SELECTION OF VARIABLES FOR REGULATORY CONTROL USING POLE VECTORS

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Abstract: This paper consider control structure design using the information given in the pole vectors. It is shown how the input and output pole vectors are related to the minimum input energy needed to stabilize a given unstable mode using one Single Input Single Output (SISO) control loop. The paper also look at stable but slow modes which need to be shifted further into the Left Half Plane (LHP) using feedback control. Moving stable slow modes are accomplished with pole placement and the results are interpreted in terms of Linear Quadratic Gaussian (LQG) control.

Keywords: Stabilization, minimum input usage, linear systems, control structure design, output selection, input selection, RHP-poles, LQG-control

1. INTRODUCTION

This paper consider control structure design and in particular the selection of inputs to be used for control and the outputs to be controlled. Our concern in this paper are plants which need to be "stabilized" in an extended meaning. That is, plants which contains one or more unstable modes and therefore need to be stabilized in the mathematical sense, or plants which contains one or more stable slow modes which need to be "stabilized" form the operator point of view. In order to provide this "stabilization" we need to select inputs to be used for control and outputs to be controlled. It is then necessary that these inputs can affect the modes which need "stabilization" and that the modes are visible in the outputs to be controlled.

The main question we will answer in this paper is:

- Given a plant G with one unstable mode p, where the measurements of the plant outputs are affected by noise. Which pair of one input and one output (u_j, y_i) stabilize the unstable mode p with minimum input energy?

In order to answer this question we need to address the following two questions:

- Which output should be controlled?
- Which input should be used for control?

For the case when the plant G has no unstable modes, minimum input energy problem is meaningless since $u = 0$ is the best solution to this problem. But we still may want to move one stable slow open-loop pole further to the left in the complex plane using a single loop controller in order to obtain satisfactory closed-loop response (i.e. speed up the open-loop response).

Some related work are given in (Wang and Davison, 1973; Benninger, 1986; Tarokh, 1985; Tarokh, 1992; Hovd, 1992; Lunze, 1992; Li *et al.*, 1994*a*; Li *et al.*, 1994*b*).

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2. POLE VECTORS AND DIRECTIONS

For a system in state-space form we compute the input and output pole vectors (non-normalized basis vectors for the input and output pole spaces) as

$$\mathrm{u}_p = B^H x_{pi}; \qquad \mathrm{y}_p = C x_{po} \tag{1}$$

where $x_{pi}, x_{po} \in \mathbb{C}^n$ are the normalized eigenvectors corresponding to the two eigenvalue problems

$$x_{pi}^H A = p x_{pi}^H \quad \text{and} \quad A x_{po} = p x_{po} \tag{2}$$

3. STABILIZING CONTROL WITH MINIMUM INPUT USAGE

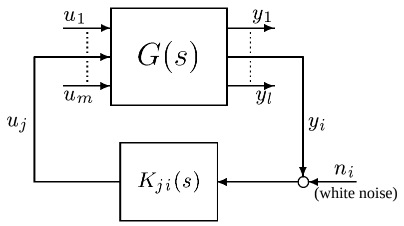


Fig. 1. Plant G and stabilizing control loop with pairing $u_j \leftrightarrow y_i$

In this section we consider the following problem, see also Figure 1:

PROBLEM 1.. *Given a plant G with minimal realization $(A,\ B,\ C,\ D)$, one unstable mode $p \in \mathbb{C}_+$ $(\mathrm{Re}\, p > 0)$ and white measurement noise n_i of unit intensity in each output y_i. Find the best pairing $u_j \leftrightarrow y_i$, such that the plant is stabilized with minimum input usage*

$$J = E\left\{ \lim_{T\to\infty} \frac{1}{T} \int_0^T u_j^2(t)dt \right\} \tag{3}$$

