Self-optimizing control: The basic idea and Taylor series analysis

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November 13, 1998

Abstract

The following important question is frequenctly overlooked: Which variables should we select to control?

It is shown that the idea of selecting the variables that achieve "self-optimizing control" provides a link between steady-state optimization, feedback control, time scale separation and uncertainty. In summary, the basic idea is to turn the optimization problem into a setpoint problem, and we show that a good candidate variable for a controlled output should have the following properties:

1. Its optimal value is insensitive to disturbances.
2. On the other hand, the variable itself should be sensitive to changes in the manipulated variables (inputs).
3. It should be easy to control accurately.

Prepared for presentation at the AIChE Annual Meeting, Miami Beach, 16-20 Nov. 1998, Paper 229c

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1 Introduction

If we formulate an optimal control problem in the usual mathematical fashion, where we define a scalar cost function J to be minimized, then we generally find that a centralized solution is the optimal choice. However, in many cases we want to decompose the control system into at least two layers: a setpoint optimizer and a feedback control layer which implements the optimal setpoints. The two parts interact through the *controlled variables* c; the optimizer computes their optimal setpoints c_s, and the control layer attempts to implement then in practice, i.e. to get $c \approx c_s$. In practice, the control system is usually divided into more than an optimization and a control layer. Typically, layers include include scheduling (weeks), site-wide optimization (day), local optimization (hour), supervisory/predictive control (minutes) and regulatory control (seconds); see Figure 1.

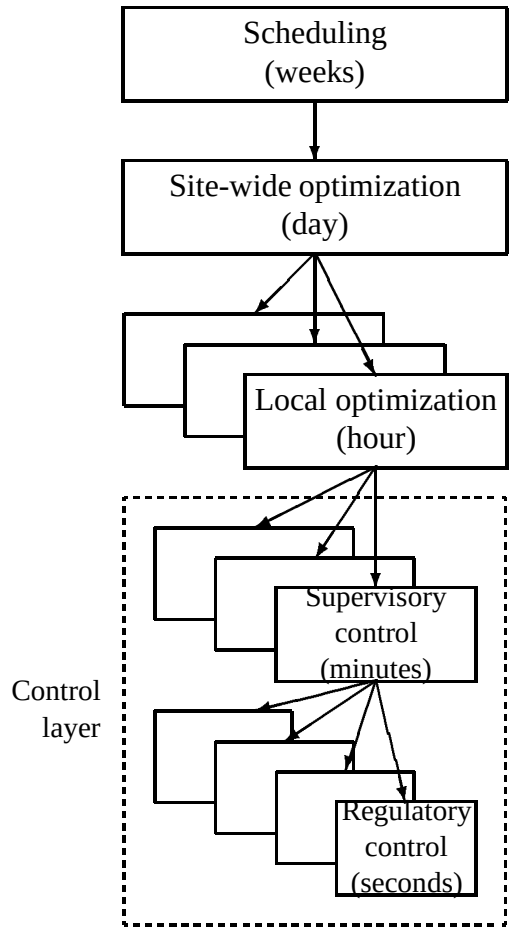

Figure 1: Typical control hierarchy in a chemical plant.

The reason for separating the control system into different layers is mainly to break the problem into more manageable subtasks. We usually do this by designing the control system in a hiearchical manner by requiring time-scale separation.

> **Definition of time-scale separation.** The control system at a given layer can be designed without the need to know the detailed controller parameters (tunings) used in the lower (faster) layers.

This means that we can assume "perfect control" in the lower layer when designing the controller for the next layer. To be able to make this assumption, the bandwith of the lower layer, ω_{B2}, must be at least a factor 2-3 higher than the bandwith in the next layer, ω_{B1}.

The loss is defined as the difference between the actual cost and the truly optimal value of the cost function.

> **Self-optimizing control** is when we can achieve acceptable loss with constant setpoint values for the controlled variables

Although our background is in process control, and we make some references to that area, we are confident that the idea of "self-optimizing" control has applicability in most other fields of control.

1.1 Control structure design

More generally, the issue of selecting controlled outputs is one of the subtasks in the **control structure design** problem (Foss 1973); (Morari 1982); (Skogestad and Postlethwaite 1996)

1. *Selection of controlled outputs* c (variables with setpoints)
2. *Selection of manipulated inputs* m
3. *Selection of measurements* v (for control purposes including stabilization)
4. *Selection of control* **configuration** (a structure interconnecting measurements/setpoints and manipulated variables, i.e. the structure of the controller K which interconnects the variables c_s and v (controller inputs) with the variables m)
5. *Selection of controller type* (control law specification, e.g., PID, decoupler, LQG, etc.).

Note that these *structural decisions* need to be made before we can start the actual design the controller. In most cases the control structure is solved by a mixture between a top-down consideration of control objectives and which degrees of freedom are available to meet these (tasks 1 and 2), combined with a bottom-up design of the control system, starting with the stabilization of the process (tasks 3,4 and 5). In most cases the problem is solved without the use of any theoretical tools.

Of course, the control field has made many advances over these years, for example, in methods for and applications of on-line optimization and predictive control. Advances has also been made in control theory and in the formulation of tools for analyzing the controllability of a plant. These latter tools can be most helpful in screening alternative control structures. However, a systematic method for generating promising alternative structures has been lacking. This is related to the fact the control structure design problem has not been well understood, has not been well defined, or even acknowledged as being an important issue.

The realization that the field of control structure design is underdeveloped is not new. In the 1970's several "critique" articles where written on the gap between theory and practice in the area of process control. The most famous is the one of (Foss 1973) who made the observation that in many areas application was *ahead* of theory, and he stated that

> The central issue to be resolved by the new theories is the determination of the control system structure. Which variables should be measured, which inputs should be manipulated and which links should be made between the two sets. ... The gap is present indeed, but contrary to the views of many, it is the theoretician who must close it.

A similar observation that applications seems to be ahead of formal theory was made by Findeisen *et al.* (1980) in their book on hierarchical systems (p. 10).

1.2 Related work

Parts of this paper are based on Chapter 10 in the book of Skogestad and Postlethwaite (1996). In addition, we have made use of some unpublished work by Skogestad and coworkers on *self-optimizing control*. The latter work is planned to be published as a series of papers with the following tentative titles:

Part 1. The basic issues in self-optimizing control (selection of controlled outputs to make implementation of the optimal solution insensitive to uncertainty).

Part 2. Taylor series analysis.

Part 3. Theoretical basis for using the minimum singular value for output selection.

Part 4. Partial and indirect control with application to selection of temperature measurements in distillation.

Part 5. Constraints and feasibility.

Except for the book of Skogestad and Postlethwaite (1996), preliminary versions of the above work are available in the Ph.D. theses of Morud (1995), Glemmestad (1997) and Havre (1998), as well as in a number of conference publications. These references are available on the Internet[1].

