Self-optimizing control:
Basis issues and Taylor series analysis

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Abstract

Which variables should we select to control? It is shown that the idea of selecting the variables that achieve "self-optimizing control" provides a link between steady-state optimization, feedback control, time scale separation and uncertainty. In summary, we will show that a good candidate variable for a controlled output should have the following properties:

1. Its optimal value should depend only weakly on disturbances.
2. On the other hand, it should be sensitive to changes in the independent variables (inputs).
3. It should be easy to control accurately.

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1 Introduction

If we formulate an optimal control problem in the usual mathematical fashion, then we generally find that a centralized solution is the optimal choice. However, in many cases we want to decompose the control system into at least two parts: a setpoint optimizer and a feedback control layer which implements the optimal setpoints. The two parts interact through the *controlled variables* c; the optimizer computes their optimal setpoints c_s, and the control layer attempts to implement then in practice, i.e. to get $c \approx c_s$. The ideal case would be if we could keep the setpoints c_s constant, thus effectively turning the optimization problem into a setpoint problem. However, this is usually not possible, but in any case we would like that the controlled variables c somehow characterize the optimal solution. This is discussed in detail in the paper.

Control structure design

More generally, the issue of selecting controlled outputs is one of the subtasks in the **control structure design** problem ((Foss 1973); (Morari 1982); (Skogestad and Postlethwaite 1996))

1. *Selection of controlled outputs* c (variables with setpoints)
2. *Selection of manipulated inputs* m
3. *Selection of measurements* v (for control purposes including stabilization)
4. *Selection of control* **configuration** (a structure interconnecting measurements/setpoints and manipulated variables, i.e. the structure of the controller K which interconnects the variables c_s and v (controller inputs) with the variables m)
5. *Selection of controller type* (control law specification, e.g., PID, decoupler, LQG, etc.).

Note that these *structural decisions* need to be made before we can start the actual design the controller. In most cases the control structure is solved by a mixture between a top-down consideration of control objectives and which degrees of freedom are available to meet these (tasks 1 and 2), combined with a bottom-up design of the control system, starting with the stabilization of the process (tasks 3,4 and 5). In most cases the problem is solved without the use of any theoretical tools.

Of course, the control field has made many advances over these years, for example, in methods for and applications of on-line optimization and predictive control. Advances has also been made in control theory and in the formulation of tools for analyzing the controllability of a plant. These latter tools can be most helpful in screening alternative control structures. However, a systematic method for generating promising alternative structures has been lacking. This is related to the fact the control structure design problem has not been well understood, has not been well defined, or even acknowledged as being an important issue.

The realization that the field of control structure design is underdeveloped is not new. In the 1970's several "critique" articles where written on the gap between theory and practice in the area of process control. The most famous is the one of (Foss 1973) who made the observation that in many areas application was *ahead* of theory, and he stated that

> The central issue to be resolved by the new theories is the determination of the control system structure. Which variables should be measured, which inputs should be manipulated and which links should be made between the two sets. ... The gap is present indeed, but contrary to the views of many, it is the theoretician who must close it.

A similar observation that applications seems to be ahead of formal theory was made by Findeisen *et al.* (1980) in their book on hierarchical systems (p. 10).

Related work

Parts of this review are based on Chapter 10 in the book of Skogestad and Postlethwaite (1996). In addition, we have made use of some unpublished work by Skogestad and coworkers on *self-optimizing control*. The latter work is planned to be published as a series of papers with the following tentative titles:

Part 1. The basic issues in self-optimizing control (selection of controlled outputs to make implementation of the optimal solution insensitive to uncertainty).

Part 2. Taylor series analysis.

Part 3. Theoretical basis for using the minimum singular value for output selection.

Part 4. Partial and indirect control with application to selection of temperature measurements in distillation.

Part 5. Constraints and feasibility.

Except for the book of Skogestad and Postlethwaite (1996), preliminary versions of the above work are available in the Ph.D. theses of Morud (1995), Glemmestad (1997) and Havre (1998), as well as in a number of conference publications. These references are available on the Internet[1].

Of the earlier work, Morari *et al.* (1980) come closest to the ideas presented above. Morari *et al.* (1980) write that *in attempting to synthesize a feedback optimizing control structure, our main objective is to translate the economic objectives into process control objectives. In other words, we want to find a function c of the process variables which when held constant, leads automatically to the optimal adjustments of the manipulated variables, and with it, the optimal operating conditions. ... This means that by keeping the function $c(u, d)$ at the setpoint c_s, through the use of the manipulated variables u, for various disturbances d, it follows uniquely that the process is operating at the optimal steady-state $J = J_{\text{opt}}$.* This is a precise description of the best self-optimizing control structure, except that they do not consider the effect of implementation error $e = c - c_s$. Unfortunately, it seems that very few people, including the authors themselves, have picked up on the idea.

Although at first sight it may seem quite different, another related work is that of Shinnar (1981) and (Arbel *et al.* 1996). Maarleveld and Rijnsdrop (1970) state that the steady-state optimum usually is constrained, and that we therefore we should control that variable. Arkun and Stephanopoulos (1980) reach the same conclusion and provide a good discussion on the advantages of active constraint control. Luyben and coworkers (e.g. Luyben (1975), Yi and Luyben (1995), Luyben (1988)) have studied unconstrained problems, and some of the examples presented point in the direction of the selection methods presented in this paper. However, Luyben proposes to select controlled ouputs which minimizes the steady-state sensitive of the inputs (m) to disturbances, i.e. to select controlled outputs (c) such that $(\partial m/\partial d)_c$ is small, whereas we really want to minimize the steady-state sensitivity of the economic loss (L) to disturbances, i.e. to select controlled outputs (c) such that $(\partial L/\partial d)_c$ is small. Fisher *et al.* (1988) discuss plant economics in relation to control. Narraway and Perkins ((Narraway *et al.* 1991), (Narraway and Perkins 1993) and (Narraway and Perkins 1994)) strongly stess the need to base the selection of the control structure on economics. Finally, Marlin and Hrymak (1997) stress the need to find a good way of implementing the optimal solution in terms how the control system should respond to disturbances, "i.e. the key constraints to remain active, variables to be maximized or minimized, priority for adjusting manipualted variables, and so forth." They suggest that an issue for improvement in today's real-time optimization systems is to select the control system that yields the highest profit for a range of disturbances that occur between each execution of the optimization.