At first sight it is not clear that the output selection problem is included at all, since the outputs do not enter into the objective (3) explicitly. However, the output selection problem is included implicitly through the measurement noise and the expectation operator E. We assume that the noise are uncorrelated zero-mean Gaussian stochastic processes with power spectral density matrix equal to the identity I. That is, each n_i are white noise processes with covariance

$$E\left\{n(t)n^T(\tau)\right\} = I\,\delta(t-\tau) \tag{4}$$

where $n = [n_1 \ \cdots \ n_l]^T$. As for the LQG design we use the Separation Theorem (Certainty Equivalence Principle) and find the best input using state feedback (LQR) under the assumption of perfect measurement of all states. The next step is to construct the optimal state observer (LQE) and find the best output so that the mean square reconstruction error

$$E\left\{(x(t)-\hat{x}(t))^T(x(t)-\hat{x}(t))\right\} \tag{5}$$

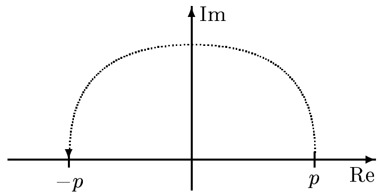


Fig. 2. Mapping of pole from RHP to LHP with state feedback and minimum input usage

is minimized using output y_i only. Let us first state the solution:

SOLUTION TO PROBLEM 1. *The minimum value of the objective J, for a specified input u_j and a specified output y_i is*

$$\boxed{J = \frac{8p^3}{\mathrm{u}_{p,j}^2 \mathrm{y}_{p,i}^2}(x_{pi}^H x_{po})^2} \tag{6}$$

where p is the pole, $\mathrm{u}_{p,j}$ is the j'th element in the input pole vector and $\mathrm{y}_{p,i}$ is the i'th element in the output pole vector. To minimize the control effort to stabilize the pole p, one should

- *use input u_j for control, where j corresponds to $\max\limits_j \mathrm{u}_{p,j}$,*
- *control output y_i, where i corresponds to $\max\limits_i \mathrm{y}_{p,i}$.*

We note that p, x_{pi} and x_{po} in (6) are independent of the input/output selection problem.

Proof of (6).

Optimal state feedback to input u_j. In this case, the problem is to minimize the input usage due to non-zero initial states x_0, i.e. minimize the deterministic cost

$$J_{LQR} = \int_0^\infty u_j^2(t)dt$$

The optimal state feedback gain K_j becomes

$$K_j = \overbrace{e_j^T \underbrace{B^T x_{pi}}_{\mathrm{u}_p}}^{\mathrm{u}_{p,j}} x_{pi}^T \frac{2p}{\mathrm{u}_{p,j}^2} = \frac{2p}{\mathrm{u}_{p,j}} x_{pi}^T \tag{7}$$

It is well-known (Kwakernaak and Sivan, 1972) that minimum input to stabilize an unstable plant with state feedback $u(t) = -Kx(t)$ mirrors the unstable poles across the imaginary axis, see Figure 2.

Kalman filter based on y_i. In this case the Kalman filter is updated by only using the information in output y_i, and in this case there is no process noise. The structure is similar to the structure in an ordinary state observer

$$\dot{\hat{x}} = A\hat{x} + Bu + K_{f,i}(y_i - e_i^T C\hat{x} - e_i^T Du) \tag{8}$$

The feedback gain $K_{f,i}$ from output y_i to the state estimate becomes

$$K_{f,i} = \frac{2p}{\mathrm{y}_{p,i}^2} x_{po} \overbrace{\underbrace{x_{po}^T C^T}_{\mathrm{y}_p^T} e_i}^{\mathrm{y}_{p,i}} = \frac{2p}{\mathrm{y}_{p,i}} x_{po} \tag{9}$$

Finally, to prove the minimum value of the objective J given in (3), we use Theorem 5.4 part (d) in Kwakernaak and Sivan (1972, page 394–395). □

4. POLE PLACEMENT

In the previous section we showed that for the special case of stabilizing one unstable mode with a single control loop and selecting the input u_j and the output y_i according to the largest elements in the pole vectors, corresponds to the best input/output combination in terms of minimum input usage. We note that an alternative interpretation to the minimum input usage is to select input u_j and output y_i which minimize $\|K_j\|_2$ and $\|K_{f,i}\|_2$, see the expressions for K_j and $K_{f,i}$ in (7) and (9).

