Fundamental limitations for control of unstable Siso plants

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Abstract

This paper examines the fundamental limitations on closed-loop performance imposed by instability in the plant (Right Half Plane (Rhp) poles). The main limitation is that instability requires active use of plant inputs, and we quantify this is terms of tight lower bounds on the input magnitudes required for disturbance and measurement noise rejection. These new bounds involve the $\mathcal{H}_\infty$-norm, which has direct engineering significance. The output performance in terms of disturbance rejection or reference tracking is *only* limited if the plant has Rhp-zeros, and for a one degree-of-freedom controller the presence of Rhp-poles further deteriorate the response, whereas there is no additional penalty for having Rhp-poles if we use a two degrees-of-freedom controller. It is important to stress that the derived bounds are controller independent and that they are tight, meaning that there exists controllers which achieve the lower bounds.

1 Introduction

An unstable plant can only be stabilized by use of feedback control which implies active use of the plant inputs. If measurement noise and/or disturbances are present (which is always the case in practical control), then the input usage may become unacceptable.

In this paper, the above statements are quantified by deriving tight lower bounds on the $\mathcal{H}_\infty$-norm of the closed-loop transfer functions SV and TV, where S and T are the sensitivity and complementary sensitivity functions. The transfer function V can be viewed as a *generalized* "weight", which for our purpose should be independent of the feedback controller K.

Some reasons for deriving such bounds are:

1) The lower bounds provide direct insights to the limitations imposed by Rhp zeros and poles in Siso systems.

2) The lower bounds derived are independent of the controller, so they can be used as controllability measures.

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3) In some cases we can show that the bounds are tight. This implies that we in these cases can find a controller K, analytically, which achieves an $\mathcal{H}_\infty$-norm of the closed-loop transfer function equal to the lower bound.

4) We can *quantify*, in terms of the $\mathcal{H}_\infty$-norm, the "best achievable" closed-loop effect of the worst case disturbance, measurement noise and references both at the input and at the output of the plant.

One important application is that we can quantify the minimum input usage for stabilization in the presence of worst case measurement noise and disturbances. It appears that *even* for SISO systems this has been a difficult task, which has not been solved analytically until now.

To give the reader some appreciation of the basis of the bounds and their usefulness, we consider as a motivating example an unstable plant with a RHP-pole p. We want to obtain a lower bound on the $\mathcal{H}_\infty$-norm of the closed-loop transfer function KS from measurement noise n to plant input u. We first rewrite $KS = G^{-1}T$, which is on the form TV with $V = G^{-1}$. The basis of our bound is the use of the maximum modulus principle and the "interpolation constraint" $T(p) = 1$, which must apply to achieve internal stability. We obtain (see Theorem 1 for details)

$$\|KS(s)\|_\infty = \|G^{-1}T(s)\|_\infty \geq |G_{ms}^{-1}(p)|$$

where G_{ms} is the "stable and minimum phase" version of G (if G also has a RHP-zero z we get the additional penalty $\frac{|z+p|}{|z-p|}$). As an example, consider the plant $G(s) = \frac{1}{s-10}$, which has an unstable pole $p = 10$. We obtain $G_{ms}(s) = \frac{1}{s+10}$. For *any* linear feedback controller K, we find that the lower bound

$$\|KS(s)\|_\infty \geq |G_{ms}^{-1}(p)| = 2p = 20$$

must be satisfied. Thus, if we require that the plant inputs are bounded with $\|u\|_\infty \leq 1$, then we cannot allow the magnitude of measurement noise to exceed $\|n\|_\infty = 1/20 = 0.05$.

The basis for our results is the *important* work by Zames (1981), who made use of the interpolation constraint $S(z) = 1$ and the maximum modulus theorem to derive bounds on the $\mathcal{H}_\infty$-norm of S for plants with one RHP-zero. Subsequently, these results were extended to unstable plants with one RHP-pole and then to plants with combined RHP zeros and poles, e.g. (Doyle, Francis and Tannenbaum, 1992, pp. 93–95) and (Skogestad and Postlethwaite, 1996).

However, these generalizations to unstable plants did *not* consider the input usage which involves the closed-loop transfer function KS. An important contribution of this paper is therefore to use the "trick" $KS = G^{-1}T$, which enable us to derive lower bounds on input usage, by using the general lower bound on $\|TV(s)\|_\infty$ with $V = G^{-1}$. But when G is unstable (with RHP-pole p), then $V = G^{-1}$ has RHP-zeros for $s = p$. A second important contribution compared to earlier work is the ability to include RHP zeros and poles in the "weight" V (under the assumption that SV and TV are stable).