Of earlier work, Morari *et al.* (1980) come closest to the ideas presented in this paper. Morari *et al.* (1980) write that *in attempting to synthesize a feedback optimizing control structure, our main objective is to translate the economic objectives into process control objectives. In other words, we want to find a function c of the process variables which when held constant, leads automatically to the optimal adjustments of the manipulated variables, and with it, the optimal operating conditions. [...] This means that by keeping the function $c(u, d)$ at the setpoint c_s, through the use of the manipulated variables u, for various disturbances d, it follows uniquely that the process is operating at the optimal steady-state* $J = J_{\mathrm{opt}}$. This is a actually precise description of the best self-optimizing control structure, except that they do not consider the effect of implementation error $e = c - c_s$. Unfortunately, they did not attempt to further formalize the idea, and it seems that very few people, including the authors themselves, have picked up on it later.

Although at first sight it may seem quite different, another related work is that of Shinnar (Shinnar 1981) (Arbel *et al.* 1996). Maarleveld and Rijnsdrop (1970) state that the steady-state optimum usually is constrained, and that we therefore should control the constrained variable. Arkun and Stephanopoulos (1980) reach the same conclusion and provide a good discussion on the advantages of *active constraint control*. Luyben and coworkers (e.g. Luyben (1975), Yi and Luyben (1995), Luyben (1988)) have studied unconstrained problems, and some of the examples presented point in the direction of the selection methods presented in this paper. However, Luyben proposes to select controlled ouputs which minimizes the steady-state sensitive of the independent variable (u) to disturbances, i.e. to select controlled outputs (c) such that $(\partial u/\partial d)_c$ is small, whereas we really want to minimize the steady-state sensitivity of the economic loss (L) to disturbances, i.e. to select controlled outputs (c) such that $(\partial L/\partial d)_c$ is small. Fisher *et al.* (1988) discuss plant economics in relation to control. Narraway and Perkins ((Narraway *et al.* 1991), (Narraway and Perkins 1993) and (Narraway and Perkins 1994)) strongly stess the need to base the selection of the control structure on economics. Finally, Marlin and Hrymak (1997) stress the need to find a good way of implementing the optimal solution in terms how the control system should respond to disturbances, "i.e. the key constraints to remain active, variables to be maximized or minimized, priority for adjusting manipulated variables, and so forth." They suggest that an issue for improvement in today's real-time optimization systems is to select the control system that yields the highest profit for a range of disturbances that occur between each execution of the optimization.

For a more detailed literature review and a more precise defintion of terms, the reader is referred to the following internal report

S. Skogestad and T. Larsson, "A review of plantwide control". Available at: `http://www.chembio.ntnu.no/us`

[1] `http://www.chembio.ntnu.no/users/skoge/`

2 Optimization and control

The focus on this paper on selection of controlled outputs (task 1), which is probably the least studied of the structural decisions. But ask the question:

> Why are we controlling hundreds of temperatures, pressures and compositions in a chemical plant, when there is no specification on most of these variables? Is it just because we can measure them or is there some deeper reason?

To answer this problem we need think more carefully about why we do control. First, there is the issue of stabilization and then of keeping the operation within given constraints. These issues may consume some degrees of freedom (e.g. to stabilize levels with no steady-state effect and to satisfy product specifications), but there will genereally be many degrees of freedom left. What should these be used for?

Loosely speaking, they should be used to "optimize the operation". There may be many issues involved, and to trade them off against each other in a systematic manner we usually quantify a scalar performance (cost) index J which should be minimized. In many cases this index is an economic measure, e.g. the operation cost. Since the economics of plant operation usually are determined mainly by steady-state issues, the analysis of how to use the remaining degrees of freedom can often be based on steady-state considerations, and their optimal values may be found using steady-state optimization. The resulting optimization problem may be very large, with hundreds of thousands of equations, and hundreds of degrees of freedom. However, with todays computers and optimization methods this problem is easily solvable, and it is indeed solved routinely in some plants, such as ethylene plants.

However, it is often much less clear how the optimal solution should actually be implemented in practice. Three alternative solutions are shown in Figure 2:

(a) Open-loop optimization.

(b) Closed-loop implementation with a separate control layer.

(c) Integrated optimization and control.

It should be stressed that in all the cases the "optimization" may be performed manually (by operators or engineers).

In Figure 2 the "process" denotes the process as seen from the optimization layer, so it may actually be a partially controlled plant. Correspondingly, the variables u denote the independent variables (degrees of freedom) as seen from the optimization layer, and typically consists of setpoint for the lower-layer controlelrs. that is different from m, and we will use the symbol u to denote this set which includes setpoints for the lower layers. Only at the lowest layer do we have $u = m$;

The open-loop implementation (a) where we directly manipulate m can generally not be used because of sensitivity to uncertainty.

From a theoretical point of view, the centralized scheme in (c) should be the best implementation. Here, the optimizing controller stabilizes the process and also updates the model (using feedback) and at the same time perfectly coordinates all the manipulated inputs based on dynamic on-line optimization. However, there are fundamental reasons why such a solution is not the best, even with todays and tomorrows computing power. The main reason is probably the cost of modeling; in the centralized controller there are no predetermined links, so the controller must rely only on the model to take the right action.

On the other hand, if we use local controllers (which use only a subset of the measurements and manipulators), then the task of each controller is well-defined (e.g. keep the temperature at its setpoint) and we can often tune the controllers with a minimum of modelling efforts. In fact, by cascading feedback loops, it is possible to control large plants with thousands of variables without the actual need to develop any models. In any case, we find that in practice the hierarchical feedback implementation (b) is preferred. It consists of