4.1 *Moving one pole*

State feedback to input u_j. The problem is to move the distinct real open-loop pole p to μ by the use of state feedback to input u_j. The solution is

$$K_j = \frac{p-\mu}{u_{p,j}} x_{pi}^T \tag{10}$$

where $u_{p,j}$ is the j'th element in the input pole vector corresponding to the pole p and x_{pi} is the corresponding state input pole direction, i.e. $x_{pi}^H A = p x_{pi}^H$. We see that the magnitude of K_j is minimized by selecting input j corresponding to the element with largest magnitude $|u_{p,j}|$ in the input pole vector u_p.

State observer base on y_i. In a similar way, we move the observer pole p to the desired location μ by adding feedback from $y_i - \hat{y}_i$ to the estimated state. The solution is

$$K_{f,i} = \frac{p-\mu}{y_{p,i}} x_{po} \tag{11}$$

where $y_{p,i}$ is the i'th element in the output pole direction corresponding to the pole p and x_{po} is the corresponding state output pole direction, i.e. $Ax_{po} = px_{po}$. We see that the magnitude of $K_{f,i}$ is minimized by selecting output i corresponding to the element with largest magnitude $|y_{p,i}|$ in the output pole vector y_p.

5. IMPLICATIONS ON INPUT/OUTPUT SELECTION

The pole input/output vectors depends on scaling, so it is crucial to scale the inputs and outputs properly. One procedure for selecting inputs and outputs to stabilize a given unstable mode is:

1) Scale the inputs so that a change in each input are of equal importance in the objective.
2) Scale outputs relative to measurement noise.
3) Use input u_j for control, where j corresponds to a large element in input pole vector u_p
4) Control output y_i, where i corresponds to a large element in output pole vector y_p.

If the plant has several unstable modes which need to be stabilized, after stabilizing one mode using one loop, the poles and the pole vectors of the partially controlled system (closed-loop system with the SISO controller included) can be recomputed. It may be that the SISO controller has "stabilized" several unstable or slow modes. If there are remaining unstable poles then new control links can be identified from the recomputed pole directions and new controllers can be included, see the Tennessee Eastman example in Section 6 for a illustration of this procedure.

6. CASE STUDIES

The first example consider the Tennessee Eastman problem, where we use the pole directions to find a stabilizing control structure.

EXAMPLE 1. TENNESSEE EASTMAN PROBLEM. The plant layout of the Tennessee Eastman problem is shown in Figure 3. For details about the Tennessee Eastman problem refer to (Downs and Vogel, 1993). In the figure both measurements y_i and manipulated variables u_j are labeled. Also given in the figure are candidate outputs (y_i) for stabilizing control. A separate numbering scheme is given for those outputs. Table 1 summarizes the selected candidate outputs for stabilizing control and the corresponding variable number in the full model (referred to as PID No.). Also given in the table is the scaling of the outputs used in this analysis. The manipulated variables are summarized in Table 2, also given in the table is the suggested scaling of the inputs used in this analysis. The linearized model in the base case (mode 1, 50/50 G/H mass ratio) is used in this example.

Table 1. Candidate outputs for stabilizing control of the Tennessee Eastman problem.

Variable name	No. [a]	PID No. [b]	Scaling
Reactor pressure	y_1	y_7	54.1 [kPa]
Reactor level	y_2	y_8	1.5 %
Reactor temperature	y_3	y_9	1.2 [°C]
Separator temperature	y_4	y_{11}	1.0 [°C]
Separator level	y_5	y_{12}	1.0 %
Separator pressure	y_6	y_{13}	52.6 [kPa]
Stripper level	y_7	y_{15}	1.0 %
Stripper pressure	y_8	y_{16}	62.0 [kPa]
Stripper temperature	y_9	y_{18}	1.0 [°C]
Reactor cooling water outlet temperature	y_{10}	y_{21}	0.2 [°C]
Separator coolingwater outlet temperature	y_{11}	y_{22}	0.2 [°C]

[a] Variable number in the smaller model used in the analysis.
[b] Variable number in the full model provided by Downs and Vogel.

The model has six unstable poles at the operating point considered

$$P_u = [\,0 \quad 0.001 \quad 0.023 \pm 0.156i \quad 3.066 \pm 5.079i\,]$$

The output pole vectors are

Fig. 3. Tennessee Eastman test problem

Table 2. Manipulated variables in the Tennessee Eastman problem.

