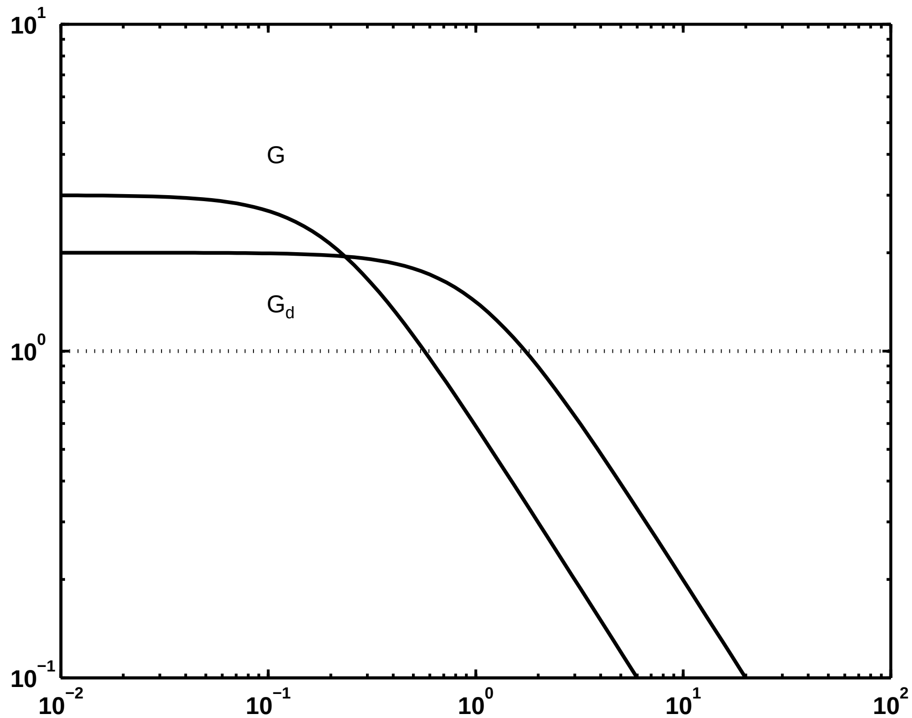


Figure 4: Magnitude of G and G_d.

Problem 2

$$G(s) = \frac{3}{5s+1} \quad G_d(s) = \frac{2}{s+1}$$



Problem 3

Given

$$G(s) = \frac{3}{5s+1} \quad G_d(s) = 7.5\frac{s-0.8}{(s+0.2)(s+20)}$$

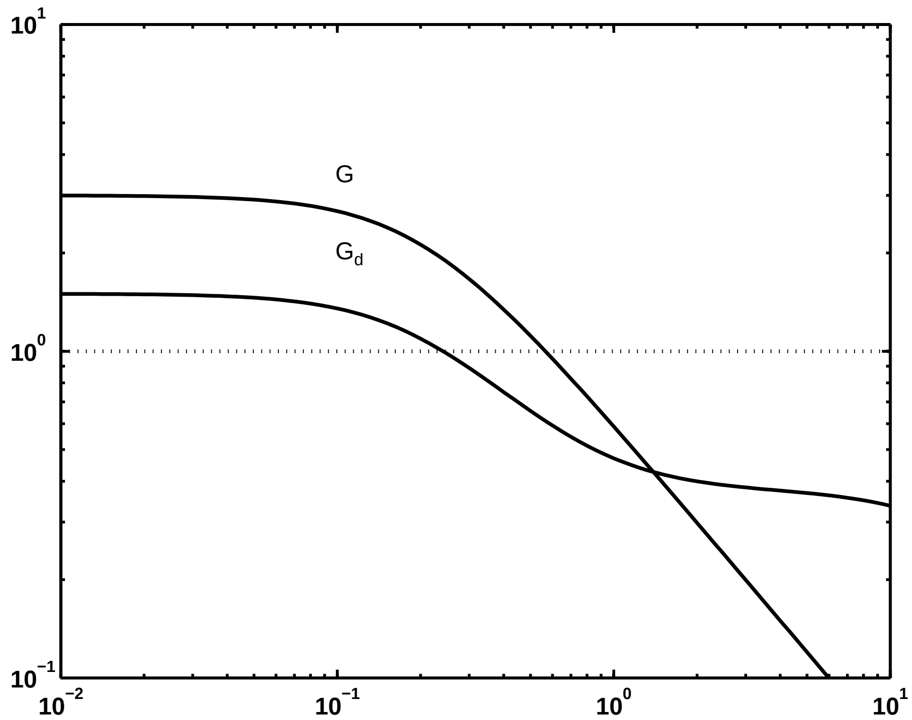
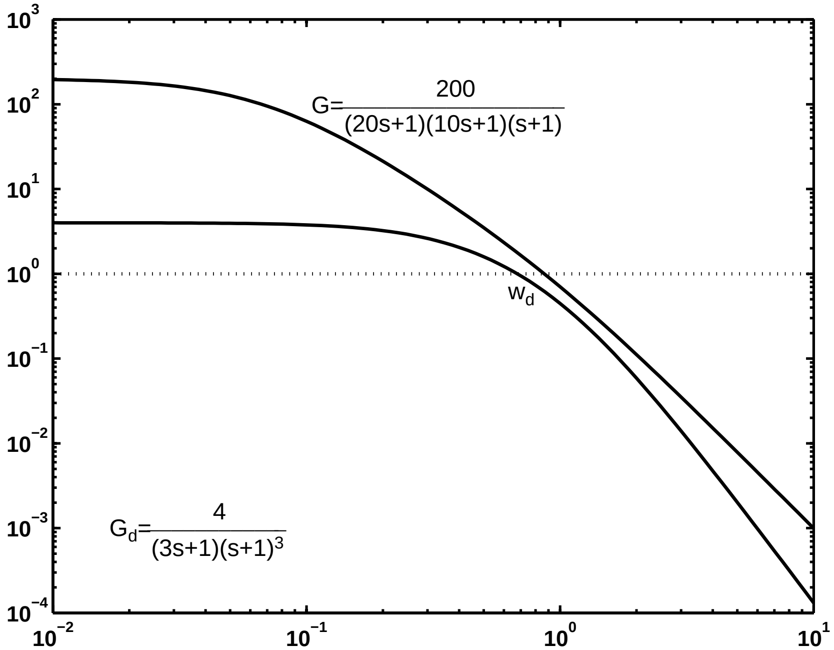