A third important contribution is that we show that the lower bounds are *tight*. That is, we give analytical expressions for stable controllers which *achieves* an $\mathcal{H}_\infty$-norm of the closed-loop transfer function which is equal to the lower bound.

Several authors, among them Kwakernaak (1995), have noted the symmetries between sensitivity and complementary sensitivity and the roles of RHP zeros and poles of G. In this paper, the symmetries are also reflected in where performance is measured. We find that RHP-zeros of G pose limitations on performance measured at the *output* of the plant whereas RHP-poles of G pose limitations on performance measured at the *input* of the plant (input usage).

The bounds on $\|S(s)\|_\infty$ for plants with RHP-zero derived by Zames (1981) are also valid for multivariable systems. It is important to note that all the results given in this paper have been generalized to multivariable systems (Havre and Skogestad, 1997b). However, the notation becomes complicated

in the multivariable case, with the result that it is difficult to understand the implications of the bounds. In the SISO case, the bounds may easily be derived by hand for a particular plant. However, in the multivariable case, we must in general evaluate the bounds numerically.

The paper is organized as follows: First we introduce the notation and present some basics from linear control theory. In Section 3 we derive the general lower bounds on $\|SV(s)\|_\infty$ and $\|TV(s)\|_\infty$, and in Section 4 we prove the tightness of the lower bounds. In Section 5 we show some applications and implications of the lower bounds both on output performance, input usage, peaks in sensitivity and complementary sensitivity, and we give some simple examples to illustrate the applications and the implications. In Section 6 we discuss briefly the relationship to stabilization with input constraints. In Section 7 we derive a lower bound applicable to two degrees-of-freedom (2-DOF) control. The proofs of the results which are not given in the main text are given in Appendix A.

2 Basics from linear control theory

We consider linear time invariant transfer function models on the form

$$y(s) = G(s)u(s) + G_d(s)d(s) \tag{1}$$

where u is the manipulated input, d is the disturbance, y is the output, G is the SISO plant model and G_d is the SISO disturbance plant model. The measured output is $y_m = y + n$ where n is the measurement noise.

The $\mathcal{H}_\infty$-norm of a stable rational transfer function $M(s)$ is defined as the peak value in the magnitude $|M(j\omega)|$ over all frequencies.

$$\|M(s)\|_\infty \triangleq \sup_\omega |M(j\omega)| \tag{2}$$

2.1 Zeros and poles

In a rational transfer function M the zeros and poles are the roots of the numerator and denominator polynomials. That is, the zeros z_j and the poles p_i are the solutions to the following equations

$$M(z_j) = 0 \quad \text{and} \quad M^{-1}(p_i) = 0 \tag{3}$$

When we refer to zeros and poles we mean the zeros and poles of the plant G unless otherwise explicitly stated.

2.2 Factorizations of RHP zeros and poles

A rational transfer function $M(s)$ with zeros z_j and poles p_i in the open RHP, $\{z_j,\, p_i\} \in \mathbb{C}_+$, can be factorized in *Blaschke products* as follows[1]

$$
\begin{align}
M(s) &= \mathcal{B}_z(M)\, M_m(s) \tag{4} \\
M(s) &= \mathcal{B}_p^{-1}(M)\, M_s(s) \tag{5} \\
M(s) &= \mathcal{B}_z(M)\, \mathcal{B}_p^{-1}(M)\, M_{ms}(s) \tag{6}
\end{align}
$$

where

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[1]Note that the notation on the all-pass factorizations of RHP zeros and poles used in this paper is reversed compared to the notation used in (Skogestad and Postlethwaite, 1996; Havre and Skogestad, 1997a). The reason to this change of notation is to get consistent with what the literature generally defines as an all-pass filter.
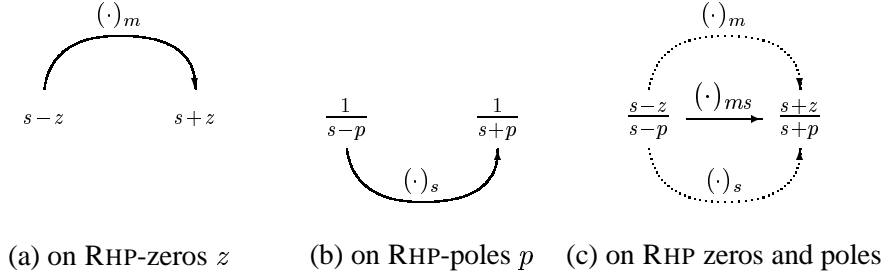
(a) on RHP-zeros z (b) on RHP-poles p (c) on RHP zeros and poles

Figure 1: Operations on RHP zeros and poles for scalar transfer functions

M_m – Minimum phase (subscript m) version of M with the RHP-zeros mirrored across the imaginary axis.