For a more detailed literature review and a more precise defintion of terms, the reader is referred to the following internal report

> S. Skogestad and T. Larsson, "A review of plantwide control". Available at:
> http://www.chembio.ntnu.no/users/skoge/transfer/plantwide_review2.ps

[1] http://www.chembio.ntnu.no/users/skoge/

2 Optimization and control

The focus on this paper on selection of controlled outputs (task 1), which is probably the least studied of the structural decisions. But ask the question:

> Why are we controlling hundreds of temperatures, pressures and compositions in a chemical plant, when there is no specification on most of these variables? Is it just because we can measure them or is there some deeper reason?

To answer this problem we need think more carefully about why we do control. First, there is the issue of stabilization and then of keeping the operation within given constraints. These issues may consume some degrees of freedom (e.g. to stabilize levels with no steady-state effect and to satisfy product specifications), but there will genereally be many degrees of freedom left. What should these be used for?

Loosely speaking, they should be used to "optimize the operation". There may be many issues involved, and to trade them off against each other in a systematic manner we usually quantify a scalar performance index J which should be minimized. In many cases this performance index is an economic measure, e.g. the operation cost. Since the economics of plant operation usually are determined mainly by steady-state issues, the analysis of how to use the remaining degrees of freedom can often be based on steady-state considerations, and their optimal values may be found using steady-state optimization. The resulting optimization problem may be very large, with hundreds of thousands of equations, and hundreds of degrees of freedom. However, with todays computers and optimization methods this problem is easily solvable, and it is indeed solved routinely in some plants, such as ethylene plants.

However, it is often much less clear how the optimal solution should actually be implemented in practice. Three alternative solutions are shown in Figure 1:

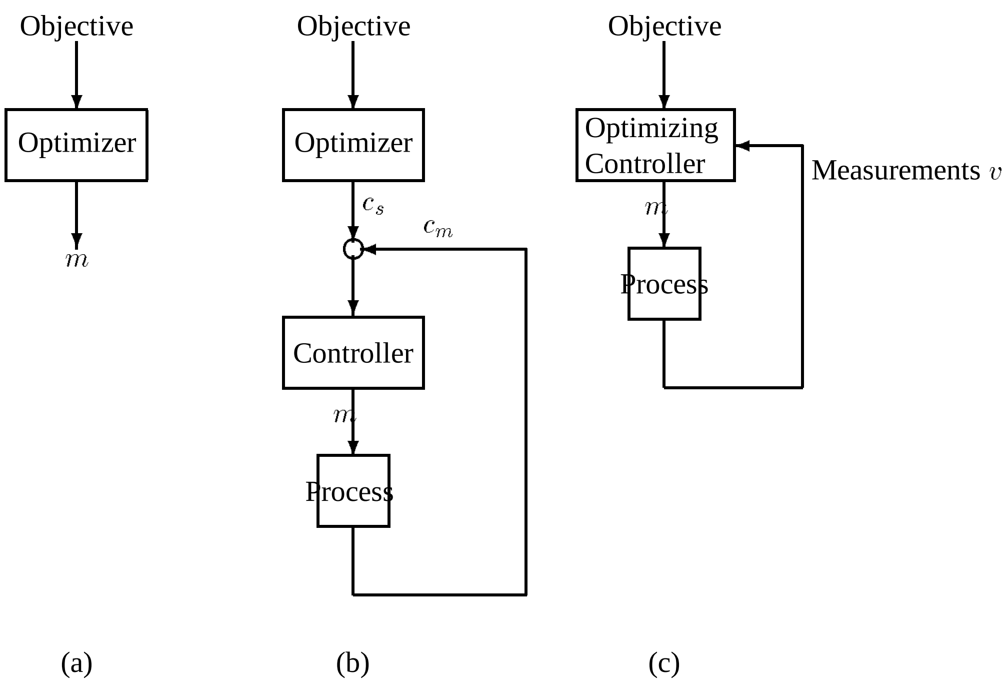

Figure 1: Alternative structures for optimization and control.

(a) Open-loop optimization.

(b) Closed-loop implementation with a separate control layer.

(c) Integrated optimization and control.

It should be stressed that in all the cases "optimization" may be performed manually be operators or engineers.

The open-loop implementation (a) can generally not be used because of sensitivity to uncertainty.

From a theoretical point of view, the centralized scheme in (c) should be the best implementation. Here, the optimizing controller stabilizes the process and at the same time perfectly coordinates all the manipulated inputs based on dynamic on-line optimization. However, there are fundamental reasons why such a solution is not the best, even with todays and tomorrows computing power. The main reason is probably the cost of modeling; in the centralized controller there are no predetermined links, so the controller must rely only on the model to take the right action. On the other hand, if we use local controllers (which use only a subset of the measurements and manipualtors), then the task of each controller is well-defined (e.g. keep the temperature at its setpoint) and we can often tune the controllers with a minimum of modelling efforts. In fact, by cascading feedback loops, it is possible to control large plants with thousands of variables without the actual need to develop any models.