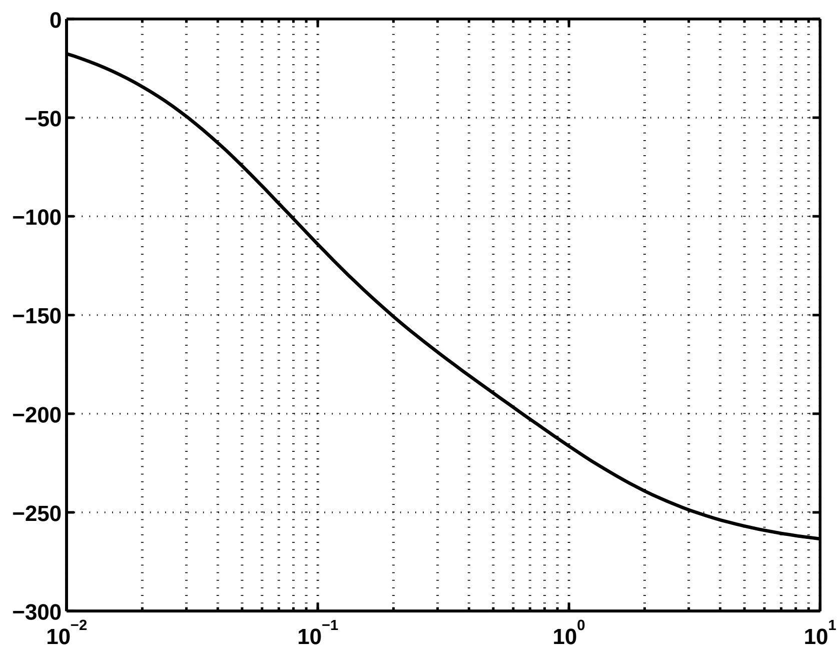
Figure 5: Magnitude of G and G_d.

Problem 4

$$G(s) = \frac{200}{(20s+1)(10s+1)(s+1)} \quad G_d(s) = \frac{4}{(3s+1)((s+1)^3}$$



(a) Magnitude of G and Gd



(b) Phase plot of G

Problem 5

$$G(s) = \frac{2.5e^{-0.1s}(1-5s)}{(3s+1)((s+1)^3} \quad G_d(s) = \frac{2}{s+1}$$

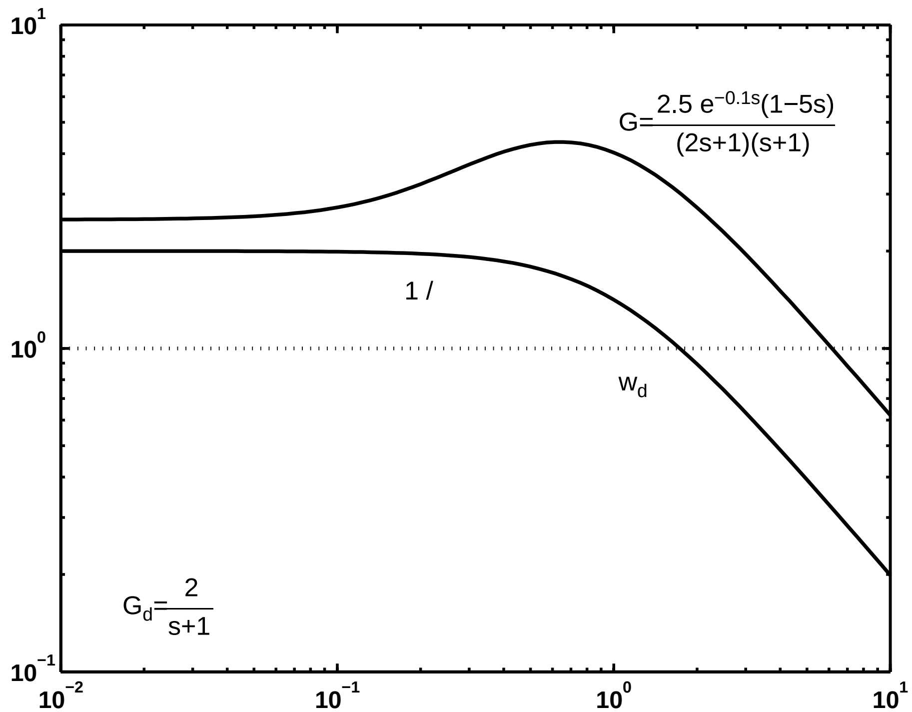


Figure 6: Magnitude of G and G_d.