Variable name	No. [a]	Str. no.	Scaling
D feed flow	u_1	2	10%
E feed flow	u_2	3	10%
A feed flow	u_3	3	10%
A and C feed flow	u_4	4	10%
Compressor recycle valve	u_5		10%
Purge valve	u_6	9	10%
Separator pot liquid flow	u_7	10	10%
Stripper liquid product flow	u_8	11	10%
Stripper steam valve	u_9	Stm	10%
Reactor cooling water flow	u_{10}	CWS	10%
Condenser cooling water flow	u_{11}	CWS	10%
Agitator speed	u_{12}		10%

[a] Variable number in both the full model and the model used in the analysis.

$$|Y_p| = \left[\begin{array}{cccc} 0.000 & 0.001 & 0.041 & 0.112 \\ 0.000 & 0.004 & 0.169 & 0.065 \\ 0.000 & 0.000 & 0.013 & 0.366 \\ 0.000 & 0.001 & 0.051 & 0.410 \\ 0.009 & 0.580 & 0.488 & 0.315 \\ 0.000 & 0.001 & 0.041 & 0.115 \\ 1.605 & 1.192 & 0.754 & 0.131 \\ 0.000 & 0.001 & 0.039 & 0.107 \\ 0.000 & 0.001 & 0.038 & 0.217 \\ 0.000 & 0.001 & 0.055 & 1.485 \\ 0.000 & 0.002 & 0.132 & 0.272 \end{array}\right] \begin{array}{l} \\ \\ \\ \\ \\ \\ \leftarrow y_{15} \\ \\ \\ \leftarrow y_{21} \\ \\ \end{array}$$

We have taken the absolute value to avoid complex numbers in the vectors. The first column corresponds to the pole $p_1 = 0$, the second column corresponds to the pole $p_2 = 0.001$, the third column corresponds to the complex conjugate pair $p_{3,4} = 0.023 \pm 0.156i$ and the fourth column corresponds to the complex conjugate pair $p_{5,6} = 3.066 \pm 5.079i$. We see that output y_{15} in the full model (row 7) has the largest component in the output pole vector for the pole $p_1 = 0$, and none of the other outputs has significant components in this vector. In a similar way output y_{21} (row 10) has a large component in the pole output vector corresponding to the complex conjugate pair $p_{5,6} = 3.066 \pm 5.079i$. The input pole vectors are

$$|U_p| = \left[\begin{array}{cccc} 6.815 & 6.909 & 2.573 & 0.964 \\ 6.906 & 7.197 & 2.636 & 0.246 \\ 0.148 & 1.485 & 0.768 & 0.044 \\ 3.973 & 11.550 & 5.096 & 0.470 \\ 0.012 & 0.369 & 0.519 & 0.356 \\ 0.597 & 0.077 & 0.066 & 0.033 \\ 0.132 & 1.850 & 1.682 & 0.110 \\ 22.006 & 0.049 & 0.000 & 0.000 \\ 0.007 & 0.054 & 0.009 & 0.013 \\ 0.247 & 0.708 & 1.501 & 2.020 \\ 0.109 & 0.976 & 1.446 & 0.753 \\ 0.033 & 0.094 & 0.201 & 0.302 \end{array}\right] \begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \leftarrow u_8 \\ \\ \leftarrow u_{10} \\ \\ \\ \end{array}$$

By considering both input and output pole vectors at the same time we arrive at the suggested pairings; $y_{15} \leftrightarrow u_8$ and $y_{21} \leftrightarrow u_{10}$ which corresponds to controlling the stripper level using the stripper liquid product flow and controlling reactor cooling water outlet temperature using the reactor cooling water flow. It can also be seen from the pole vectors that these two loop will interact very little since the common elements in the two vectors are almost zero. It is worth noting that both of these loops were also included by McAvoy and Ye (1994) in their study.