- *optimization layer* — computes setpoints c_s for the controlled variables c

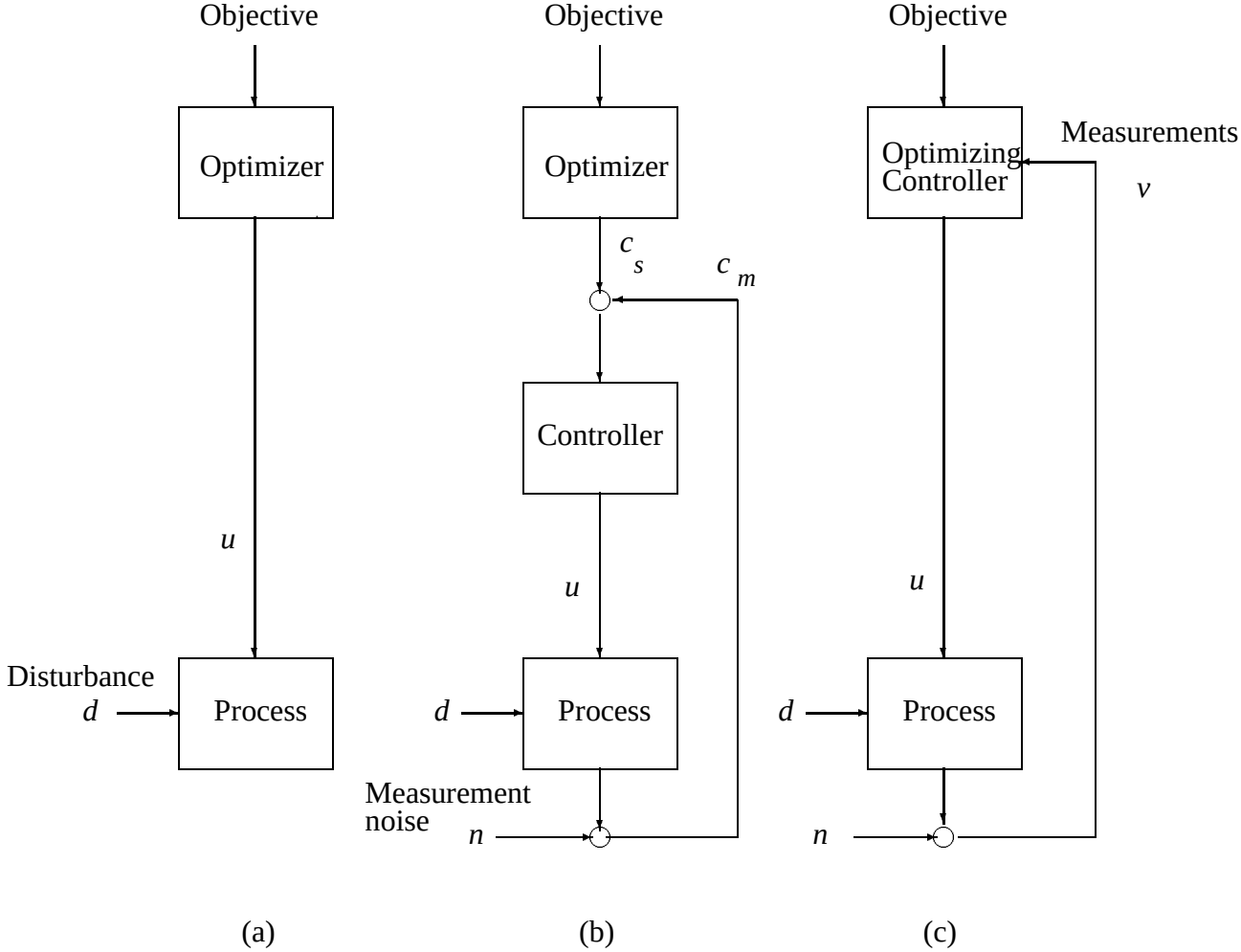

Figure 2: Alternative structures for optimization and control.

- *control layer* — implements this in practice, with the aim of achieving $c \approx c_s$ (in practice we achieve $c = c_s + e$ where e is the control error; at steady-state and with integral action $e = n$ where n is the measurement noise).

In process control applications, the optimization layer typically recomputes new setpoints c_s only about every hour or so, whereas the feedback layer operates continuously. Since the data and model used by the optimizer are uncertain and there are disturbances entering the plant between each re-optimization, the objective of the feedback layer is to keep the plant close to its optimal operating point in spite of this uncertainty.

Why do we select a particular set c of controlled variables? (e.g., why specify (control) the top composition in a distillation column, which does not produce final products, rather than just specifying its reflux?) The answer to this question is not obvious, because at first it seems like it does not really matter which variables we specify (as long as all degrees of freedom are consumed, because the remaining variables are then uniquely determined). However, this is true only when there is no uncertainty (disturbances, noise or model uncertainty). When there is uncertainty then it does make a difference how the solution is implemented, that is, which variables we select to control at their setpoints.

We also stress that the analysis below is based on steady-state considerations. The main justification for this is, as mentioned above, that the economic performance is mainly determined by steady-state considerations. Of course, one could extend the analysis on a frequency-by-frequency basis, and include in the variable e information about how well a variable can be controlled at a given frequency. However, this would complicate the analysis, and should therefore be used only when needed, and is not considered

any further here.

3 Selecting controlled variables for optimal operation

3.1 The performance index (cost) J

We assume that the optimal operation problem can be quantified in terms of a scalar performance index (cost) J, such that the objective of the operation is to minimize J with respect to the available degrees of freedom. J may be a purely economic objective, but is more generally a weighted sum of the various control objectives. For the optimization itself it does not really matter which variables we use as degrees of freedom as long as they form an independent set. Let the "base set" for the degrees of freedom be denoted u (these may consist, for example, of a subset the physical manipulators m). In addition, the cost will depend on the unknown disturbances d (which here is assumed to include uncertainty in the model and uncertainty in the optimizer). We can then write $J(u, d)$. The nominal value of the disturbances is denoted d_0, and we can solve the nominal operating problem and obtain $u_{opt}(d_0)$ for which

$$min_u J(u, d_0) = J_{opt}(d_0) = J(u_{opt}(d_0), d_0)$$

From this we can obtain a table with the corresponding optimal value of any other dependent variable, including $c_{opt}(d_0)$.

The issue is now to decide how to best implement the optimal policy in the presence of uncertainty by selecting the right set of controlled variables c with constants setpoints $c_s = c_{opt}(d_0)$. Here it is assumed that the number of controlled variables y equals the number of independent variables u, or more exactly that we starting from $c = f(u, d)$ can derive the inverse relationship

$$u = f^{-1}(c, d)$$

where the function f^{-1} exists and is unique.

Instead of evaluating the mean value of the performance index, it may be better to evaluate the always positive loss function. The loss function expresses the difference between the actual operating costs (e.g. obtained when we adjust u in order to keep c at a given setpoint) and the optimal operating cost (obtained with $u = u_{opt}(d)$),

$$L(u, d) = J(u, d) - J_{opt}(d) \tag{1}$$

The objective of the operation is to minimize J (or some average of J), or equivalently to minimize the loss L. The loss function is zero if we use the optimal policy $u = u_{opt}(d)$. The loss has the advantage of providing a better "absolute scale" on which to judge whether a given set of controlled variables c is "good enough", and thus is self-optimizing.

3.2 Open-loop implementation

Let us first consider an open-loop implementation where we attempt to keep u constant at the value u_s. With this implementation the operation may be non-optimal (with a positive loss) due to the following reasons

1. The value of u_s is different from the optimal value $u_{opt}(d)$.
2. The actual value of u is different from u_s (due to an implementation error caused by imperfect control).

This can be seen more clearly if we write the actual input as

$$u = u_s + \underbrace{u - u_s}_{e_u} \tag{2}$$

where e_u is the implementation error for u. In process control, u is often a flowrate, and it is difficult in practice to obtain exactly the desired value u_s, so e_u may be large. [2]

Introduce the optimization error

$$e_{u,opt}(d) = u_s - u_{opt}(d)$$

Then the difference between the actual and optimal input, which causes a positive loss, can be written

$$u - u_{opt}(d) = u_s - u_{opt}(d) + e_u = e_{u,opt}(d) + e_u \tag{3}$$

i.e. it is the sum of the optimization error and the control error. In summary, the open-loop policy is often poor; both because the optimal input value often depends strongly on the disturbance (so $e_{u,opt}$ is large), and because we are not able to implement u accurately (so e_u is large).

3.3 Closed-loop implementation

As already mentioned, in theory, the truly optimal solution would be to use some "optimizing controller" which uses the measurements information (feedback) to correct the model and estimate the disturbance d, and based on this computes a new optimal value $u_{opt}(d)$. The main problem with this approach is the modelling effort, and the lack of theoretical tools to ensure robustness (insensitivity to uncertainty).