M_s – Stable (subscript s) version of M with the RHP-poles mirrored across the imaginary axis.

M_{ms} – Minimum phase, stable (subscript ms) version of M with the RHP zeros and poles mirrored across the imaginary axis.

$\mathcal{B}_z(M)$ – Stable all-pass rational transfer function $(|\mathcal{B}_z(j\omega)| = 1, \forall\omega)$ containing the RHP-zeros (subscript z) of M.

$\mathcal{B}_p(M)$ – Stable all-pass rational transfer function $(|\mathcal{B}_p(j\omega)| = 1, \forall\omega)$ containing the RHP-poles (subscript p) of M as RHP-zeros.

The all-pass filters are

$$\mathcal{B}_z(M(s)) = \prod_{j=1}^{N_z} \frac{s - z_j}{s + \bar{z}_j} \tag{7}$$

$$\mathcal{B}_p(M(s)) = \prod_{i=1}^{N_p} \frac{s - p_i}{s + \bar{p}_i} \tag{8}$$

where N_z is the number of RHP-zeros $z_j \in \mathbb{C}_+$ and N_p is the number of RHP-poles $p_i \in \mathbb{C}_+$ in M.

In most cases $M = G$ and to simplify the notation we often omit to show that the all-pass filters are dependent on G, i.e. we write $\mathcal{B}_p(s)$ and $\mathcal{B}_z(s)$ in the meaning of $\mathcal{B}_p(G(s))$ and $\mathcal{B}_z(G(s))$.

Figure 1 demonstrates the operations, $(\cdot)_m$ for RHP-zeros, $(\cdot)_s$ for RHP-poles and the combined operator $(\cdot)_{ms}$ of scalar transfer functions. The order of the two operations $(\cdot)_m$ and $(\cdot)_s$ in the combined operator $(\cdot)_{ms}$ is arbitrary. It also follows that

$$(G^{-1})_{ms} = (G_{ms})^{-1} = G_{ms}^{-1} \tag{9}$$

And we note that

$$\|M(s)\|_\infty = \|M_m(s)\|_\infty = \|M_{ms}(s)\|_\infty \tag{10}$$

The first identity follows since $|\mathcal{B}_z(M(j\omega))| = 1, \forall\omega$, and the latter identity follows since M is stable, i.e. $M_{ms} = M_m$ and $\mathcal{B}_p(M_m) = \mathcal{B}_p(M) = 1$.

To prove the main results in this paper we make use of the following Lemma.

LEMMA 1. *Consider a stable* SISO *transfer function* AB *which can be expressed by the product of the* SISO *transfer functions* A *and* B, *where both* A *and* B *may be unstable. Then*

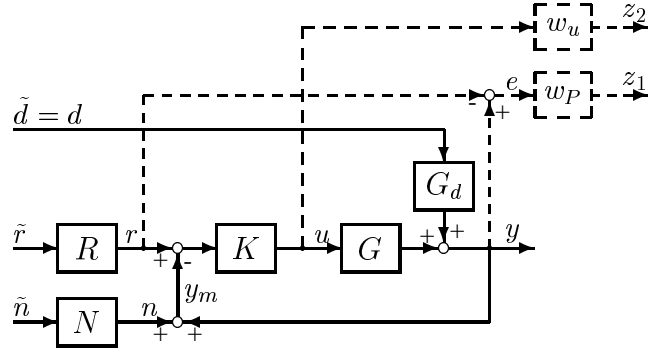$$\|AB\|_\infty = \|(AB)_m\|_\infty = \|A_{ms}B_{ms}\|_\infty \tag{11}$$

Figure 2: One degree-of-freedom control configuration

2.3 Closing the loop

A typical control problem is shown in Figure 2. In the figure possible performance weights are given in dashed lines. Mainly for simplicity, but also because it is most practically relevant, we assume that the performance weights w_P and w_u are stable and minimum phase. If intergrators (poles at $s = 0$) are present in w_P and w_u then we need the same number of integrators in $L = GK$, to have a stable closed-loop transfer function. In Figure 2 we have included both the reference r and the measurement noise n, in addition to disturbances d as external inputs. The transfer functions, G_d, R and N can be viewed as weights on the inputs, and the inputs: $\tilde{d}$, $\tilde{r}$ and $\tilde{n}$ are normalized in magnitude. Normally, N is the inverse of signal to noise ratio. For most practical purposes, we can assume that R and N are stable. However from the technical point of view it suffice that the unstable modes in N or R can be stabilized through the input u.