In any case, we find that in practice the hierarchical feedback implementation (b) is preferred. It consists of

- *optimization layer* — computes setpoints c_s for the controlled variables c
- *control layer* — implements this in practice, with the aim of achieving $c \approx c_s$.

In process control applications, the optimization layer may typically recomputes new setpoints c_s only every hour or so, whereas the feedback layer operates continuously. However, the data and model used by the optimizer are uncertain and there are disturbances entering the plant between each re-optimization, and the objective of the feedback layer is therefore to keep the plant close to its optimal operating point in spite of this uncertainty.

The resulting control system is usually divided into more than an optimization and a control layer. Typically, layers include include scheduling (weeks), site-wide optimization (day), local optimization (hour), supervisory/predictive control (minutes) and regulatory control (seconds); see Figure 2.

Why do we select a particular set c of controlled variables? (e.g., why specify (control) the top composition in a distillation column, which does not produce final products, rather than just specifying its reflux?) The answer to this question is not obvious, because at first it seems like it does not really matter which variables we specify (as long as all degrees of freedom are consumed, because the remaining variables are then uniquely determined). However, this is true only when there is no uncertainty caused by disturbances and noise (signal uncertainty) or model uncertainty. When there is uncertainty then it does make a difference how the solution is implemented, that is, which variables we select to control at their setpoints.

We also stress that the analysis below is based on steady-state considerations. The main justification for this is that, under the assumption that reasonable control performance can be achieved, the economic performance is mainly determined by steady-state considerations. Of course, one could extend the analysis on a frequency-by-frequency basis, and include in the variable e_c information about how well a variable can be controlled at a given frequency. However, this would complicate the analysis, and should therefore be used only when needed, and is not considered any further here.

3 Selecting controlled variables: Mathematical formulation

3.1 The performance index (cost) J

We assume that the optimal operation problem can be quantified in terms of a scaler performance index (cost) J, such that the objective of the operation is to to be minimize J with respect to the available degrees of freedom. J may be a purely economic objective, but is more generally a weighted sum of the various control objectives. For the optimization itself it does not really matter which variables we use as degrees of freedom as long as they form an independent set. Let the "base set" for the degrees of freedom be denoted u (these may consist, for example, of a subset the physical manipulators m). In addition, the cost will depend on the unknown disturbances d (which here is assumed to include uncertainty in the

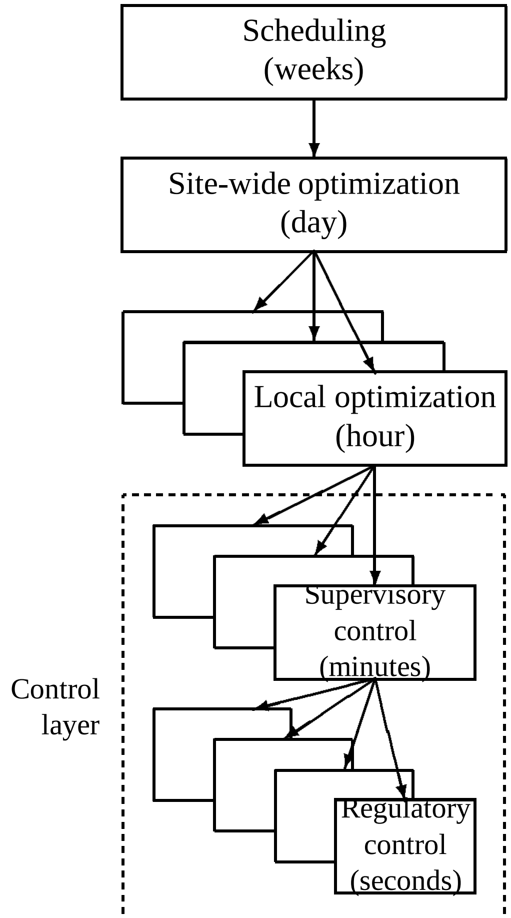

Figure 2: Typical control hierarchy in a chemical plant.

model and uncertainty in the optimizer). We can then write $J(u, d)$. The nominal value of the disturbances is denoted d_0, and we can solve the nominal operating problem and obtain $u_{opt}(d_0)$ for which

$$min_u J(u, d_0) = J_{opt}(d_0) = J(u_{opt}(d_0), d_0)$$

From this we can obtain a table with the corresponding optimal value of any other dependent variable, including $c_{opt}(d_0)$.

The issue is now to decide how to best implement the optimal policy in the presence of uncertainty by selecting the right set of controlled variables c with constants setpoints $c_s = c_{opt}(d_0)$.

It is assumed that the number of controlled variables y equals the number of independent variables u, or more exactly that we starting from $c = f(u, d)$ can derive the inverse relationship

$$u = f^{-1}(c, d)$$

where the function f^{-1} exists and is unique.

The scalar objective function to be minimized can then be written

$$J = J(u, d) = J(c, d) \tag{1}$$

(This is an abuse of notation; it would be more correct to write $J = J_1(u, d) = J_2(c, d)$ and we will do this sometimes if needed to avoid ambiguity). We further have that

$$J_{opt}(d) = min_u J_1(u, d) = J_1(u_{opt}(d), d) = \min_c J_2(c, d) = J_2(c_{opt}(d), d)$$

We assume that the controlled variables can be controlled within accuracy e (where e is at least as large as the measurement noise for the variable c). Then the set of variables c we are looking for are the ones which minimize some mean value of the performance index

$$J(\underbrace{c_s + e}_{c}, d)$$

for the expected set of disturbances $d \in \mathcal{D}$, and expected set of control error $e \in \mathcal{E}$.