In practice, a simpler closed-loop implementation is preferred if it yields acceptable operation (loss). This is to use directly the measurements c_m of the selected controlled variables and adjust u in an inner feedback loop to achieve $c_m \approx c_s$, where in most cases $c_s = c_{opt}(d_0)$, i.e. c_s comes from solving the nominal optimization problem. The idea is that by keeping $c_m \approx c_s$ we achieve an operation where the deviation $u - u_{opt}(d)$ is smaller than for the open-loop policy (in the open-loop policy we keep u constant, but this is not optimal in the face of disturbances). This may happen because $c_{opt}(d)$ is relatively insensitive to d and/or because c may more accurately controlled. We next formalize these ideas.

We here rewrite the problem with the variables c as independent variables rather than the original independent variables (inputs) u. However, note that we as a special case may choose $c = u$, or some of the elements in the vector c may be the original input variables. Thus, the open-loop implementation is included as a special case.

If we compare the open-loop and closed-loop policies then the question is:

> *Is it best adjust the input variables u such that $u = u_s + e_u$ (where e_u is the implementation error for the input u), or is it better to adjust $u = f^{-1}(c, d)$ in feedback fashion such that $c = c_s + e$ (where e is the implementation error for control of c) ?*

More generally, if there are many alternative sets of variables c which can be measured and controlled, which set should be used? If we let y_m represent all the candidate measured variables then we can write

$$c = g(y_m, u) \tag{4}$$

where the function g is free to select. An open-loop policy is obtained with $g(y_m, u) = u$. Linearized in terms of deviation variables (4) becomes

$$\Delta c = C_1 \Delta y_m + C_2 \Delta u \tag{5}$$

The issue is then to find the optimal choice for the matrices C_1 and C_2, but under the restriction that the number of controlled variables (c's) equals the number of independent inputs (u's). If we use only feedback then $C_2 = 0$. If we disallow "combined" controlled variables, then the matrix $C = \begin{pmatrix} C_1 & C_2 \end{pmatrix}$ is a a "selection matrix" with only one nonzero element (a 1) in each row.

[2] The implementation error e_u may be reduced in some cases if we measure the variable u and implement an inner control loop with setpoint u_s. However, also in this case there will be a control and a measurement error; if we use integral action then at steady-state e_u will equal the steady-state measurement error (noise).

To compare the alternative choices we then evaluate the objective function, or equivalently the loss function, for alternative values of the disturbance d and the implementation error e_c. **The optimal choice for of controlled variables** c (i.e. optimal choice of the matrix C) is then the one that minimizes some average value of the loss

$$L(u, d) = L(f^{-1}(c_s + e, d), d) \tag{6}$$

for the expected set of disturbances $d \in \mathcal{D}$, and expected set of implementation (control) errors $e \in \mathcal{E}$. In the simplest case we select the setpoints as $c_s = c_{opt}(d_0)$, but the value of c_s may also be the subject to an optimization.

The difference between the actual and optimal outputs, which causes a positive loss, can be written

$$c - c_{opt}(d) = c_s + e - c_{opt}(d) = e_{opt}(d) + e \tag{7}$$

i.e. it is the sum of the optimization error $e_{opt}(d) = c_s - c_{opt}(d)$ and the control error e.

As already mentioned, if there where no uncertainty (i.e. $d = d_0$ and $e_c = 0$), then it would make no difference which variable c that were chosen.

3.4 A procedure for output selection (Method 1)

We are now in a position to formulate a procedure for selecting controlled outputs c. Preferably, one should find several candidate sets of candidate outputs, which could be further analyzed to see if they are adequate with respect to other criteria that may be relevant, such that the input-output controllability (including the presence of RHP-zeros).

1. Define the optimal operation problem (including specifying the cost function J to be minimized)
2. Solve the optimization problem at a given nominal operating point. That is, find $u_{opt}(d_0)$ by solving the nominal optimization problem

   $$min_u J(u, d_0)$$

   where

   u – "base set" for the N_u degrees of freedom

   d_0 – nominal value of the parameters (disturbances)
3. This yields a table with the nominal optimal values of all variables, $c_{opt}(d_0)$).
4. Define the uncertainty:
   (a) For the optimization: Define the magnitude or set of unknown disturbances ($d \in \mathcal{D}$) (including any changes that occur between each reoptimization). Treat also errors in the data and model for the optimizer as disturbances.
   (b) For each candidate output variable (y): Define the magnitude or set of control error ($e \in \mathcal{E}$ (e.g. due to measurement error)
5. Repeat for each *candidate set* of N_u output variables (y's)
   (a) Evaluate the cost function $J(c, d)$ with fixed setpoints

   $$c = c_s + e$$

   where $c_s = c_0 \equiv c_{opt}(d_0)$ is taken from the above table.

   Do this for all disturbances ($d \in \mathcal{D}$) and all control errors ($e \in \mathcal{E}$).
   (b) Compute the "mean" cost, J_{mean} (or equivalently, the mean loss)
6. Select as the controlled outputs the candidate set with the lowest "average" cost (or retain all the sets with an acceptable loss for further screening)

Comments

1. Instead of evaluating the cost J we may equivalently evaluate the loss L.
2. There are many possibilites for defing the **mean cost ,** J_{mean}., for example,
   (a) Mean cost over a finite set (e.g., max, nominal, min for each disturbance and each control error)
   (b) Mean cost from "Monte-Carlo" evaluation of given distribution of d and e
   (c) Worst-case loss (compared to true optimal)
3. The computation load can be significantly reduced if we use a local analysis based on a Taylor series expansion of J or L. This is discussed in section 4.7.

4 Taylor series analysis

In this section we study the problem of selecting controlled outputs by expanding the cost function around a nominal optimal operating point. To this end, we here assume that the cost function J is smooth, or more precicely twice differentiable, at the operating point we are considering.

We assume that the nominal disturbance is d_0 and that the nominal operating point is optimal, i.e.

$$u_0 = u_{opt}(d_0) \quad \text{and} \quad c_0 = c_{opt}(d_0)$$

so that we have $J(u_0, d_0) = J_{opt}$. We next consider a disturbance and input change so that the new disturbance is

$$d = d_0 + \Delta d$$

and the new input is

$$u = u_0 + \Delta u$$

where Δu is the input change. The input u will generally be different from the optimal input, $u_{opt}(d)$, and we define the deviation from the optimal value as

$$\Delta u' = u - u_{opt}(d)$$

(Similarly, we define $\Delta c' = c - c_{opt}(d)$, etc.). The issue is now what effect a nonzero value of $\Delta u'$ will have on the operation (as quantified by the value of the loss function L).