APPLICATION. Neutralization process.

Acid
pH = -1
$c_H = 10$ mol /l

Base
pH = 15
$c_{OH} = 10$ mol/l

V = 10000 l; Salt water (10 l/s); pH=7 ± 1; $c_H = 0.0000001$ mol/l

$y =$ concentration of product (meas. delay θ=10 s)

$u = \text{Flow}_{base}$

$d = \text{Flow}_{acid}$

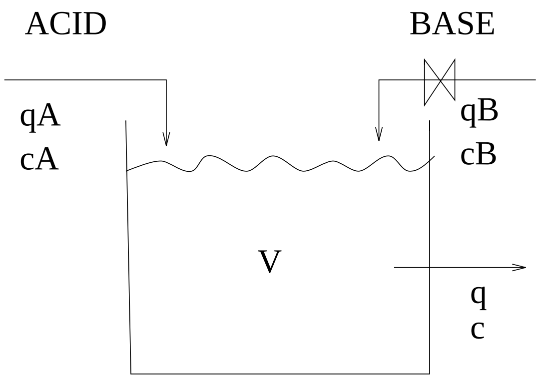
Introduce excess of acid $c = c_H - c_{OH}$ [mol/l].

In terms of c the dynamic model is a simple mixing process !!

$$\frac{d}{dt}(Vc) = q_A c_A + q_B c_B - qc$$

- Increase acid flow by 50% (max disturbance): Takes 0.2 millisecond for pH to drop from 7 to 6.

Model for pH Example



$$H_2O = H^+ + OH^-; \quad r \; [\mathrm{mol/s}, m^3]$$

Material balances:

c_H [mol/l], c_{OH} [mol/l] : conc. of H^+ and OH^--ions.

$$\frac{d}{dt}(Vc_H) = q_A c_{H,A} + q_B c_{H,B} - qc_H + rV$$

$$\frac{d}{dt}(Vc_{OH}) = q_A c_{OH,A} + q_B c_{OH,B} - qc_{OH} + rV$$

Introduce excess acid, $c = c_H - c_{OH}$ and add equations:

$$\frac{d}{dt}(Vc) = q_A c_A + q_B c_B - qc$$

(Material balance for mixing tank without reaction !!)

Linearization and Laplace transform

$$c(s) = \frac{1}{1+\tau s}\left(\frac{c_A^* - c^*}{q^*} q_A(s) + \frac{c_B^* - c^*}{q^*} q_B(s)\right); \quad \tau = V/q^*$$

Excess acid c [mol/l]

- $c = 0$ [mol/l] ⇔ pH=7
- $c = 10^{-6}$ [mol/l] ⇔ pH=6
- $c = -10^{-6}$ [mol/l] ⇔ pH=8
- Want $c = 0 \pm c_{max}$ where $c_{max} = 10^{-6}$

Scaled variables

$$y = \frac{c}{10^{-6}}; \quad u = \frac{q_B}{5[l/s]}; \quad d = \frac{q_A}{2.5[l/s]}$$

Scaled linear model

$$y = \frac{k_d}{\tau s + 1}(-2u + d); \quad k_d = 2.5 \cdot 10^6 \qquad \tau = V/q = 1000s$$

EXTREMELY SENSITIVE TO DISTURBANCES.

Controllability analysis

$$G = -2\frac{k_d}{\tau s + 1}, \quad G_d = \frac{k_d}{\tau s + 1}$$

- Input constraints: No problem since $|g| = 2|G_d|$ at all frequencies.
- Main control problem: High disturbance sensitivity. The frequency up to which feedback is needed

  $$\omega_d \approx k_d/\tau = 5000 \text{ rad/s}$$

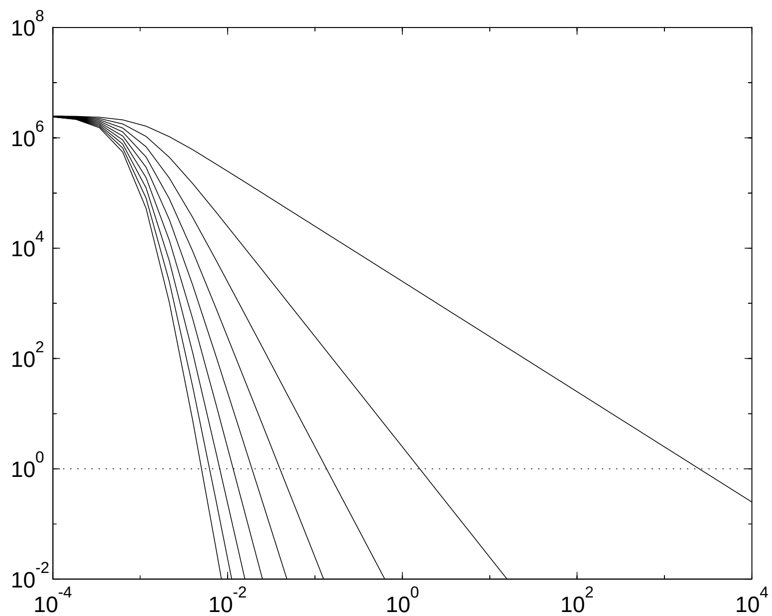
  Requires a response time $\tau_r < 1/5000 = 0.2$ millisecond.