We apply negative feedback control

$$u = K(r - y_m) = K(r - y - n) \tag{12}$$

The closed-loop transfer function F from

$$v = \begin{bmatrix} \tilde{r} \\ \tilde{d} \\ \tilde{n} \end{bmatrix} \quad \text{to} \quad z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} w_P(y - r) \\ w_u u \end{bmatrix}$$

is

$$F(s) = \begin{bmatrix} -w_P SR & w_P SG_d & -w_P TN \\ w_u SKR & -w_u KSG_d & -w_u KSN \end{bmatrix} \tag{13}$$

where the sensitivity S and the complementary sensitivity T are defined as

$$S \triangleq (1 + GK)^{-1} = \frac{1}{1 + GK} \tag{14}$$

$$T \triangleq 1 - S = \frac{GK}{1 + GK} \tag{15}$$

To have good control performance (keep z_1 small) with a small input usage (keep z_2 small) we need to have $\|F(s)\|_\infty$ small. That is we want all the SISO transfer functions in (13) small. In addition, there are robustness issues. For example, we wish to have $\|w_{\mathrm{unc}} T(s)\|_\infty$ small, where w_{unc} is the magnitude of the relative plant uncertainty.

The first requirement for being able to satisfying all these objectives (e.g. having all seven transfer functions mentioned above small), is that the weights w_P, w_u and w_{unc} are such that the objectives

can be achieved. For example, since $S + T = 1$ we cannot have $w_P R$ and w_{unc} large *at the same frequency* if we want to have $\|w_P SR(s)\|_\infty$ (tight control of setpoint changes) and $\|w_{\text{unc}} T(s)\|_\infty$ (the closed-loop response is insensitive to plant uncertainty) small. However, the presence of RHP zeros and poles in the plant G provide additional limitations, which is the focus of this paper.

2.4   Interpolation constraints

If G has a RHP-zero z or a RHP-pole p then for internal stability of the feedback system the following interpolation constraints must apply (e.g. Skogestad and Postlethwaite, 1996):

$$T(z) = 0; \qquad\qquad S(z) = 1 \qquad\qquad\qquad (16)$$

$$S(p) = 0; \qquad\qquad T(p) = 1 \qquad\qquad\qquad (17)$$

3   Lower bounds on the $\mathcal{H}_\infty$-norm of closed-loop transfer functions

In this section we will give the main results, which are lower bounds on the $\mathcal{H}_\infty$-norm of closed-loop transfer functions which can be written on the form TV or SV. The generalized "weight" V is assumed to be independent of the feedback controller K. V may be unstable but TV and SV must be stable. That is, it must be possible to stabilize all transfer functions by controlling the output y using the input u (this implies that all unstable modes of V also are modes of G).

Some examples. Consider the six transfer functions in (13). The first two can be written on the form SV by selecting $V_{11} = w_P R$ and $V_{12} = w_P G_d$. The remaining four can be written on the form TV by selecting $V_{13} = w_P N$, $V_{21} = w_u G^{-1} R$, $V_{22} = w_u G^{-1} G_d$ and $V_{23} = w_u G^{-1} N$. From this we see that the "weight" V may be unstable (if one or both of G_d and G^{-1} are unstable) and may contain RHP-zeros (if one or both of G_d and G^{-1} contain RHP-poles).

In the first result, which is the lower bound on $\|TV(s)\|_\infty$, we consider any number of RHP-zeros in the plant G and one RHP-pole at a time. Then by maximizing over all RHP-poles in the plant G we find the largest lower bound on $\|TV(s)\|_\infty$ which takes into account one RHP-pole and all RHP-zeros.