4.1 Expansion of the cost function

A second order Taylor series expansion of the cost function gives

$$\begin{aligned} J(u,d) &= J(u_0,d_0) + J_u^T(u-u_0) + J_d^T(d-d_0) + \frac{1}{2}(u-u_0)^T J_{uu}(u-u_0) \\ &\quad + \frac{1}{2}(d-d_0)^T J_{dd}(d-d_0) + (d-d_0)^T J_{ud}(u-u_0) + \mathcal{O}^3 \end{aligned} \tag{8}$$

where all derivatives are evaluated at the optimal nominal operating point (with $d = d_0$ and $u = u_0 = u_{opt}(d_0)$), as indicated by using the subscript $_0$. We have

$$J_u = \left(\frac{\partial J}{\partial u}\right)_0 = 0$$

$$J_d = \left(\frac{\partial J}{\partial d}\right)_0$$

$$J_{uu} = \left(\frac{\partial^2 J}{\partial u^2}\right)_0$$

$$J_{dd} = \left(\frac{\partial^2 J}{\partial d^2}\right)_0$$

$$J_{ud} = \left(\frac{\partial^2 J}{\partial u \partial d}\right)_0$$

Note that $J_u = 0$ because the Jacobian with respect to the independent variables must be zero at the optimum. We can write the expansion in (8) more compactly as

$$J(u,d) = J(u_0,d_0) + \begin{pmatrix} \underbrace{J_u}_{=0} & J_d \end{pmatrix}^T \begin{pmatrix} \Delta u \\ \Delta d \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \Delta u \\ \Delta d \end{pmatrix}^T H \begin{pmatrix} \Delta u \\ \Delta d \end{pmatrix} \tag{9}$$

where H is the Hessian matrix of J with respect to $\begin{pmatrix} \Delta u \\ \Delta d \end{pmatrix}^T$,

$$H = \begin{pmatrix} J_{uu} & J_{ud} \\ J_{du} & J_{dd} \end{pmatrix}$$

The Hessian matrix is always symmetric, so J_{uu} and J_{dd} are symmetric and $J_{ud} = J_{du}^T$. Since the expansion is performed at the point where J has a minimum, we have that $\Delta u^T J_{uu} \Delta u$ is positive for any nonzero vector Δu, i.e. J_{uu} is positiv definite, $J_{uu} > 0$ (if the minimum is a saddle the $\Delta u^T J_{uu} \Delta u$ is zero in some direction and J_{uu} is positive semidefinite, $J_{uu} \geq 0$).

4.2 The optimal input

The nominal operating point (u_0, d_0) is assumed to be optimal se we have $u_0 = u_{opt}(d_0)$, and as noted the Jacobian must be zero,

$$J_u = \frac{\partial J}{\partial u}(u_0, d_0) = 0$$

Next, consider a disturbance and input change so that the new operating point is (u, d) and the new Jacobian is

$$J'_u = \frac{\partial J}{\partial u}(u, d)$$

An first-order expansion of the Jacobian gives

$$J'_u = J_u + J_{uu}(u - u_0) + J_{ud}^T(d - d_0)$$

We assume that we change the input so that also the new operating point is optimal, i.e. $u = u_{opt}(d)$. Then we must also here have that the Jacobian is zero, i.e. $J'_u = 0$, and we get

$$0 = J_{uu}(u_{opt}(d) - u_{opt}(d_0)) + J_{ud}^T(d - d_0)$$

and we find that that a first-order accurate approximation of the optimal input iwhen there is a disturbance change is

$$\boxed{u_{opt}(d) = \underbrace{u_{opt}(d_0)}_{u_0} - J_{uu}^{-1} J_{du}(d - d_0)} \tag{10}$$

4.3 Expansion of the loss function

Let us now consider the loss function

$$L(u,d) = J(u,d) - J(u_{opt}(d), d)$$

Expand $J(u,d)$ around the perturbed point $(u_{opt}(d),d)$ (which is close to but not the same as the nominal point $(u_{opt}(d_0),d_0)$. We get

$$J(u,d) \approx J(u_{opt}(d),d) + J_u'^T(u-u_{opt}(d)) + \frac{1}{2}(u-u_{opt}(d))^T J_{uu}'(u-u_{opt}(d)) \tag{11}$$

where the $'$ denotes that the derivatives are evaluated at the perturbed point. However, it turns out this does not matter. First, since also the perturbed point is optimal, we must have that $J_u' = 0$. Second, J_{uu}' can be expanded in terms of J_{uu},

$$J_{uu}' = J_{uu} + J_{uud}^T(d-d_0) + J_{uuu}^T(u_{opt}(d)-u_{opt}(d_0))$$

but the resulting third order terms can be neglected upon substitution into (11), which then gives the following second order accurate expansion for the loss function in terms of the deviation from the optimal input, $\Delta u' = u - u_{opt}(d)$,

$$\boxed{L = \frac{1}{2}(u-u_{opt}(d))^T J_{uu}(u-u_{opt}(d)) = \frac{1}{2}\Delta u'^T J_{uu}\Delta u'} \tag{12}$$

which is a very useful expression.

Comment. To confirm that the approach taken when deriving (12) is acceptable, we shall rederive (8) by expanding in only "one" variable (u or d) at a time. Let here a double prime ($''$) denote that the derivative is evaluated at the point (u_0,d). We then have by first expanding in u (with d constant)

$$J(u,d) = J(u_0,d) + J_u''^T(u-u_0) + \frac{1}{2}(u-u_0)^T J_{uu}''(u-u_0) \tag{13}$$

We then expand in d the terms that were not at the nominal operating point,

$$J(u_0,d) \approx J(u_0,d_0) + J_u^T(d-d_0) + \frac{1}{2}(d-d_0)^T J_{dd}(d-d_0)$$

$$J_u'' = J_u(u_0,d) \approx J_u + J_{ud}^T(d-d_0)$$

and with $J_{uu}'' \approx J_{uu}$ and substituting into (13) we rederive (8).

4.4 Loss with constant inputs

Assume there is a disturbance change, but we attempt to keep the input fixed at its nominally optimal value u_0, i.e.

$$u_s = u_0$$

where $u_0 = u_{opt}(d_0)$. We use the word "attempt", since in practice there will be an implementation error so the actual input will be

$$u = u_s + e_u$$

where e_u is the implementation error for the input. Then from (10) the deviation from the optimal input is

$$\boxed{\Delta u' = u - u_{opt} = J_{uu}^{-1}J_{du}\Delta d + e_u} \tag{14}$$

and we can evaluate the resulting loss from (12).

4.5 Loss with constant controlled outputs

As already mentioned, the outputs c are related to the inputs and disturbances by the relationship

$$c = f(u,d)$$

The corresponding linearized relationship in terms of deviation variables ($\Delta c = c - c_0$, etc.) is

$$\Delta c = G\Delta u + G_d \Delta d \tag{15}$$

where $G = (\partial f/\partial u)^T$ and $G_d = (\partial f/\partial d)^T$.

Assume there is a disturbance change, but we attempt to keep the control output fixed at its nominally optimal value c_0, i.e.

$$c_s = c_0$$

where $c_0 = c_{opt}(d_0)$. We use the word "attempt", because, in practice, there will be an implementation error so the actual controlled output will be

$$c = c_s + e \tag{16}$$

where e is the implementation error (typically, the sum of the measurement error and the control error). We have in this case $\Delta c = e$, so from (15) the corresponding input change is

$$\Delta u = -G^{-1} G_d \Delta d + G^{-1} e$$

and from (10) the resulting deviation from the optimal input is

$$\boxed{\Delta u' = u - u_{opt} = \left(J_{uu}^{-1} J_{du} - G^{-1} G_d\right) \Delta d + G^{-1} e} \tag{17}$$

The optimal choice for the controlled outputs is the one that minimizes the "mean" value of the loss

$$L = \frac{1}{2} \Delta u'^T J_{uu} \Delta u'$$

for the expected disturbances (as expressed by the magnitude Δd) and the expected control error (as expressed by the magnitude of e). Note that the matrices J_{uu} and $J_{uu}^{-1} J_{du}$ are independent of the choice of controlled outputs.