Conclusion: Process is impossible to control irrespective of controller design.

IMPROVE CONTROLLABILITY
BY REDESIGN OF PROCESS

- Use several similar tanks in series with gradual adjustment
- Similar to golf

With n tanks: $G_d(s) = k_d/(1+\tau s)^n$.
τ: residence time in each tank.



To reject disturbance must require

$$|G_d(j\frac{1}{\theta})| < 1$$

where θ is the measurement delay. Gives

$$\tau > \theta\sqrt{(k_d)^{2/n} - 1}$$

Total volume : $V_{tot} = n\tau q$ where $q = 0.01$ m^3/s.

With $\theta = 10$ s the following designs have the same controllability:

No. of tanks n	Total volume V_{tot} $[m^3]$	Volume each tank $[m^3]$
1	250000	250000
2	316	158
3	40.7	13.6
4	15.9	3.98
5	9.51	1.90
6	6.96	1.16
7	5.70	0.81

Minimum total volume: 3.66 m^3 (18 tanks of 203 l each).

Economic optimum: 3 or 4 tanks.

Agrees with engineering rules.

Conclusion pH-example

- Used frequency domain controllability procedure
- Heuristic design rules follow directly
- Key point: Consider disturbances and scale variables
- Example illustrates design of buffer tank for composition/temperature changes
- Can use same ideas to design buffer tank for flowrate changes (there we must also consider the level controller)

MIMO CONTROLLABILITY ANALYSIS

- Most of the SISO rules generalize.
- Main difference: Directionality.

  Important tool to understand gain directionality: Singular Value Decomposition (SVD)

MIMO CONTROLLABILITY ANALYSIS

1. Scale all variables
2. SVD of G (and possibly also G_d)
3. Check if all outputs can be controlled independtly.
   (a) At least as many inputs as outputs
   (b) "Worst-case" gain sufficiently large.

   $$\underline{\sigma}(G) > 1, \omega < \omega_B$$

   Smallest singular value larger than 1 up to the desired bandwidth (otherwise we cannot make independent ± 1 changes in all outputs)
4. Check for multivariable RHP-zeros (which generally are **not** related to the lement zeros. Compute their associated output *directions* to find which outputs may be difficult to control.

5. Unstable plant. Compute the associated directions for the RHP-poles. Can also be used to assist in selecting a stablizing control structure (see Tennessee Eastman example).

6. Compute relative gain array

$$\mathrm{RGA} = G \times {G^{\dagger}}^{T}$$

   as a function of frequency (bandwidth frequencies most important!).

   Large RGA-elements means that the plant is fundamentally difficult to control (use pseudo-inverse $G^{\dagger}$ so also applies to non-square plant).

7. **Disturbances** Consider elements in

$$G^{\dagger} G_d$$

   Should all be less than 1 to avoid input saturation.

Partial control

Close loop involving u_2 and y_2 using controller K_2:

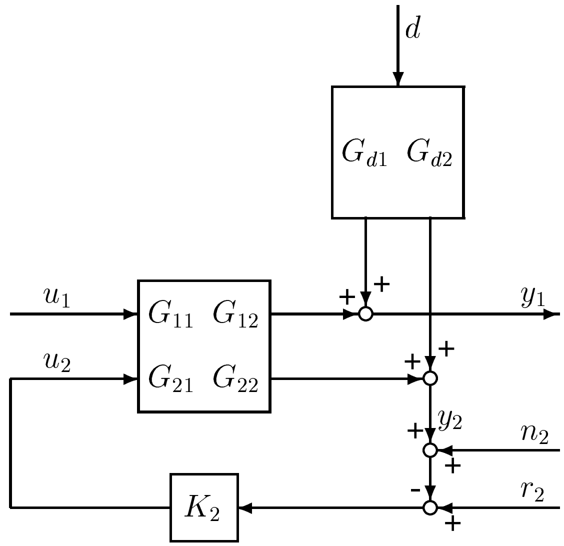


Figure 7: Block diagram of a partial control system

IMPORTANT

- Closing a loop does not imply a loss of degrees of freedom (DOFs) (since the setpoint r_2 replaces u_2 as a DOF), **BUT** we usually “use up” some of the dynamic range.