THEOREM 1 (LOWER BOUND ON $\|TV(s)\|_\infty$). *Consider the* SISO *plant G with $N_z \geq 0$* RHP-*zeros $z_j \in \mathbb{C}_+$ and $N_p \geq 1$* RHP-*poles p_i. Let V be a rational transfer function, and assume that TV is (internally) stable. Then the following lower bound on $\|TV(s)\|_\infty$ applies:*

$$\|TV(s)\|_\infty \geq \max_{\text{RHP-poles, } p_i} |\mathcal{B}_z^{-1}(p_i)| \cdot |V_{ms}(p_i)| \qquad\qquad (18)$$

REMARK 1. With $|\mathcal{B}_z(p_i)|$ we mean $|\mathcal{B}_z(G(s))|$ evaluated at $s = p_i$.
REMARK 2. The assumption that TV is internally stable, means that TV is stable and we have no RHP zero/pole cancellations between G and K.

The lower bound (18) is independent of the controller K, if the weight V is independent of K. The factor $|\mathcal{B}_z^{-1}(p_i)|$ takes into account the interactions between all the RHP-zeros $z_j \in \mathbb{C}_+$ and the single RHP-pole p_i of G. As we shall see this factor can be quite large if G contains one or more RHP-zeros close to the RHP-pole p_i.

Proof of Theorem 1.

1) **Factor out RHP zeros and poles in T and V.** Lemma 1 gives

$$\|TV(s)\|_\infty = \|T_{ms}V_{ms}(s)\|_\infty = \|T_mV_{ms}(s)\|_\infty$$

where the last equality holds since T is stable, i.e. $T_{ms} = T_m$.

2) **Introduce the stable scalar function $f(s) = T_mV_{ms}(s)$.**

3) **Apply the maximum modulus theorem to $f(s)$ at the RHP-poles p_i of G.**

$$\|f(s)\|_\infty \geq |f(p_i)|$$

4) **Resubstitute the factorization of RHP-zeros in T**, i.e. use $T_m(p_i) = T(p_i)\mathcal{B}_z^{-1}(p_i)$ to get

$$f(p_i) = T_m(p_i)V_{ms}(p_i) = T(p_i)\mathcal{B}_p^{-1}(p_i)V_{ms}(p_i)$$

5) **Use the interpolation constraint (17) for RHP-poles p_i in G**, i.e. use $T(p_i) = 1$.

6) **Evaluate the lower bound.**

$$|f(p_i)| = |\mathcal{B}_z^{-1}(p_i)| \cdot |V_{ms}(p_i)| \tag{19}$$

Note that $f(p_i)$ is independent of the controller K if V is independent of K.

Since these steps holds for all RHP-poles p_i, Theorem 1 follows. $\qquad\square$

In the next result, which is the lower bound on $\|SV(s)\|_\infty$, we consider any number of RHP-poles in the plant G and one RHP-zero at a time. Then by maximizing over all RHP-zeros in the plant G we find the largest lower bound on $\|SV(s)\|_\infty$ which takes into account one RHP-zero and all RHP-poles.

THEOREM 2 (LOWER BOUND ON $\|SV(s)\|_\infty$). *Consider the SISO plant G with $N_z \geq 1$ RHP-zeros z_j and $N_p \geq 0$ RHP-poles $p_i \in \mathbb{C}_+$. Let V be a rational transfer function, and assume that SV is (internally) stable. Then the following lower bound on $\|SV(s)\|_\infty$ applies:*

$$\|SV(s)\|_\infty \geq \max_{\text{RHP-zeros, } z_j} |\mathcal{B}_p^{-1}(z_j)| \cdot |V_{ms}(z_j)| \tag{20}$$

Remarks on Theorems 2 and 1:

1) The lower bounds on $\|TV(s)\|_\infty$ and $\|SV(s)\|_\infty$ involve V_{ms}. Thus, we get the same result if the "weight" V is replaced by its stable minimum phase counterpart with the same magnitude V_{ms}. Note that for $V = V^1V^2$ we have

$$\|TV(s)\|_\infty = \|T_mV_{ms}^1V_{ms}^2\|_\infty \tag{21}$$

Which means that we can treat the different factors of V independently.

2) The bound on $\|TV(s)\|_\infty$ is caused by the RHP-poles p_i in G, and the term $|\mathcal{B}_z^{-1}(p_i)| \geq 1$ gives an additional penalty for plants which also have RHP-zeros. For the case when G has *no* RHP-zeros, then $\mathcal{B}_z^{-1}(p_i) = 1$.

3) The bound on $\|SV(s)\|_\infty$ is caused by the RHP-zeros z_j in G, and the term $|\mathcal{B}_p^{-1}(z_j)| \geq 1$ gives an additional penalty for plants which also have RHP-poles. For the case when G has *no* RHP-poles, then $\mathcal{B}_p^{-1}(z_j) = 1$.