4.5.1 Alternative form

An alternative form is to express the loss directly in terms of the controlled outputs. A similar derivation as for the inputs, see (12), gives

$$L = \frac{1}{2} \Delta c'^T J_{cc} \Delta c' \tag{18}$$

where $\Delta c' = c - c_{opt}(d)$ and

$$J_{cc} = G^{-1^T} J_{uu} G^{-1}$$

(the latter follows from $\Delta c' = G\Delta u'$)., We see that J_{cc} depends directly on the choice of the controlled outputs through the matrix G^{-1}), and to keep $J_c c$ and thus L small, we want G^{-1} small. The deviation between the actual and optimal output, $\Delta c'$, will be nonzero due to the presence of two generally independent terms,

$$\boxed{\Delta c' = e_{opt} + e} \tag{19}$$

where $e_{opt} = c_s - c_{opt}(d)$ is the optimization error (introduced by attempting to keep c at c_s rather than at $c_{opt}(d)$), and $e = c - c_s$ is the implementation or control error (introduced by incorrect measurement and poor control of c).

We may also express the optimization error directly in terms of the disturbance. Using the linearized model in (15)

$$-e_{opt} = \underbrace{c_{opt}(d) - c_0}_{\Delta c} = G \underbrace{\left(u_{opt}(d) - u_0\right)}_{\Delta u} + G_d \Delta d$$

where from (10)

$$u_{opt}(d) - u_0 = -J_{uu}^{-1} J_{du} \underbrace{(d - d_0)}_{\Delta d}$$

and we find

$$\boxed{e_{opt}(d) = c_s - c_{opt}(d) = \left(GJ_{uu}^{-1} J_{du} - G_d\right) \Delta d} \tag{20}$$

where $c_s = c_0$. We will return to this expression shortly.
Remark. Obviously, substitution of (20) and (19) into (18) gives the same expression for the loss L as a function of e and Δd, as the one we obtain by substituting (17) into (12).

4.6 "Ideal" choice of controlled outputs

If we for the moment disregard the control error e, then the ideal choice of controlled outputs would be to have $e_{opt}(d) = c_s - c_{opt}(d) = 0$ for any value of d. Here $c_s = c_0$ is constant, so to achieve this, we need the optimal value of output to be independent of the disturbance. An example of such an ideal output would be to have a direct measurement of the gradient of the cost function with respect to the input (since it is optimal for any disturbance to have this gradient zero, we could directly specify its setpoint at zero). In particular, consider the following output

$$c = f(u, d) = c_1 \frac{\partial J(u, d)}{\partial u} + c_0 = c_1 J_u + c_0 \tag{21}$$

where c_0 and c_1 are constants. To see that this output would be "ideal", we linearize (21) to get

$$\Delta c = J_{uu} \Delta u + J_{ud}^T \Delta d$$

i.e. we find that $G = J_{uu}$ and $G_d = J_{ud}^T = J_{du}$, which upon substitution into (20) gives $e_{opt} = 0$.

However, as we see when studying, for example, selection of measurement locations in a distillation column, the implementation error e may be a very important factor, and the "ideal" output may not be the best after all.

4.7 A procedure for output selection (Taylor-version of Method 1)

The computations for the procedure (method 1) given in the previous section can be very time-consuming, but they can be reduced significantly if we use the local Taylor series approximations derived above.

Upon substitution of (17) into (12) we can write

$$L = \frac{1}{2} z^T z = \frac{1}{2} \|z\|_2^2 \tag{22}$$

where $\|z\|_2$ denotes the 2-norm of the vector z and by

$$z = J_{uu}^{1/2} \left[\left(J_{uu}^{-1} J_{du} - G^{-1} G_d \right) \Delta d + G^{-1} e \right] \tag{23}$$

Let the elements in the positive diagonal matrices W_d and W_e represent the expected magnitudes of the disturbances and the control errors, i.e. let

$$\Delta d = W_d \tilde{\Delta d}$$

$$e = W_e \tilde{e}$$

where we assume

$$\|\tilde{\Delta d}\|_2 \leq 1; \quad \|\tilde{e}\|_2 \leq 1$$

Minimization of the loss L for any (worst-case) combination of disturbances and control errors is then equivalent to minimizing induced 2-norm (maximimum singular value) of the matrix

$$M = \begin{pmatrix} M_1 & M_2 \end{pmatrix} \tag{24}$$

where

$$M_1 = J_{uu}^{1/2} \left(J_{uu}^{-1} J_{du} - G^{-1} G_d \right) W_d$$

$$M_2 = J_{uu}^{1/2} G^{-1} W_e$$

where $J_{uu}^{1/2} J_{uu}^{1/2} = J_{uu}$

Thus, if we assume that the disturbances and control errors are two-norm bounded then we have that

$$L = \frac{1}{2} \bar{\sigma}(M) \tag{25}$$

and the procedure becomes

1. Define the optimal operation problem (specify the cost function J).
2. Solve the optimization problem at the given nominal operating point and find the second-order derivatives of the cost, J_{uu} and J_{ud}, at this nominal optimal operation point.
3. For each candidate set of controlled variables obtain the linear model $\Delta c = G\Delta u + G_d \Delta d$.
4. Define the uncertainty:
   (a) The elements in the diagonal matrix W_d represents the magnitude of each disturbance
   (b) The elements in the diagonal matrix W_e represents the magnitude of the control error for each output c (e.g. due to measurement error)
5. For each *candidate set* compute the singular value of the matrix M, $\bar{\sigma}(M)$.
6. Select as the controlled outputs the candidate set with the lowest value of the loss $L = \frac{1}{2}\bar{\sigma}(M)$.

Comment. We can easily use this approach to search for the best linear combination of measurements y_m and independent inputs u to control,

$$\Delta c = C_1 \Delta y_m + C_2 \Delta u$$

where the matrices C_1 and C_2 are free to choose, but we make the restriction that the number of controlled variables (c's) equals the number of independent inputs (u's) (recall the comments following (5)).

We first identify all the candidate measurements y_m and obtain the linear model

$$\Delta y_m = G^m \Delta u + G_d^m \Delta d$$

We also need to identify (or at least estimate) the control error (measurement noise) associated with controlling the measurements and inputs, and collect these in the diagonal matrices W_{em} and W_{eu}.

The matrices used in the procedure above then become

$$G = C_2 + C_1 G^m$$

$$G_d = C_1 G_d^m$$

and the j'th diagonal element in the matrix control error matrix W_e is given by

$$W_{e,jj} = \sqrt{\sum_i C_{ij}^2 W_{m,ii}^2}$$

where $C = \begin{pmatrix} C_1 & C_2 \end{pmatrix}$ and $W_m = \mathrm{diag}\{W_{em}, W_{eu}\}$.

It is then possible to find the choice for C which minimizes L.

5 Relationship to indirect and partial control

Here we consider a special problem which from the outset is a setpoint problem

$$J = \frac{1}{2}(y_1 - y_{1s})^T W (y_1 - y_{1s}) = \frac{1}{2} e_1^T W e_1 \tag{26}$$

where $W > 0$ is a weighting matrix, and y_1 are the primary output variables. In this case the optimal cost for any disturbanc is to have $J_{opt} = 0$, so we have that the loss equals the cost, $L = J$.