Instead of evaluating the mean value of the performance index, it may be better to evaluate the always positive loss function. The loss function expresses the difference between the actual operating costs (obtained when we adjust u in order to keep $c = c_{opt}(d_0) + e$) and the optimal operating cost (obtained with $u = u_{opt}(d)$),

$$L(u, d) = J(u, d) - J_{opt}(d) \tag{2}$$

The objective of the operation is to minimize J (or some average of J), or equivalently to minimize the loss L. The loss function is zero if we use the optimal policy $u = u_{opt}(d)$. The loss has the advantage of providing a better "absolute scale" on which to judge whether a given set of controlled variables c is "good enough", and thus is self-optimizing.

3.2 Open-loop implementation

Let us first consider an open-loop implementation where we attempt to keep u constant at the value u_s. With this implementation the operation may be non-optimal (with a positive loss) due to the following reasons

1. The value of u_s is different from the optimal value $u_{opt}(d)$.
2. The actual value of u is different from u_s (due to an implementation error caused by imperfect control).

This can be seen more clearly if we write the actual input as

$$u = u_s + \underbrace{u - u_s}_{e_u} \tag{3}$$

where e_u is the implementation error for u. In process control, u is usually a flowrate, and it is diffuclt in practice to obtain exactly the desired flow u_s. This error may be reduced in some cases if we measure the variable u and implement an inner control loop with setpoint u_s. However, also in this case there will be a control and a measurement error; if we use integral action then at steady-state e_u will equal the steady-state measurement error (noise n).

The difference between the actual and optimal inputs, which causes a positive loss, can then be written

$$u - u_{opt}(d) = u_s - u_{opt}(d) + e_u = e_{u,opt}(d) + e_u \tag{4}$$

i.e. it is the sum of the optimization error $e_{u,opt}(d) = u_s - u_{opt}(d)$ and the control error e_u.

The open-loop policy is often poor; both because the optimal input value often depends strongly on the disturbance (so e_{opt} is large), and because we are not able to implement u accurately (so e_u is large).

The solution to the problem is to find some measurements which characterizes the optimality of operation, and apply feedback.

As already mentioned, in theory, the optimal solution would be to use some "optimizing controller" which uses the measurements to correct the value of u, and at the same uses the measurement information to correct the model (estimate d) and compute a new optimal value $u_{opt}(d)$. The main problem with this approach is the modelleing effort, and the lack of theoretical tools to ensure robustness (insensitivity to uncertainty).

In practice, a simpler solution solution is preferred if it yields acceptable operation. This is to use directly the measurements c_m of the selected controlled variables in adjust u in an inner feedback loop to achieve $c_m \approx c_s$, Here the optimal setpoint c_s (which is kept constant) is found by solving the same optimization problem that resulted in c_s, e.g. $c_s = c_{opt}(d_0)$ if we base the optimization on the nominal operating point. The idea is obviously that by keeping $c_m \approx c_s$ we achieve an operation where $u - u_{opt}(d)$ is smaller than for the open-loop policy. This may happen because $c_{opt}(d)$ is relatively insensitive to d and/or because c may more accurately controlled. We next formalize these ideas more exactly.

3.3 Closed-loop implementation

We here rewrite the problem with the variables c as independent variables rather than the original independent variables (inputs) u. However, note that we as a special case may choose $c = u$, or some of the elements in the vector y may be the original input variables. Thus, the open-loop implementation is included as a special case.

If we compare the open-loop and closed-loop policies then the question is: Is it best adjust the input variables u such that $u = u_s + e_u$, or is it better to adjust the input variables u such that $u = f^{-1}(c, d)$ where $c = c_s + e_c$ (where e_c is the implementation error for control of c) ?

More generally, if there are many alternative sets of variables c which can be measured and controlled, which set should be used? If we let y_m represent all the candidate measured variables then we can write

$$c = C \begin{pmatrix} y_m \\ u \end{pmatrix} \tag{5}$$

The issue is then to find the optimal choice for the matrix C, but under the restriction that the number of controlled variables (c's) equals the number of independent inputs (u's). Note that we often impose restrictions on C, e.g. that C is a "selection matrix" with only one nonzero element (a 1) in each row.

To compare the alternative choices we then evaluate the objective function, or equivalently the loss function, for alternative values of the disturbance d and the implementation error e_c. **The optimal choice for of controlled variables** c (i.e. optimal choice of the matrix C) is then the one that minimizes some average value of the loss

$$L(u, d) L(f^{-1}(c_s + e_c, d), d) \tag{6}$$

for the expected set of disturbances $d \in \mathcal{D}$, and expected set of implementation (control) errors $e_c \in \mathcal{E}$. In the simplest case we select the setpoints as $c_s = c_{opt}(d_0)$, but the value of c_s may also be the subject to an optimization.

The difference between the actual and optimal outputs, which causes a positive loss, can be written

$$c - c_{opt}(d) = c_s - c_{opt}(d) + e = e_{opt}(d) + e \tag{7}$$

i.e. it is the sum of the optimization error $e_{opt}(d) = c_s - c_{opt}(d)$ and the control error e.

As already mentioned, if there where no uncertainty (i.e. $d = d_0$ and $e_c = 0$), then it would make no difference which variable c that was chosen.

4 Taylor series analysis

In this section we study the problem of selecting controlled outputs by considering a second order accurate expansion of the cost function. To this end, we assume the cost function J is smooth, or more precicely twice differentiable, around the point we are considering.