4) In all the lower bounds which follows from Theorems 2 and 1, one of the following two factors appears

$$|\mathcal{B}_z^{-1}(p_i)| = \frac{\displaystyle\prod_{j=1}^{N_z} |p_i + \bar{z}_j|}{\displaystyle\prod_{j=1}^{N_z} |p_i - z_j|} \geq 1 \tag{22}$$

$$|\mathcal{B}_p^{-1}(z_j)| \quad = \quad \frac{\prod\limits_{i=1}^{N_p} |z_j + \bar{p}_i|}{\prod\limits_{i=1}^{N_p} |z_j - p_i|} \geq 1 \tag{23}$$

The factor $|\mathcal{B}_z^{-1}(p_i)|$ is a measure for the interactions between all RHP-zeros $z_j \in \mathbb{C}_+$ of G and the single RHP-pole p_i of G. If one or more RHP-zeros are close to the RHP-pole p_i, then $|\mathcal{B}_z^{-1}(p_i)|$ is much larger than one. In a similar way $|\mathcal{B}_p^{-1}(z_j)|$ combines all RHP-poles $p_i \in \mathbb{C}_+$ of G together with the single RHP-zero z_j of G. Clearly, $|\mathcal{B}_p^{-1}(z_j)|$ is much larger than one if one or more RHP-poles are located close to the RHP-zero z_j.

4 Tightness of lower bounds

Theorems 1 and 2 provide lower bounds on $\|TV(s)\|_\infty$ and $\|SV(s)\|_\infty$. The question is whether these bounds are tight, meaning that there actually exist controllers which achieve the bounds? The answer is "yes" if there is only one RHP-zero or one RHP-pole. Specifically, we find that the bound $\|TV(s)\|_\infty$ is tight if the plant G has one RHP-pole and any number of RHP-zeros, and that the bound on $\|SV(s)\|_\infty$ is tight if the plant G has one RHP-zero and any number of RHP-poles. We prove tightness of the lower bounds by constructing controllers which achieve the bounds.

First, we consider the controller which minimizes $\|TV(s)\|_\infty$.

THEOREM 3 (K WHICH MINIMIZE $\|TV(s)\|_\infty$). *Consider the* SISO *plant G with one* RHP-*pole p and $N_z \geq 0$* RHP-*zeros $z_j \in \mathbb{C}_+$. Then the feedback controller K which minimize $\|TV(s)\|_\infty$ is given by*

$$K(s) = G_{ms}^{-1} K_o(s), \quad K_o(s) = PQ^{-1}(s) \tag{24}$$

where

$$P(s) \quad = \quad \mathcal{B}_z^{-1}(p) \, V_{ms}(p) \, V_{ms}^{-1}(s) \tag{25}$$

$$Q(s) \quad = \quad (1 - \mathcal{B}_z(s) \, P(s))_m = \mathcal{B}_p^{-1}(s) \, (1 - \mathcal{B}_z(s) \, P(s)) \tag{26}$$

With this controller we have

$$\|TV(s)\|_\infty = |\mathcal{B}_z^{-1}(p)| \cdot |V_{ms}(p)| \tag{27}$$

which shows that the bound given in Theorem 1 is tight when the plant has one RHP-*pole.*

We stress that the bound given in Theorem 1 is generally *not* tight if the plant has more than one RHP-pole. The controller in Theorem 3 yields a constant ("flat") frequency response $|TV(j\omega)|$ for all ω. We note that no properness restriction has been imposed on the controller, so the controller given in Theorem 3 may be improper. Also note that the controller $K(s)$ in Theorem 3 is always stable and minimum phase. This may seem surprising since it is known that some plants with RHP zeros and poles require an unstable controller (Youla, Bongiorno and Lu, 1974) to achieve closed-loop stability. However, these results assume that the loop transfer function GK is proper or strictly proper, and does therefore not apply in our case where K may be improper. In practice, controllers are often made proper by adding high-frequency dynamics, e.g. by multiplying with $1/(\varepsilon s + 1)^m$ where ε is small, and m is some integer. This works in most cases. However, it will not work for plants which needs RHP zeros or poles in the controller to make the closed-loop transfer function TV stable, and we therefore conclude that our lower bound on TV may not be tight in such cases. *Remark*: We have written "*may* not be tight" since numerical results using standard state-space $\mathcal{H}_\infty$-synthesis in

MATLAB, have shown that the bounds *are* tight even for plants which result in an unstable state-space $\mathcal{H}_\infty$-controller. Thus for some plants, in addition to the stable improper controller given by (24), there may exist an unstable proper controller which yields the same minimum value of $\|TV(s)\|_\infty$.