Note that $y_{1,opt} = y_{1s}$, and we may write

$$e_1 = y_1 - y_{1s} = (y_1 - y_{10}) - (y_{1s} - y_{10}) = \Delta y_1 - \Delta y_{1s}$$

where we have selected the nominal operating point such that $y_{10} = y_{1s}$. To make the problem interesting we assume that the "ideal" choice of outputs $c = y_1$ can or should not be used because direct control of y_1 is difficult or impossible. We therefore instead consider controlling the secondary outputs y_2 (i.e. we choose $c = y_2$). The idea is to find a set of variables y_2, such that keeping y_2 close to the setpoint y_{2s}, indirectly achieves good control of y_1 (i.e. y_1 is kept close to y_{1s}).

The linear model relating the variables is

$$\Delta y_1 = G_1 \Delta u + G_{d1} \Delta d \tag{27}$$

$$\Delta y_2 = G_2 \Delta u + G_{d2} \Delta d \tag{28}$$

where $\Delta u = u - u_0$, etc. We assume that the nominal operating point (u_0, d_0) is optimal, i.e. $y_{10} = y_{1s}$.

1. Let us first use our derived relationships to confirm that the outputs $c = y_1$ would be ideal (this is really just a check of our derived formulas). We assume here that the setpoints y_{1s} are constant (since we assumed in the derivation above that c_s is constant), i.e. we have $\Delta y_{1s} = 0$ and $e_1 = \Delta y_1$. We get

   $$J = \frac{1}{2} \Delta y_1^T W \Delta y_1 = (G_1 \Delta u + G_{d1} \Delta d)^T W (G_1 \Delta u + G_{d1} \Delta d)$$

   and we get that

   $$J_u = (G_1 \Delta u + G_{d1} \Delta d)^T W G_1$$
   $$J_{uu} = G_1^T W G_1$$
   $$J_{ud}^T = G_1^T W G_d$$

   and from (20) we get as expected

   $$e_{opt} = c_s - c_{opt}(d) = (G_1 J_{uu}^{-1} J_{du} - G_{d1}) \Delta d = 0$$

2. Let us next consider the more interesting case of selecting $c = y_2$, where we keep the setpoints constant, $y_{2s} = y_{20}$. Rewriting the linear model gives

   $$\boxed{e_1 = \underbrace{G_1 G_2^{-1}}_{P_y} e_2 + \underbrace{(G_{d1} - G_1 G_2^{-1} G_{d2})}_{P_d} \Delta d - \Delta y_{1s}} \tag{29}$$

   where P are called the partial control gains. (To derive this we first solve (28) with respect to u

   $$\Delta u = G_2^{-1} \Delta y_2 - G_2^{-1} G_{d2} \Delta d$$

   and then substitute this into (27) and use the fact that $\Delta y_2 = e_2$ and $\Delta y_1 = e_1 + \Delta y_{1s}$ to get (29)). To minimize the cost function J we want $e_1 = y_1 - y_{1s}$ small. (29) shows how e_1 is affected by disturbances d, by the control error for the secondary variables, e_2, and by changes in the setpoints y_{1s}.

   Let us here disregard setpoint changes for the primary outputs, i.e. let $\Delta y_{1s} = 0$. If we furthermore

- scale the outputs y_1 such that $W = I$
- scale the outputs y_2 sich that the expected control error e_2 is of magnitude 1
- scale the disturbances such that the expected disturbance change Δd is of magnitude 1

then we see from (29) that to minimize e_1 (and J) we should attempt to minimize the combined norm of the matrices P_y and P_d (appropriately scaled).

This simple approach has been used on a distillation case study (Havre 1998). Here we find that we can not use temperature measurements located at the end of the column because of sensitivity to control error e_2 (measurement noise) (as seen since the scaled matrix P_y is large), and we can not use measurements close to the middle at the column yield because of sensitivity to disturbances (as seen since the scaled matrix P_d is large). The best balance between sensitivity to measurement noise and disturbances is found when the measurements are located somewhere between the end and the middle of the column.

3. An alternative form of (29) is

$$\boxed{e_1 = y_1 - y_{1s} = P_y(e_2 + e_{2,opt})} \tag{30}$$

where by definition

$$e_{2,opt} = y_{2s} - y_{2,opt}(d, y_{1s})$$

is the difference between the chosen constant setpoint for y_2, $y_{2s} = y_{20}$, and the optimal value $y_{2,opt}(d, y_{1s})$ that corrsponds to $e_1 = y_1 - y_{1s} = 0$. We may obtain $e_{2,opt}$ by setting $e_1 = 0$ in (29) and solving for e_2. We get

$$P_y e_{2,opt} = \Delta y_{1s} - P_d \Delta d$$

and substituting this into (29) gives (30).

Expression (30) is rather obvious, but it is nevertheless very useful in some cases, and forms the basis for the common rule of minimizing the minimum singular value (see below).

Comment: Another way of deriving (30) is to use (for any d)

$$y_1 - y_{1opt} = G_1(u - u_{opt})$$

$$y_2 - y_{2opt} = G_2(u - u_{opt})$$

which since $y_1 - y_{1opt} = y_1 - y_{1s} = e_1$ and $y_2 - y_{2opt} = (y_2 - y_{2s}) + (y_{2s} - y_{2opt}) = e_2 - e_{2opt}$ directly gives (30).

4. By replacing P_y and P_d by the corresponding transfer function matrices, $P_y(s)$ and $P_d(s)$, we can extend these results to nonzero frequencies.

6 Maximizing the minimum singular value (Method 2)

Let the matrix G represent the effect on the controlled variables c of a small change in the "base set" of independent variables u, i.e..

$$\Delta c = G \cdot \Delta u$$

Then, a common criterion (rule) in control structure design is to select the set of *controlled outputs* that maximizes the minimum singular value of the gain matrix, $\underline{\sigma}(G)$ (Yu and Luyben (1986) refer to this as the "Morari Resiliency Index"). Previously, this rule has had little theoretical justification, and it has not been clear how to scale the variables. However, as indicated by Skogestad and Postlethwaite (1996) the rule may be derived by considering a local approximation of the loss function.

It is desirable to select the controlled variables such that the loss is minimized. For a given disturbance d, a Taylor series expansion of the loss at the optimal point (where $u = u_{opt}(d)$) gives

$$\Delta L = J(u,d) - J(u_{opt},d) = \frac{1}{2}(u - u_{\text{opt}})^T \left(\frac{\partial^2 J}{\partial u^2}\right)_{\text{opt}} (u - u_{\text{opt}}) \tag{31}$$

(where we have assumed that the problem is unconstrained, so that the first-order term $\partial J/\partial u$ is zero.) Thus, the loss depends on the quantity $u - u_{opt}$ which we obviously want as small as possible. Now, for small deviations from the optimal operating point we have that the candidate output variables are related to the independent variables by $c - c_{opt} = G(u - u_{opt})$, or

$$u - u_{opt} = G^{-1}(c - c_{opt}) \tag{32}$$

Since we want $u - u_{opt}$ as small as possible, it therefore follows that we should select the set of controlled outputs c such that the product of G^{-1} and $c - c_{opt}$ is as small as possible. Thus, the correct statement of the rule is:

> *Assume we have scaled each output c such that the expected $c - c_{opt}$ is of magnitude 1 (including the effect of both disturbances and control error), then select the output variables c which minimize the norm of G^{-1}, which in terms of the two-norm is the same as maximizing the minimum singular value of G, $\underline{\sigma}(G)$.*

Interestingly, we note that this rule does not depend on the actual expression for the objective function J, but it does enter indirectly through the variation of c_{opt} with d, which enters into the scaling. Also note that in the multivariable case we should scale the inputs u such that the Hessian $\left(\frac{\partial^2 J}{\partial u^2}\right)$ is close to unitary (Skogestad and Postlethwaite, 1996). Also note that use of the rule may be computationally much simpler than evaluating the mean value of J or the loss function.