We assume that the nominal disturbance is d_0 and that the nominal input is optimal,

$$u_0 = u_{opt}(d_0)$$

so that we have $J(u_0, d_0) = J_{opt}$. We next consider a disturbance and input change so that the new disturbance is

$$d = d_0 + \Delta d$$

and the new input is

$$u = u_0 + \Delta u$$

where Δu is the input change. The input u will generally be different from the optimal input, $u_{opt}(d)$, and we define the difference as

$$\Delta u' = u - u_{opt}(d)$$

4.1 Expansion of the cost function

A second order Taylor series expansion of the cost function gives

$$\begin{aligned} J(u,d) &= J(u_0,d_0) + J_u^T(u-u_0) + J_d^T(d-d_0) + \frac{1}{2}(u-u_0)^T J_{uu}(u-u_0) \\ &\quad + \frac{1}{2}(d-d_0)^T J_{dd}(d-d_0) + (d-d_0)^T J_{ud}(u-u_0) + \mathcal{O}^{\ni} \end{aligned} \quad (8)$$

where all derivatives are evaluated at the optimal nominal operating point denoted wuth the subscript $_0$ (with $d = d_0$ and $u = u_0 = u_{opt}(d_0)$),

$$J_u = \left(\frac{\partial J}{\partial u}\right)_0 = 0$$

$$J_d = \left(\frac{\partial J}{\partial d}\right)_0$$

$$J_{uu} = \left(\frac{\partial^2 J}{\partial u^2}\right)_0$$

$$J_{dd} = \left(\frac{\partial^2 J}{\partial d^2}\right)_0$$

$$J_{ud} = \left(\frac{\partial^2 J}{\partial u \partial d}\right)_0$$

Note that $J_u = 0$ because the Jacobian with respect to the independent variables must be zero at the optimum. We can write the expansion in (8i)more compactly as

$$J(u,d) = J(u_0,d_0) + \begin{pmatrix} \underbrace{J_u}_{=0} & J_d \end{pmatrix}^T \begin{pmatrix} \Delta u \\ \Delta d \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \Delta u \\ \Delta d \end{pmatrix}^T H \begin{pmatrix} \Delta u \\ \Delta d \end{pmatrix} \quad (9)$$

where H is the Hessian matrix of J with respect to $\begin{pmatrix} \Delta u \\ \Delta d \end{pmatrix}^T$,

$$H = \begin{pmatrix} J_{uu} & J_{ud} \\ J_{du} & J_{dd} \end{pmatrix}$$

Note that the Hessian is symmetric, so J_{uu} and J_{dd} are symmetric and $J_{ud} = J_{du}^T$.

4.2 The optimal input

The nominal operating point (u_0, d_0) is assumed to be optimal se we have $u_0 = u_{opt}(d_0)$, and we must have that the Jacobian is zero,

$$J_u = \frac{\partial J}{\partial u}(u_0, d_0) = 0$$

Next, consider a disturbance and input change so that the new operating point is (u, d) and the new Jacobian is

$$J_u' = \frac{\partial J}{\partial u}(u, d)$$

An first-order expansion of the Jacobian gives

$$J'_u = J_u + J_{uu}(u - u_0) + J_{ud}^T(d - d_0)$$

If also the new operating point is optimal, i.e. $u = u_{opt}(d)$, then we must also here have that the Jacobian is zero, i.e. $J'_u = 0$, so we get to the first-order approximation

$$0 = J_{uu}(u_{opt}(d) - u_{opt}(d_0)) + J_{ud}^T(d - d_0)$$

and we find that the optimal input is

$$\boxed{u_{opt}(d) = \underbrace{u_{opt}(d_0)}_{u_0} - J_{uu}^{-1} J_{du}(d - d_0)} \tag{10}$$

4.3 Expansion of the loss function

Let us now consider the loss function

$$L(u, d) = J(u, d) - J(u_{opt}(d), d)$$

Expand $J(u, d)$ around the perturbed point $(u_{opt}(d), d)$ (which is close to but not the same as the nominal point $(u_{opt}(d_0), d_0)$. We get

$$J(u, d) \approx J(u_{opt}(d), d) + {J'_u}^T(u - u_{opt}(d)) + +\frac{1}{2}(u - u_{opt}(d))^T J'_{uu}(u - u_{opt}(d)) \tag{11}$$

where the $'$ denotes that the derivatives are evaluated at the perturbed point. However, it turns out this does not matter. First, since also the perturbed point is optimal, we must have that $J'_u = 0$. Second, J'_{uu} can be expanded in terms of J_{uu},

$$J'_{uu} = J_{uu} + J_{uud}^T(d - d_0) + J_{uuu}^T(u_{opt}(d) - u_{opt}(d_0))$$

but the resulting third order terms can be neglected upon substitution into (11), which finally gives the following second order accurate expansion for the loss function in terms of the optimization error $\Delta u' = u - u_{opt}(d)$,

$$\boxed{L(u, d) = \frac{1}{2}(u - u_{opt}(d))^T J_{uu}(u - u_{opt}(d)) = \Delta u'^T J_{uu} \Delta u'} \tag{12}$$

Comment. To confirm that the approach taken in the last derivation is acceptable, we shall rederive (8) by expanding in only "one" variable (u or d) at a time. Let here a double prime ($''$) denote that the derivative is evaluated at the point (u_0, d). We then have

$$J(u, d) = J(u_0, d) + {J''_u}^T(u - u_0) + \frac{1}{2}(u - u_0)^T J''_{uu}(u - u_0) \tag{13}$$

We further expand the terms which were not at the nominal operating point,

$$J(u_0, d) \approx J(u_0, d_0) + J_u^T(d - d_0) + \frac{1}{2}(d - d_0)^T J_{dd}(d - d_0)$$

$$J''_u \approx J_u + J_{ud}^T(d - d_0)$$

and with $J''_{uu} \approx J_{uu}$ and substituting into (13) we rederive (8).