Set $y_2 = r_2 - n_2$

$$y_1 = \underbrace{(G_{11} - G_{12}G_{22}^{-1}G_{21})}_{\stackrel{\text{def}}{=} P_u} u_1 + \underbrace{(G_{d1} - G_{12}G_{22}^{-1}G_{d2})}_{\stackrel{\text{def}}{=} P_d} d + \underbrace{G_{12}G_{22}^{-1}}_{\stackrel{\text{def}}{=} P_e}(r_2 - n_2)$$

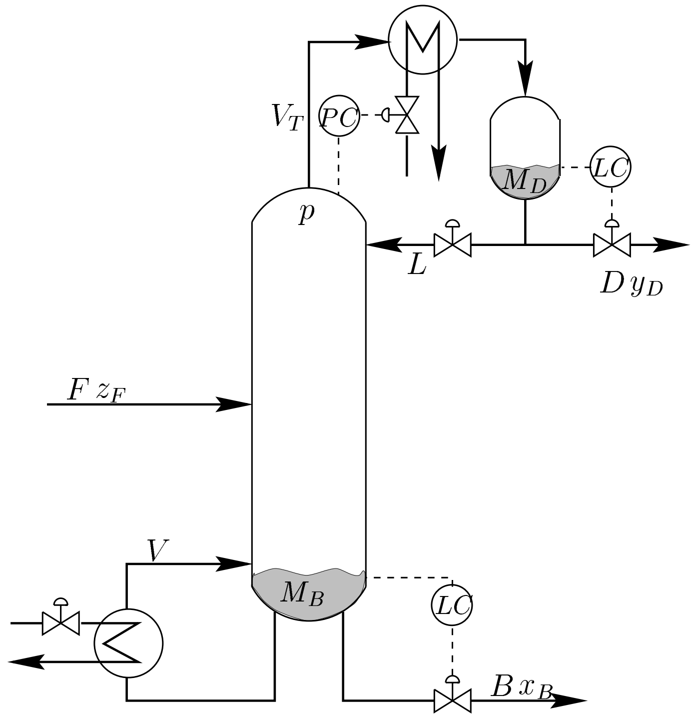
Some criteria for selecting u_2 and y_2 in lower-layer:

1. Lower layer must quickly implement the setpoints from higher layers, i.e., controllability of subsystem u_2/ y_2 should be good. (G_{22})
2. Provide for local disturbance rejection. (partial disturbanve gain P_d should be small)
3. Impose no unnecessary control limitations on problem involving u_1 and/or r_2 to control y_1. (P_u or P_e)

   - Avoid negative RGA for pairing u_2/y_2 – otherwise P_u likely has RHP-zero

"Unnecessary": Limitations (RHP-zeros, ill-conditioning, etc.) not in original problem involving u and y

DISTILLATION EXAMPLE



$$u = \begin{pmatrix} L \\ V \end{pmatrix}; \quad y = \begin{pmatrix} y_D \\ x_B \end{pmatrix} [\text{mol} - \%\ \text{light}]$$

Steady-state gains $y = Gu$

$$G(o) = \begin{bmatrix} 87.8 & -86.4 \\ 108.2 & -109.6 \end{bmatrix}$$

SINGULAR VALUE DECOMPOSITION

Look at directions is descending order by decomposing G into three parts (matrices)

$$G = U\Sigma V^T$$

- V - columns are input singular vectors
- U - columns are output singular vectors
- Σ - diagonal entries are corresponding gains

Distillation example:

```
>> g = [87.8 -86.4; 108.2 -109.6]

>> [u,s,v]=svd(g)

u =
    0.6246    0.7809
    0.7809   -0.6246

s =
  197.2087         0
         0    1.3914

v =
    0.7066    0.7077
   -0.7077    0.7066
```