Example

To give a simple "toy example", let $J = (u - d)^2$ where nominally $d_0 = 0$. For this problem we always have $J_{opt}(d) = 0$ corresponding to $u_{opt}(d) = d$. Let is now consider three alternative choices for the controlled output (e.g. we can assume they are three alternative measurements)

$$c_1 = 0.1(u - d); \quad c_2 = 20u; \quad c_3 = 10u - 5d$$

For the nominal case with $d_0 = 0$ we have in all three cases that $c_{opt}(d_0) = 0$ so we select in all three cases $c_s = 0$. Since in all cases $u_{opt}(d) = d$, the optimal value of the controlled variable for the three cases are $c_{1opt}(d) = 0$, $c_{2opt}(d) = 20d$ and $c_{3opt} = 5d$.

Method 1. The losses can for this example be evaluated analytically, and we find for the three cases

$$L_1 = (10e_1)^2; \quad L_2 = (0.05e_2 - d)^2; \quad L_3 = (0.1e_3 - 0.5d)^2$$

(For example, in case 3, we have $u = (c_3 + 5d)/10$ and with $c_3 = c_{3s} + e_3 = e_3$ we get $J = (u - d)^2 = (0.1e_3 + 0.5d - d)^2$). If we further assume that the variables have been scaled such that $|d| \leq 1$ and $|e_i| \leq 1$ then the worst-case values of the losses are $L_1 = 100$, $L_2 = 1.05^2 = 1.1025$ and $L_3 = 0.6^2 = 0.36$, and we find that *output c_3 is the best overall choice for self-optimizing control*. However, with no control error c_1 is the best, and with no disturbances c_2 is the best.

Method 2. For the three choices of controlled outputs we have $G_1 = 0.1$, $G_2 = 20$ and $G_3 = 10$, and $\underline{\sigma}(G_1) = 0.1$, $\underline{\sigma}(G_2) = 20$ and $\underline{\sigma}(G_3) = 10$. This would indicate that c_2 is the best choice, but this is only correct with no disturbances. The reason for the error is that we have not scaled the output variables properly; in particular, we have not take into account the effect of the disturbances on the magnitude of $c - c_{opt}(d)$.

Let us now scale the variables properly. We have $u_{opt} = d$, so we have $c_{1,opt} = 0, c_{2,opt} = 20d$ and $c_{3,opt} = 5d$. For c_1 we then have that $|c_1 - c_{1,opt}| = 1 + 0$ (the control error is 1 plus the variation in $c_{1,opt}(d)$ due to disturbances is 0), and we find that

$$|G_1^{-1}(c_1 - c_{1,opt})| = \frac{1}{0.1} \cdot (1 + 0) = 10$$

Similarly,

$$|G_2^{-1}(c_2 - c_{2,opt})| = \frac{1}{20} \cdot (1 + 20) = 1.05$$

$$|G_3^{-1}(c_3 - c_{3,opt})| = \frac{1}{10} \cdot (1 + 5) = 0.6$$

and we find as expected that c_3 is the best choice. Thus, the two methods agree.

In general, method 1 is more accurate that method 2. The main limitation with method 2, is that for the multivariable case the particular value of $c - c_{opt}(d)$ corresponding the direction of the minimum singular value of G may not occur in practice, that is, there is no disturbance in this direction. Method 2 may therefore eliminate some viable control structures.

7 Discussion

7.1 Constraint problems

The approach outlined above may be extended to include problems with constraints,

$$\begin{array}{lll} \min\limits_{u} & J(u,d) & \\ \text{subject to} & g_1(u,d) & = 0 \\ & g_2(u,d) & \leq 0 \end{array} \tag{33}$$

Problems with equality constraints are relatively straightforward, especially if we can identify a single variable (manipulated or measured) directly related to the constraint; this should then be included as a controlled variables c ("active constraint control" (Arkun and Stephanopoulos 1980)). The main effect is then that each constraint removes a degree of freedom for the optimization. The same argument holds for inequality constraints where the optimal policy is always to keep the same constraint active (i.e. satisfy them as equalities for any disturbance).

The more difficult problems are when we have a inequality constraint which is active only under certain conditions (disturbances), and this constrained variable is not included as a controlled variable. For such cases one must be careful to avoid infeasibility during implementation, for example, there may be a disturbance such that the specified value of the controlled variable can only be achieved with a nonphysical value of the input (e.g. a negative flowrate). The on-line optimization is usually for simplicity based on the nominal disturbance (d_0), and two approaches to avoid infeasibility are then

1. to use back-offs for the controlled variables during implementation, or
2. to add safety margins to the constraints during the optimization (Narraway *et al.* (1991); Glemmestad (1997)).

Alternatively, one may solve the "robust optimization problem", where one also optimizes c_s for all the possible disturbances. A fourth, and better approach in terms of minimizing the loss, is to track the active constraint, but this requires a more complex control system. In particular, model predictive control is very well suited and much used for tracking active constraints.

7.2 Controllability issues

Of course, steady-state issues related to the cost J are not the only ones to be considered when selecting controlled outputs. It may happen that the "optimal" controlled outputs from a steady-state point of view, may result in a difficult control problem, so that dynamic control performance is poor. This may analyzed using an input-output controllability analysis. For example, in distillation column control it is well-known (Skogestad 1997) that controlling both product compositions may be difficult due to strong two-way interactions. In such cases, one may decide to control only one composition ("one-point control") and use, for example, constant reflux L/F (although this may not be optimal from a steady-state point of view). Alternatively, one may choose to over-purify the products to make the control problem easier (reducing the sensitivity to disturbances).

7.3 Why separate into optimization and control

Why is the controller decomposed? (1) The first reason is that it requires less computation. This reason may be relevant in some decision making systems where there is limited capacity for transmitting and handling information (like in most systems where humans are involved), but it does not hold in todays chemical plant where information is centralized and computing power is abundant. Two other reasons often given are (2) failure tolerance and (3) the ability of local units to act quickly to reject disturbances (e.g. Findeisen et al., 1980). These reasons may be more relevant, but, as pointed out by Skogestad and Hovd (1995) there are probably even more fundamental reasons. The most important one is probably (4) to reduce the cost involved in defining the control problem and setting up the detailed dynamic model which is required in a centralized system with no predetermined links. Also, (5) decomposed control systems are much less sensitive to model uncertainty (since they often use no explicit model). In other words, by imposing a certain control configuration, we are implicitly providing information about the behavior of the process, which we with a centralized controller would need to supply explicitly through the model.

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