4.4 Loss with constant inputs

Assume there is a disturbance change, but we attempt to keep the input fixed at its nominally optimal value u_0, i.e.

$$u_s = u_0$$

where $u_0 = u_{opt}(d_0)$. We use the word "attempt", since in practice there will be an implementation error so the actual input will be

$$u = u_0 + e_u$$

where e_u is the implementation error (i.e., we have in this case $\Delta u = e_u$). From (10)

$$\boxed{\Delta u' = u - u_{opt} = J_{uu}^{-1} J_{du} \Delta d + e_u} \tag{14}$$

and from (12) the loss is

$$L(u_0, d) = \frac{1}{2} \Delta u'^T J_{uu} \Delta u') \tag{15}$$

4.5 Loss with constant controlled outputs

The outputs c are related to the inputs and disturbances by the relationship

$$c = f(u, d)$$

For a small disturbance and input change, the corresponding output change $\Delta c = c - c_0$ is given by the linearized model

$$\Delta c = G \Delta u + G_d \Delta d \tag{16}$$

where $G = (\partial f / \partial u)^T$ and $G_d = (\partial f / \partial d)^T$.

We here consider attemting to keep the outputs constant at their nominal value c_0, i.e.

$$c_s = c_0$$

where $c_0 = c_{opt}(d_0)$. We use the word "attempt", because, in practice, there will be an implementation error so the actual controlled output will be

$$c = c_0 + e \tag{17}$$

where e is the implementation error (typically, the sum of the measurement error and the control error). Note that we in this case have that $\Delta c = e$. From (16) the input change required to keep $c = c_0 + e$ where there is a disturbance change Δd is given by

$$\Delta u = u - u_0 = -G^{-1} G_d \Delta d + G^{-1} e$$

From (10) the resulting difference between the actual and optimal input is

$$\boxed{\Delta u' = u - u_{opt} = \left(-G^{-1} G_d + J_{uu}^{-1} J_{du} \right) \Delta d + G^{-1} e} \tag{18}$$

and from (12) the resulting loss is

$$L(u_0, d) = \frac{1}{2} \Delta u'^T J_{uu} \Delta u')$$

The optimal choice for the controlled outputs is then one that minimizes the value of the loss $L(u, d)$ for the expected disturbances (as expressed by the magnitude Δd) and the expected control error (as expressed by the magnitude of e). Note that the matrix J_{uu} is independent of the choice of the controlled output.

4.5.1 Alternative form

An alternative form is to express the loss directly in terms of the output. A similar derivation as for the inputs, see (12), gives

$$L(c,d) = \frac{1}{2}(c - c_{opt}(d))^T J_{cc}(c - c_{opt}(d)) = \frac{1}{2}\Delta c'^T J_{cc}\Delta c' \tag{19}$$

With d constant the linearized model yields $\Delta c' = G\Delta u'$, and we find

$$J_{cc} = G^{-1^T} J_{uu} G^{-1}$$

and we see clearly that J_{cc} depends on the choice of the controlled outputs c (to keep it small we want G^{-1} small). In (19) $\Delta c'$ is the difference between the actual and optimal output

$$\Delta c' = c - c_{opt}(d)$$

It will be nonzero due to the presence of two generally independent terms

$$\Delta c' = e_{opt} + e$$

where $e_{opt} = c_0 - c_{opt}(d)$ is the optimization error (introduced by attempting to keep c constant at c_0) and $e = c - c_0$ is the implementation error (introduced by incorrect measurement and poor control of c).

We may also express the optimization error directly in terms of the disturbance. From the linearized model in (16)

$$-e_{opt} = \underbrace{c_{opt}(d) - c_0}_{\Delta c} = G\,\underbrace{(u_{opt}(d) - u_0)}_{\Delta u} + G_d\Delta d$$

where from (10)

$$u_{opt}(d) - u_0 = -J_{uu}^{-1} J_{du}\underbrace{(d - d_0)}_{\Delta d}$$

and we find

$$\boxed{e_{opt}(d) = c_0 - c_{opt}(d) = \left(GJ_{uu}^{-1}J_{du} - G_d\right)\ \Delta d} \tag{20}$$

We will return to this expression shortly.

4.6 Ideal choice of controlled outputs

If we for the moment disregard the control error e, then the ideal choice of controlled outputs would be to have $e_{opt} = 0$, that is, we want the optimal value of output to be independent of the disturbance. An example of such a kind of ideal output would be to have a direct measurement of the gradient of the cost function (since it is optimal to have this gradient zero and we could then directly specify this as a setpoint). In particular, the following output satisfies this

$$c = f(u,d) = c_1\frac{\partial J(u,d)}{\partial d} + c_0 = c_1 J_u + c_0 \tag{21}$$

where c_1 is a constant. To see this, we linearize (21) to get

$$\Delta c = J_{uu}\Delta u + J_{ud}^T\Delta d$$

i.e. we have $G = J_{uu}$ and $G_d = J_{ud}^T = J_{du}$, which upon substitution into (20) gives $e_{opt} = 0$.

However, as we see when studying selection of measurement location in a distillation column, the implementation error may be a very important factor, and the “ideal” output may not be the best after all.

4.7 Relationship to partial control

Here we consider a problem which from the outset is a setpoint problem

$$J = \frac{1}{2}(y_1 - y_{1s})^T W (y_1 - y_{1s}) \tag{22}$$

where $W > 0$ is a weighting matrix, and y_{1s} is constant and equal to y_0 so we have that

$$\Delta y = y_1 - y_{1s}$$

To make the problem interestinmg we assume that the "ideal" choice of outputs $c = y_1$ can not be used because direct control of y_1 is difficult or impossible. We therefore consider controlling secondary outputs y_2 (i.e. $c = y_2$).