Using two PI-controllers with tunings given in Table 3, we manage

Table 3. Tunings of PI-controllers

Loop	k_p	T_i
$y_{15} \leftrightarrow u_8$	-0.1 [1/°C]	1 [min]
$y_{21} \leftrightarrow u_{10}$	-0.05 [m^3/h]	300 [min]
$y_{12} \leftrightarrow u_7$	-0.0025 [m^3/h]	200 [min]

to stabilize all the unstable modes except the mode $p_2 = 0.001$. By recomputing the pole vectors with the controllers included we get

$$Y_p = \left[\begin{array}{c} -0.001 \\ -0.005 \\ 0.000 \\ 0.001 \\ -0.867 \\ -0.001 \\ 0.000 \\ -0.001 \\ 0.001 \\ 0.000 \\ -0.002 \end{array}\right] \begin{array}{l} \\ \\ \\ \\ \leftarrow y_{12} \\ \\ \\ \\ \\ \\ \\ \end{array} \quad \text{and} \quad U_p = \left[\begin{array}{c} -7.363 \\ -7.536 \\ 1.410 \\ 11.515 \\ -0.346 \\ -0.065 \\ 2.465 \\ 0.000 \\ -0.062 \\ 0.008 \\ 0.901 \\ -0.078 \end{array}\right] \begin{array}{l} \\ \\ \\ \\ \\ \\ \leftarrow u_7 \\ \\ \\ \\ \\ \\ \end{array}$$

We see that the output pole vector has a large element in y_{12} and only small elements in the other outputs. From the input direction

V_T

PC

M_D

p L $D,\ y_D$

LC

$F,\ z_F$

LC V

M_B

$B,\ x_B$

Fig. 5. Distillation column DB-configuration

with $F = 1$ [kmol/min] results in distillate D and bottom B product flows of 0.5 [kmol/min].

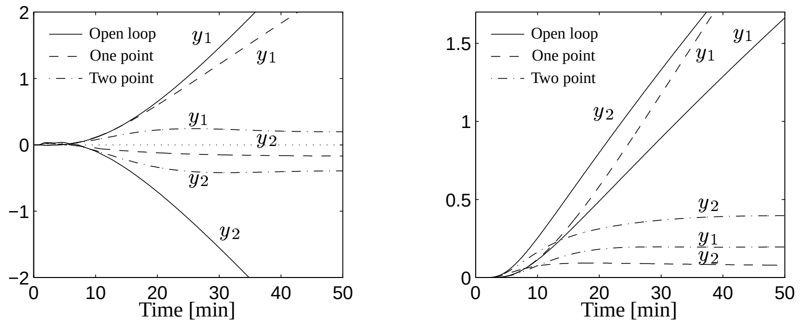
The DB-configuration is stable except for two poles close [3] to zero, and experience shows that two control loops need to be included to "stabilize" these two modes. This is in contrast to the other distillation column configurations, for example LV, LB, DV, and the configurations with single and double ratios, which have only one pole close to zero and therefore only need one "stabilizing" control loop.

In this example the objective is to predict the fact that the DB-configuration needs two control loops to be "stabilized", by looking at the poles, the pole directions/vectors and the zeros of the transfer function elements. The model is linearized in the nominal operating point and we consider G where the inputs and outputs are

$$u = [\,D \quad B\,]^T \quad \text{and} \quad y = [\,y_D \quad x_B\,]^T$$

In addition we have a disturbance model where the disturbances are feed flow rate $(d_1 = F)$ and feed composition $(d_2 = z_F)$.

[3] In this case we consider a pole p to be close to zero if $|p| < 0.01$.



(a) Step change in F

(b) Step change in z_F

Fig. 6. Responses to step in disturbances for distillation column DB-configuration. Solid line: open-loop. Dashed line: One point control. Dash-dot line: Two point control

Scalings. The variables in the linear model have been scaled such that a magnitude of 1 corresponds to a change in x_B and y_D of 0.01 mole fraction units, and a change in D and B of 0.5 [kmol/min].

We compute the poles and find that the numerical values of the two poles close to zero are $p_{1,2} = \{-5.15 \cdot 10^{-3}, 0\}$. The corresponding input and output pole directions are

$$U_p = \begin{bmatrix} -0.718 & 0.707 \\ 0.696 & 0.707 \end{bmatrix} \quad \text{and} \quad Y_p = \begin{bmatrix} 0.627 & -0.707 \\ 0.779 & 0.707 \end{bmatrix}$$

and the corresponding input and output pole vectors are

$$U_p = \begin{bmatrix} -0.066 & 0.078 \\ 0.064 & 0.078 \end{bmatrix} \quad \text{and} \quad Y_p = \begin{bmatrix} 1.283 & -1.943 \\ 1.594 & 1.943 \end{bmatrix}$$