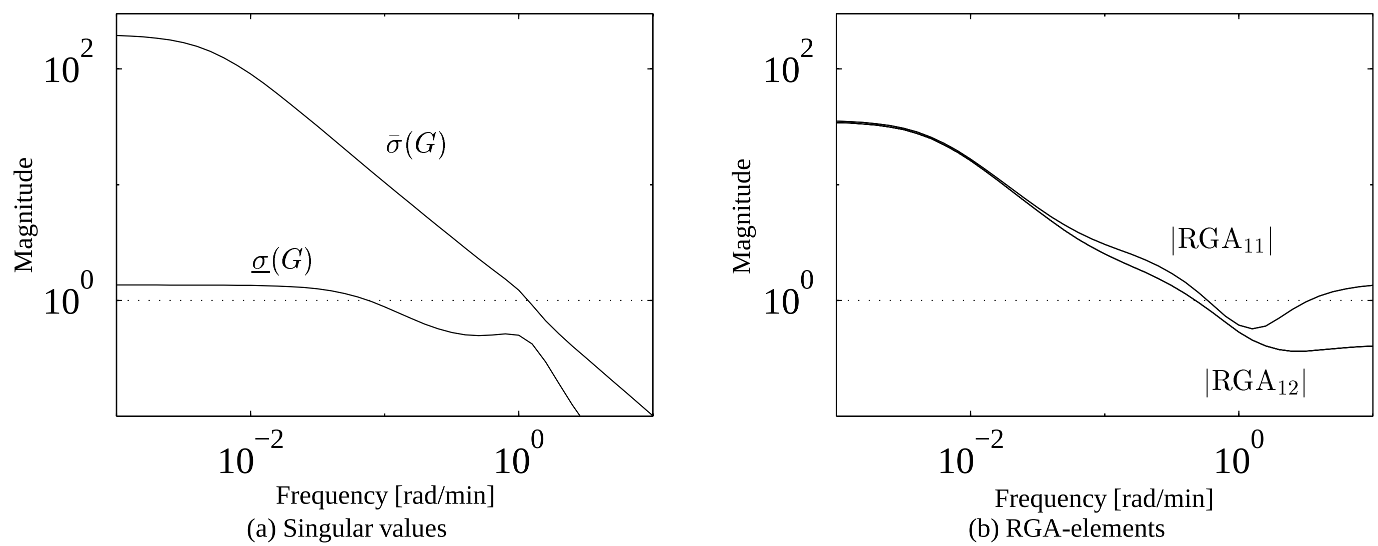
- Most sensitive input directions is $v_1 = \begin{pmatrix} 0.71 \\ -0.71 \end{pmatrix}$ (increase L and decrease V).
  - Physically, this corresponds to changing the external flow split from top to bottom
  - Its effect on the compositions is $\sigma_1 u_1 = 197.2 \begin{pmatrix} 0.63 \\ 0.78 \end{pmatrix}$, i.e. increase y_D (purer) and also x_B (less pure).
  - The effect is large because the compositions are sensitive to the ratio D/B
- The least sensitive input directions is $v_2 = \begin{pmatrix} 0.71 \\ 0.71 \end{pmatrix}$ (increase L while decreasing V by the *same* amount
  - Physically, this corresponds to increasing the *internal flows* (with no change in the external flows spit)
  - Its effect on the compositions is $\sigma_2 u_2 = 1.4 \begin{pmatrix} 0.78 \\ -0.63 \end{pmatrix}$
  - As expected, this makes both products purer and has a much smaller effect.
  - σ_2 is the minimum singular value; usually denoted $\underline{\sigma}$

- **Condition number**, $\gamma = \sigma_1/\underline{\sigma} = 197.2/1.4 = 141.7$
- A large condition number shows that some directions have a much larger gain than others, **but** does not necessarily imply that the process is difficult to control
- **Minimum singular value**. BUT if $\underline{\sigma}(G)$ is small (less than 1) then we may encounter problems with input saturation.
- For example, assume the variables have been scaled and $\underline{\sigma}(G) = 0.1$. Then in the "worst direction" a unit change (maximum allowed) in the inputs only gives a change of 0.1 in the outputs.
- **Relative Gain Array** (RGA)

$$RGA = G \times (G^{-1})^T = \begin{bmatrix} 35.07 & -34.07 \\ -34.07 & 35.07 \end{bmatrix}$$

- RGA yields sensitivity to gain uncertainty in the input channels. If the RGA-elements are large then the proces is fundamentally difficult to control

NOTE: Due mainly to liquid flow dynamics the process is much less interactive at high frequencies
⇒ Control is not so difficult if the loops are tuned tightly



(a) Singular values

(b) RGA-elements

OVERALL DISTILLATION PROBLEM

Typically, overall control problem has 5 inputs

$$u = \begin{pmatrix} L & V & D & B & V_T \end{pmatrix}$$

(flows: reflux L, boilup V, distillate D, bottom flow B, overhead vapour V_T)
and 5 outputs

$$y = \begin{pmatrix} y_D & x_B & M_D & M_B & p \end{pmatrix}$$

(compositions and inventories: top composition y_D, bottom composition x_B, condenser holdup M_D, reboiler holdup M_B, pressure p)

Without any control we have a 5×5 model

$$y = Gu + G_d d$$

(which generally has some large RGA-elements at steady-state)

DISTILLATION CONFIGURATIONS

There are usually three "unstable" outputs with no or little steady-state effect

$$y_2 = \begin{pmatrix} M_D & M_B & p \end{pmatrix}$$

Remaining outputs

$$y_1 = \begin{pmatrix} y_D & x_B \end{pmatrix}$$

Many possible choices for the three inputs for stabilization. For example, with

$$u_2 = \begin{pmatrix} D & B & V_T \end{pmatrix}$$

we get the *partially controlled* LV-configuration where

$$u_1 = \begin{pmatrix} L & V \end{pmatrix}$$

are left for composition control.
Another configuration is the DV-configuration (has small RGA-elements) where

$$u_1 = \begin{pmatrix} D & V \end{pmatrix}$$

After closing the stabilizing loops ($u_2 \leftrightarrow y_2$) we get a 2×2 model for the remaining "partially controlled" system

$$y_1 = \underbrace{G^{u_1}}_{P_u} u_1 + i \underbrace{G_d^{u_1}}_{P_d} d$$

Which configurations is the best?