The linear model relating the variables is

$$\Delta y_1 = G_1 \Delta u + G_{d1} \Delta d \tag{23}$$

$$\Delta y_2 = G_2 \Delta u + G_{d2} \Delta d \tag{24}$$

where $\Delta u = u - u_0$, etc. We assume that the nominal operating point (u_0, d_0) is optimal, i.e. $y_{10} = y_{1s}$.

1. Let us first use our derived relationships to confirm that the outputs $c = y_1$ would be ideal. Write

   $$J = \frac{1}{2}\Delta y_1^T W \Delta y_1 = (G_1 \Delta u + G_{d1}\Delta d)^T W (G_1 \Delta u + G_{d1} \Delta d)$$

   and we get that

   $$J_u = (G_1 \Delta u + G_{d1} \Delta d)^T W G_1$$
   $$J_{uu} = G_1^T W G_1$$
   $$J_{ud}^T = G_1^T W G_d$$

   and from (20) we get as expected

   $$e_{opt} = c_0 - c_{opt}(d) = (G_1 J_{uu}^{-1} J_{du} - G_{d1})\Delta d = 0$$

2. Let us next consider selecting $c = y_2$. Rewriting the linear model gives

   $$\boxed{\Delta y_1 = \underbrace{G_1 G_2^{-1}}_{P_y} \Delta y_2 + \underbrace{(G_{d1} - G_1 G_2^{-1} G_{d2})}_{P_d} \Delta d} \tag{25}$$

   where P are called the partial control gains. To derive this we first solve (24) with respect to u

   $$\Delta u = G_2^{-1} \Delta y_2 - G_2^{-1} G_{d2} \Delta d$$

   and subsitute this into (23) to get (25). Furthermore, we have

   $$\Delta y_2 = y_2 - y_{20} = e_2$$

   To minimize the cost function J we want $\|\Delta y_1\|$ small. If we have scaled the outputs y_1 such that $W = I$, the ouutputs y_2 sich that the expected control error e_2 is of magnitude 1, and scaled the disturbances such that the expected disturbance change Δd is of magnitude 1, then we see from (25) that we should attempt to minimize the combined norm of the matrices P_y and $P - d$ (appropriately scaled).

   This simple approach has been used on a distillation case study (Havre 1998). Her we find that we can use temperature measurements located at the end of the column because of sensitivity to control error e_2 (measurement noise) (as seen since the scaled matrix P_y is large), whereas measurements closer to the middle at the column yield sensitibity to disturbances (as seen since the scaled matrix P_d is large). The best balance between sensitivity to measurement noise and disturbances is found when the measurements are located

   The approach outlined above involving evaluating the matrices $P - y$ and P_d can also be extended to nonzero frequencies.

5 Constraint problems

The approach outlined above may be extended to include problems with constraints,

$$\begin{array}{lll} \min\limits_{u} & J(u,d) & \\ \text{subject to} & g_1(u,d) & = 0 \\ & g_2(u,d) & \leq 0 \end{array} \tag{26}$$

Problems with equality constraints are relatively straightforward, especially if we can identify a single variable (manipulated or measured) directly related to the constraint; this should then be included as a controlled variables c ("active constraint control" (Arkun and Stephanopoulos 1980)). The main effect is then that each constraint removes a degree of freedom for the optimization. The same argument holds for inequality constraints where the optimal policy is always to keep them active (i.e. satisfy them as equalities for any disturbance).

The more difficult problems are when we have a inequality constraint which is active only under certain conditions (disturbances), and we have chosen not to use this constrained variable as a controlled variable. For such cases one must be careful to avoid infeasibility during implementation, e.g., there may be a disturbance such that the specified value of the controlled variable can only be achieved with a negative flowrate. The on-line optimization is usually for simplicity based on the nominal disturbance (d_0), and two approaches to avoid infeasibility are (1) to use back-offs for the controlled variables during implementation, or (2) to add safety margins to the constraints during the optimization (Narraway *et al.* (1991); Glemmestad (1997)). Alternatively, one may solve the "robust optimization problem", where on also optimizes c_s for all the possible disturbances. A different approach is to track the active constraint. In particular, model predictive control is very well suited and much used for tracking active constraints.

6 Method 2: Maximizing the minimum singular value

Let the matrix G represent the effect of a small change in the "base set" of independent variables (u; often the manipulated inputs) on the selected set of controlled variables (c), i.e.

$$\Delta c = G \cdot \Delta u$$

Then, a common criterion (rule) in control structure design is to select the set of outputs which maximizes the minimum singular value of the gain matrix, $\underline{\sigma}(G)$ (Yu and Luyben (1986) refer to this as the "Morari Resiliency Index") Previously, this rule has had little theoretical justification, and it has not been clear how to scale the variables. However, as shown by Skogestad and Postlethwaite (1996) the rule may be derived by considering a local approximation of the loss function.

It is desirable to select the controlled variables such that the loss is minimized. For a given disturbance d, a Taylor series expansion of the loss around the optimal value $u_{opt}(d)$ gives

$$\Delta L = J(u,d) - J(u_{opt},d) = \frac{1}{2}(u - u_{\text{opt}})^T \left(\frac{\partial^2 J}{\partial u^2}\right)_{\text{opt}} (u - u_{\text{opt}}) \tag{27}$$

(where we have assumed that the problem is unconstrained, so that the first-order term $\partial J/\partial u$ is zero.) Thus, the loss depends on the quantity $u - u_{opt}$ which we obviously want as small as possible. Now, for small deviations from the optimal operating point we have that the candidate output variables are related to the independent variables by $c - c_{opt} = G(u - u_{opt})$, or

$$u - u_{opt} = G^{-1}(c - c_{opt}) \tag{28}$$

Since we want $u - u_{opt}$ as small as possible, it therefore follows that we should select the set of controlled outputs c such that the product of G^{-1} and $c - c_{opt}$ is as small as possible. Thus, the correct statement of the rule is:

Assume we have scaled each output c such that the expected $c - c_{opt}$ is of magnitude 1 (including the effect of both disturbances and control error), then select the output variables c which minimize the norm of G^{-1}, which in terms of the two-norm is the same as maximizing the minimum singular value of G, $\underline{\sigma}(G)$.