We see that the two input directions and the two output directions are nearly orthogonal. The relative angle between the two input directions is $89.1°$, and the relative angle between the two output pole directions is $83.8°$. In addition the two poles are very close so we find as expected that all transfer function elements have a RHP-zero close to the poles. Thus, we may conclude that it is in *practice* impossible to move *both* poles $p_{1,2}$ by controlling one output using one input. Indeed, we find that this is confirmed by the closed-loop simulations in Figure 6, where one-point control is with the pairing $x_B \leftrightarrow B$, and two-point control is with the pairings $x_B \leftrightarrow B$ and $y_D \leftrightarrow D$. In both cases we use simple proportional controllers (with gains -1 and 0.5).

SUMMARY

Input and output pole vectors and directions are introduced, and it is shown how to compute these in terms of eigenvalue problems.

The input and output pole vectors are related to the minimum input energy needed to stabilize one unstable pole using a single loop controller. Furthermore, it is shown that the best input and the best output to stabilize an unstable mode with a single SISO control loop corresponds to the input and output with largest elements in the pole vectors. Here the term "best" is in the meaning of minimizing the input energy to stabilize the plant.

In (Havre, 1998) it is shown that quantifying the minimum input usage in terms of the $\mathcal{H}_\infty$-norm rather than the $\mathcal{H}_2$-norm (equivalent to the input energy), yields the same choice for the best input and the best output. The results also shows that the two norms are closely related. This may be surprising, since the value of the two norms in general may be arbitrary far apart.

In a similar way, it is shown that the best input and the best output to move a pole from one location to a different location further to the left in the complex plane with single SISO control loop corresponds to the input and output with the largest elements in the pole vectors. Here the term "best" is in the meaning of minimizing the gain form the outputs to the states in the observer and the gain from the states to the inputs.

REFERENCES

Benninger, N. F. (1986). Proper choice of input and output variables by means of new consistent structure measures. *Large Scale Systems:Theory and Applications* **1**, 161–166.

Downs, J. J. and E. F. Vogel (1993). A plant-wide industrial process control problem. *Computers Chem. Engng.* **17**(3), 245–255.

Havre, K. (1998). Studies on Controllability Analysis and Control Structure Design. PhD thesis. Norwegian University of Science and Technology, Trondheim. See also http://www.chembio.ntnu.no/users/skoge/publications/thesis/.

Hovd, M. (1992). Studies on Control Structure Selection and Design of Robust Decentralized and SVD Controllers. PhD thesis. Norwegian University of Science and Technology, Trondheim.

Kwakernaak, H. and R. Sivan (1972). *Linear Optimal Control Systems*. Wiley Interscience. New York.

Li, K., Y. Xi and Z. Zhang (1994*a*). A new method for selection of variables in industrial process control systems. In: *Proceedings of the Asian Control Conference*. Tokyo. pp. 219–222.

Li, K., Y. Xi and Z. Zhang (1994*b*). Selection of manipulated variables for industrial process control. In: *Proceedings of the Asian Control Conference*. Tokyo. pp. 223–226.

Lunze, J. (1992). *Feedback Control of Large-Scale Systems.* Prentice-Hall. Englewood Cliffs.

McAvoy, T. J. and N. Ye (1994). Base control of Tennessee Eastman problem. *Computers Chem. Engng.* **18**(5), 383–413.

Skogestad, S. (1997). Dynamics and control of distillation columns – a tutorial introduction. *Plenary lecture at Distillation and Absorption '97, Maastricht, September 1997.*

Skogestad, S. and M. Morari (1988). LV-control of a high-purity distillation column. *Chemical Engineering Science* **43**(1), 33–48.

Tarokh, M. (1985). Fixed modes in multivariable systems using constrained controllers. *Automatica* **21**(4), 495–497.

Tarokh, M. (1992). Measures for controllability, observability, and fixed modes. *IEEE Transactions on Automatic Control* **37**(8), 1268–1273.

Wang, S. H. and E. J. Davison (1973). On the stabilization of decentralized control systems. *IEEE Transactions on Automatic Control* **18**(5), 473–478.