Next, we consider the controller which minimizes $\|SV(s)\|_\infty$.

THEOREM 4 (K WHICH MINIMIZE $\|SV(s)\|_\infty$). *Consider the* SISO *plant G with one* RHP-*zero z and $N_p \geq 0$* RHP-*poles $p_i \in \mathbb{C}_+$. Then the feedback controller K which minimize $\|SV(s)\|_\infty$ is given by*

$$K(s) = G_{ms}^{-1} K_o(s), \quad K_o(s) = PQ^{-1}(s) \tag{28}$$

where

$$Q(s) = \mathcal{B}_p^{-1}(z) V_{ms}(z) V_{ms}^{-1}(s) \tag{29}$$

$$P(s) = (1 - \mathcal{B}_p(s) Q(s))_m = \mathcal{B}_z^{-1}(s) (1 - \mathcal{B}_p(s) Q(s)) \tag{30}$$

With this controller we have

$$\|SV(s)\|_\infty = |\mathcal{B}_p^{-1}(z)| \cdot |V_{ms}(z)| \tag{31}$$

which shows that the bound given in Theorem 2 is tight when the plant has one RHP-*zero.*

The comments following Theorem 3 also apply to the bound in Theorem 2 and to the controller given in Theorem 4.

5   Applications of lower bounds

The lower bounds on $\|TV(s)\|_\infty$ and $\|SV(s)\|_\infty$ in Theorems 1 and 2 can be used to derive a large number of interesting and useful bounds.

5.1   Bounds on important closed-loop transfer functions

Consider again the six transfer functions in (13), and the weighted complementary sensitivity function $w_{\mathrm{unc}}T$. For simplicity we assume that $w_P, w_u, w_{\mathrm{unc}}, R$ and N are all stable minimum phase (or have been replaced by the stable minimum phase counterparts with same magnitude). From Theorems 1 and 2 we obtain:

Output performance, reference tracking:

$$\|w_P SR(s)\|_\infty \geq \max_{\text{RHP-zeros, } z_j} |w_P(z_j)| \cdot |\mathcal{B}_p^{-1}(z_j)| \cdot |R(z_j)| \tag{32}$$

Output performance, disturbance rejection:

$$\|w_P SG_d(s)\|_\infty \geq \max_{\text{RHP-zeros, } z_j} |w_P(z_j)| \cdot |\mathcal{B}_p^{-1}(z_j)| \cdot |(G_d)_{ms}|_{s=z_j} \tag{33}$$

Output performance, measurement noise rejection:

$$\|w_P TN(s)\|_\infty \geq \max_{\text{RHP-poles, } p_i} |w_P(p_i)| \cdot |\mathcal{B}_z^{-1}(p_i)| \cdot |N(p_i)| \tag{34}$$

9

Input usage, reference tracking:

$$\|w_u KSR(s)\|_\infty = \|w_u TG^{-1}R(s)\|_\infty \geq \max_{\text{RHP-poles},\, p_i} |w_u(p_i)| \cdot |\mathcal{B}_z^{-1}(p_i)| \cdot |G_{ms}^{-1}R(p_i)| \tag{35}$$

Input usage, disturbance rejection:

$$\|w_u KSG_d(s)\|_\infty = \|w_u TG^{-1}G_d(s)\|_\infty \geq \max_{\text{RHP-poles},\, p_i} |w_u(p_i)| \cdot |\mathcal{B}_z^{-1}(p_i)| \cdot |G_{ms}^{-1}(G_d)_{ms}|_{s=p_i} \tag{36}$$

Input usage, measurement noise rejection:

$$\|w_u KSN(s)\|_\infty = \|w_u TG^{-1}N(s)\|_\infty \geq \max_{\text{RHP-poles},\, p_i} |w_u(p_i)| \cdot |\mathcal{B}_z^{-1}(p_i)| \cdot |G_{ms}^{-1}N(p_i)| \tag{37}$$

Closed-loop sensitivity to plant uncertainty:

$$\|w_{\text{unc}}T(s)\|_\infty \geq \max_{\text{RHP-poles},\, p_i} |w_{\text{unc}}(p_i)| \cdot |\mathcal{B}_z^{-1}(p_i)| \tag{38}$$

Note that we mainly have inherent limitations on (output) performance when the plant has RHP-zeros. The exception is for measurement noise, where the requirement of stabilizing an unstable pole may give poor performance.