Interestingly, we note that this rule does not depend on the actual expression for the objective function J, but it does enter indirectly through the variation of c_{opt} with d, which enters into the scaling. Also note that in the multivariable case we should scale the inputs u such that the Hessian $\left(\frac{\partial^2 J}{\partial u^2}\right)$ is close to unitary; see Skogestad and Postlethwaite (1996) for details. Also note that use of the rule may be computationally much simpler than evaluating the mean value of J or the loss function.

Example

To give a simple "toy example", let $J = (u - d)^2$ where nominally $d_0 = 0$. For this problem we always have $J_{opt}(d) = 0$ corresponding to $u_{opt}(d) = d$. Let is now consider three alternative choices for the controlled output (e.g. we can assume they are three alternative measurements)

$$c_1 = 0.1(u - d); \quad c_2 = 20u; \quad c_3 = 10u - 5d$$

For the nominal case with $d_0 = 0$ we have in all three cases that $c_{opt}(d_0) = 0$ so we select in all three cases $c_s = 0$. Since in all cases $u_{opt}(d) = d$, the optimal value of the controlled for the three cases are $c_{1opt}(d) = 0$, $c_{2opt}(d) = 20d$ and $c_{3opt} = 5d$.

Method 1. The losses can for this example be evaluated analytically, and we find for the three cases

$$L_1 = (10e_1)^2; \quad L_2 = (0.05e_2 - d)^2; \quad L_3 = (0.1e_3 - 0.5d)^2$$

(For example, in case 3, we have $u = (c_3 + 5d)/10$ and with $c_3 = c_{3s} + e_3 = e_3$ we get $J = (u - d)^2 = (0.1e_3 + 0.5d - d)^2$). If we further assume that the variables have been scaled such that $|d| \leq 1$ and $|e_i| \leq 1$ then the worst-case values of the losses are $L_1 = 100$, $L_2 = 1.05^2 = 1.1025$ and $L_3 = 0.6^2 = 0.36$, and we find that *output c_3 is the best overall choice for self-optimizing control*. However, with no control error c_1 is the best, and with no disturbances c_2 is the best.

Method 2. For the three choices of controlled outputs we have $G_1 = 0.1, G_2 = 20$ and $G_3 = 10$, and $\underline{\sigma}(G_1) = 0.1$, $\underline{\sigma}(G_2) = 20$ and $\underline{\sigma}(G_3) = 10$. This would indicate that c_2 is the best choice, but this is only correct with no disturbances. The reason for the error is that we have not scaled the output variables properly; in particular, we have not take into account the effect of the disturbances on the magnitude of $c - c_{opt}(d)$.

Let us now scale the variables properly. We have $u_{opt} = d$, so we have $c_{1,opt} = 0, c_{2,opt} = 20d$ and $c_{3,opt} = 5d$. For c_1 we then have that $|c_1 - c_{1,opt}| = 1 + 0$ (the control error is 1 plus the variation in $c_{1,opt}(d)$ due to disturbances is 0), and we find that

$$|G_1^{-1}(c_1 - c_{1,opt})| = \frac{1}{0.1} \cdot (1 + 0) = 10$$

Similarly,

$$|G_2^{-1}(c_2 - c_{2,opt})| = \frac{1}{20} \cdot (1 + 20) = 1.05$$

$$|G_3^{-1}(c_3 - c_{3,opt})| = \frac{1}{10} \cdot (1 + 5) = 0.6$$

and we find as expected that c_3 is the best choice. Thus, the two methods agree.

In general, method 1 is more accurate that method 2. The main limitation with method 2, is that for the multivariable case the particular value of $c - c_{opt}(d)$ corresponding the direction of the minimum singular value of G may not occur in practice, that is, there is no disturbance in this direction. Method 2 may therefore eliminate some viable control structures.

7 Discussion

7.1 Controllability issues

Of course, steady-state issues related to the cost J are not the only ones to be considered when selecting controlled outputs. It may happen that the "optimal" controlled outputs from a steady-state point of view, may result in a difficult control problem, so that dynamic control performance is poor. This may analyzed using an input-output controllability analysis. For example, in distillation column control it is well-known (Skogestad 1997) that controlling both product compositions may be difficult due to strong two-way interactions. In such cases, one may decide to control only one composition ("one-point control") and use, for example, constant reflux L/F (although this may not be optimal from a steady-state point of view). Alternatively, one may choose to over-purify the products to make the control problem easier (reducing the sensitivity to disturbances).

7.2 Why separate into optization and control

Why is the controller decomposed? (1) The first reason is that it requires less computation. This reason may be relevant in some decision making systems where there is limited capacity for transmitting and handling information (like in most systems where humans are involved), but it does not hold in todays chemical plant where information is centralized and computing power is abundant. Two other reasons often given are (2) failure tolerance and (3) the ability of local units to act quickly to reject disturbances (e.g. Findeisen et al., 1980). These reasons may be more relevant, but, as pointed out by Skogestad and Hovd (1995) there are probably even more fundamental reasons. The most important one is probably (4) to reduce the cost involved in defining the control problem and setting up the detailed dynamic model which is required in a centralized system with no predetermined links. Also, (5) decomposed control systems are much less sensitive to model uncertainty (since they often use no explicit model). In other words, by imposing a certain control configuration, we are implicitly providing information about the behavior of the process, which we with a centralized controller would need to supply explicitly through the model.

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