Distillation example

4×4 model with constant pressure:

$$\begin{bmatrix} x_D \\ x_B \\ M_D \\ M_B \end{bmatrix} = \begin{bmatrix} g_{yL}(s) & g_{yV}(s) & 0 & 0 \\ g_{xL}(s) & g_{xV}(s) & 0 & 0 \\ -\frac{1}{s} & \frac{1}{s} & -\frac{1}{s} & 0 \\ \frac{1}{s} & -\frac{1}{s} & 0 & -\frac{1}{s} \end{bmatrix} \begin{bmatrix} L \\ V \\ D \\ B \end{bmatrix}$$

At steady state

$$\begin{bmatrix} g_{yL} & g_{yV} \\ g_{xL} & g_{xV} \end{bmatrix} = \begin{bmatrix} 87.8 & -86.4 \\ 108.2 & -109.6 \end{bmatrix}; \quad RGA = \begin{bmatrix} 35.1 & -34.1 & 0 & 0 \\ -34.1 & 35.1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6}$$

Theorem. Large RGA values implies that decoupling control is impossible due to sensitivity to uncertainty.

LV-configuration:

$$\underbrace{\begin{bmatrix} x_D \\ x_B \end{bmatrix}}_{y_1} = \underbrace{\begin{bmatrix} g_{yL} & g_{yV} \\ g_{xL} & g_{xV} \end{bmatrix}}_{P_{11}} \underbrace{\begin{bmatrix} L \\ V \end{bmatrix}}_{u_1} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{P_{12}} \underbrace{\begin{bmatrix} M_{Ds} \\ M_{Bs} \end{bmatrix}}_{r_2}$$

with RGA of 35.1.

BUT: DV-configuration:

$$\underbrace{\begin{bmatrix} x_D \\ x_B \end{bmatrix}}_{y_1} = \underbrace{\begin{bmatrix} -g_{yL} & g_{yV} + g_{yL} \\ -g_{xL} & g_{xV} + g_{xL} \end{bmatrix}}_{P_{11}} \underbrace{\begin{bmatrix} D \\ V \end{bmatrix}}_{u_1} + \underbrace{\begin{bmatrix} g_{yL}s & 0 \\ g_{xL}s & 0 \end{bmatrix}}_{P_{12}} \underbrace{\begin{bmatrix} M_{Ds} \\ M_{Bs} \end{bmatrix}}_{r_2}$$

RGA:

$$\Lambda\left(P_{11}^{DV}\right) = \begin{bmatrix} 0.46 & 0.54 \\ 0.54 & 0.46 \end{bmatrix}$$

A PARADOX:

Distillation columns have large RGA-elements
$\Rightarrow$ Fundamental control problems (cannot have decoupling control)

BUT: DV **configuration has small RGA-elements and we can decouple the compositions loops**
How is this possible?

Solution to paradox: DV configuration has coupling between composition and level loops
(whereas LV has decoupling between level and composition)

Analyze $P_u = G^{u_1}$ and $P_d = G_d^{u_1}$ with respect to

1. No composition control
   - Consider disturbance gain $G_d^{u_1}$ (e.g. effect of feedrate on compositions)
2. Close one composition loop ("one-point control")
   - Consider partial disturbance gain (e.g. effect of feedrate on y_D with constant x_B)
3. Close two composition loops ("two-point control")
   - Consider interactions in terms of RGA
   - Consider "closed-loop disturbance gains" (CLDG) for single-loop (decentralized) control. CLDG is the product of PRGA and G_d:

     $$CLDG = PRGA \cdot G_d(s) = G_{diag}(s)G^{-1}(s)G_d(s)$$

     $$PRGA = G_{diag}G^{-1}$$

     where PRGA has same diagonal elements as RGA.
   - See

     `http://www.chembio.ntnu.no/users/skoge/book/matlab_m/cola/paper/`

Problem:

- No single best configuration
- Generally, get different conclusion on each of the three cases

PLANTWIDE DYNAMICS

- Poles are affected by recycle of energy and mass and by interconnections
- Parallel paths may give zeros - possible control problems
- Recycle yields positive feedback and often large *open-loop* time constants
- This does *not* necessarily mean that *closed-loop* must be slow
- See MYTH on distillation contol where open-loop time constant for compositions is long because of positive feedback from reflux and boilup
- Luyben's "snowball effect" is mostly a steady-state design problem (do not feed more than the system can handle...)

PLANTWIDE CONTROL

- Where is the production rate set?
- Degrees of freedom - local “tick-off” can be useful
- Extra inputs
- Extra measurements
- Selection of variables for control
- Configuration for stabilizing control may effect layers above (including easy of model predictive control)
- One tool for stabilizing control: Pole vectors (see Tennessee Eastman example)

Alt.1 ”Cascade of SISO loops” - Control structure design

- Local feedback
- Close loop - same number of DOFs but uses up dynamic range
- Cascades - extra measurements,
- Cascades - extra inputs
- Selectors
- RGA

Alt.2 ”Optimization”: Multivariable predictive control

- Model-based
- Mostly feedforward based
- Excellent for extra inputs and changes in active constraint
- Feedback somewhat indirectly through model update.

Alt.3 Usually: A combination of feedback and models.

- How to find the right balance

CONCLUSION

- Steps in controllability analysis

  1. Find model and linearize it (G, G_d)
  2. Scale all variables within ± 1
  3. Analysis using controllability measures

- Have derived rigorous measures for controllability analysis, e.g.

$$|G_d(j\frac{1}{\theta})| < 1$$

- Use controllability analysis for:

  – What control performance can be expected?

  – What control strategy should be used?

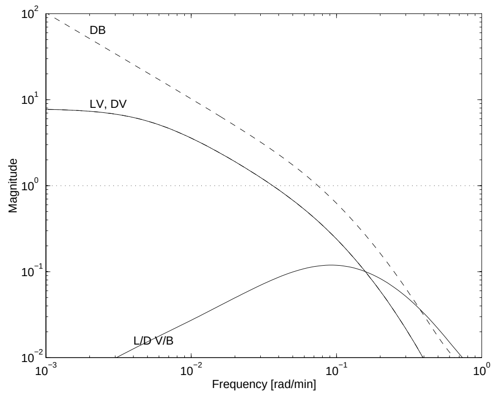
    * What to measure, what to manipulate, how to pair?

  – How should the process be changed to improve control?

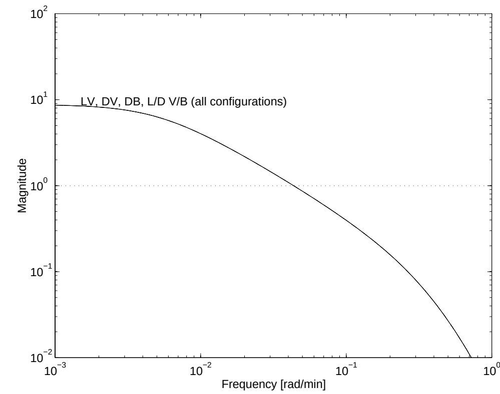
  – Tools are available in MATLAB (see my book on Multivariable control and its home page)

CONTROLLABILITY ANALYSIS OF VARIOUS DISTILLATION CONFIGURATIONS

- S. Skogestad, "Dynamics and control of distillation columns: A tutorial introduction", *Trans IChemE* (UK), **75**, Part A, 1997, 539 - 561.



(a) Effect of feed flow F on x_D



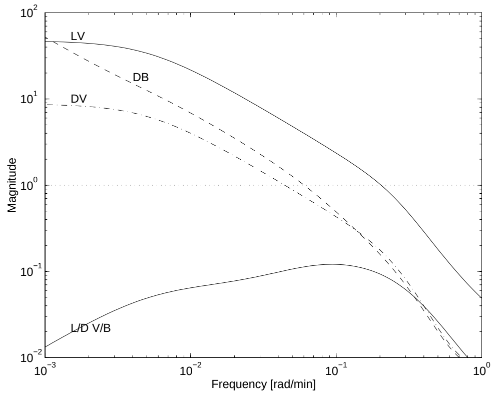
(b) Effect of feed composition z_F on x_D

Figure 8: Open-loop: Effect of disturbances on top composition x_D

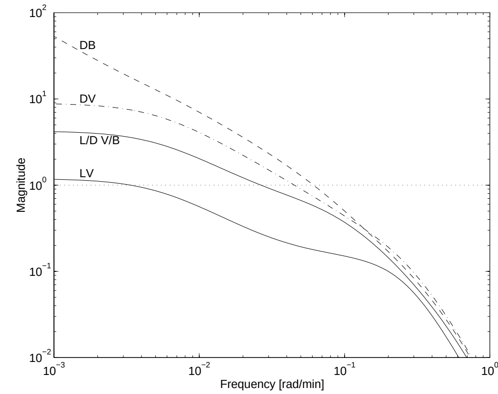
(a) Effect of feed flow F on x_D

(b) Effect of feed composition z_F on x_D

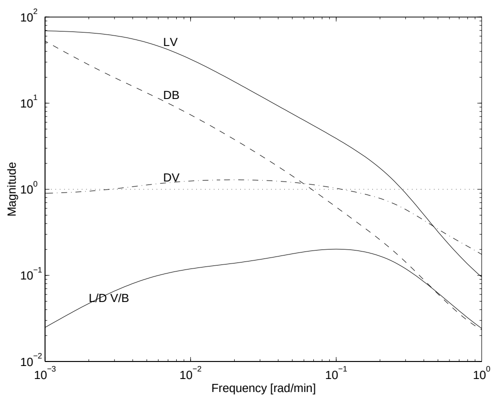
Figure 9: One-point control of x_B: Effect of disturbances on top composition x_D



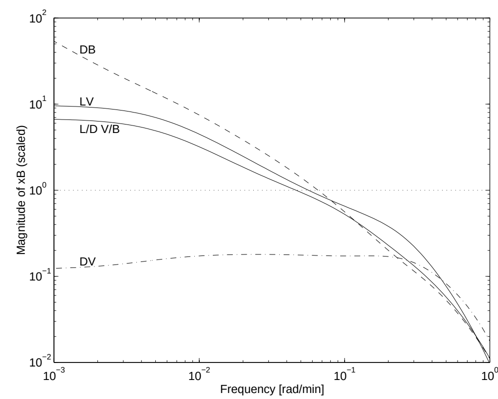
(a) Effect of feed flow F on x_D



(b) Effect of feed composition z_F on x_D



(c) Effect of feed flow F on x_B



(d) Effect of feed composition z_F on x_B

Figure 10: Two-point decentralized control (CLDG): Effect of disturbances on product compositions