On the other hand, all the bounds on input usage are caused by the presence of RHP-poles. This is reasonable since we need active use of the input in order to stabilize the plant. This is considered in more detail in the next section.

5.2 Implications for stabilization with bounded inputs

Our bounds involve the $\mathcal{H}_\infty$-norm, and their large engineering usefulness may not be immediate. In the following we will concentrate on the bounds involving input usage and we will use the lower bounds to derive and *quantify* the conclusion:

- *Bounded inputs combined with disturbances and noise may make stabilization impossible.*

The input signal for a one degree-of-freedom (1-DOF) controller due to disturbance d, measurement noise n of magnitude N and reference r of magnitude R is

$$u = KS(R\tilde{r} - G_d d - N\tilde{n}) \tag{39}$$

Measurement noise. The transfer function from normalized measurement noise $\tilde{n}$ to the input u is KSN. Then from (37) with $w_u = 1$

$$\|u\|_\infty = \|KSN(s)\|_\infty \geq \max_{\text{RHP-poles},\, p_i} |\mathcal{B}_z^{-1}(p_i)| \cdot |G_{ms}^{-1}(p_i)N(p_i)| \tag{40}$$

Thus, to have $\|u\|_\infty \leq 1$ for $\|\tilde{n}\|_\infty = 1$, we must require

$$|G_{ms}(p_i)| \geq |\mathcal{B}_z^{-1}(p_i)| \cdot |N(p_i)| \quad \text{for the worst case pole } p_i \tag{41}$$

(we have here assumed that N is minimum phase). That is:

- *To keep the input magnitude less than one ($\|u\|_\infty \leq 1$) we must require that the plant gain is larger than the measurement noise at frequencies corresponding to the unstable poles.*

To better understand this statement, we will make use of the interpretation of the $\mathcal{H}_\infty$-norm in terms of steady-state sinusoids. Consider the case when $\|\tilde{n}\|_\infty = 1$ and assume that the lower bound in terms of $\|u\|_\infty = \|KSN(s)\|_\infty$ in (40) is larger than one (i.e. (41) is *not* satisfied). In this case, no matter what linear controller we design, there will always be a sinusoidal noise signal

$$n(t) = n_{\max} \sin(\omega_0 t), \quad n_{\max} = |N(j\omega_0)|$$

such that the resulting input signal

$$u(t) = u_{\max} \sin(\omega_0 t + \varphi)$$

has $u_{\max} > 1$ (the value of φ is not of interest here). For a given controller K, the worst case frequency ω_0 may be chosen as the frequency ω where $|KSN(j\omega)|$ has its peak value, i.e. $|KSN(j\omega_0)| = \|KSN(s)\|_\infty$.

Disturbances. Similar results as those for measurement noise apply to disturbances if we replace N by G_d. From (36) with $w_u = 1$ we obtain

$$\|u\|_\infty = \|KSG_d(s)\|_\infty \geq \max_{\text{RHP-poles},\, p_i} |\mathcal{B}_z^{-1}(p_i)| \cdot |G_{ms}^{-1}(G_d)_{ms}|_{s=p_i} \tag{42}$$

To have $\|u\|_\infty \leq 1$ for $\|d\|_\infty = 1$ we must require

$$|G_{ms}(p_i)| \geq \left|\mathcal{B}_z^{-1}(p_i)\right| \cdot \left|(G_d)_{ms}\right|_{s=p_i} \quad \text{for the worst case pole } p_i \tag{43}$$

That is:

- *To keep the input magnitude less than one ($\|u\|_\infty \leq 1$) we must require that the plant gain is larger than the gain of the disturbance plant at frequencies corresponding to the unstable poles.*

References. For reference changes with $\|\tilde{r}\|_\infty = 1$, we find the same bound (42), but with G_d replaced by R. However, the implications are less severe since we may choose *not* to follow the references (e.g. set $R = 0$). Also, in the case of reference changes we may use a 2-DOF controller, such that the "burden" on the feedback part of the controller K is less. This is discussed in Section 7.

5.3   Combined RHP zeros and poles

It is well known that the combination of RHP zeros and poles imply peaks in the sensitivity and complementary sensitivity for SISO systems which are larger than one. This has previously been quantified by Freudenberg and Looze (1988) in terms of sensitivity integral relations, but *not* directly in terms of $\|S(s)\|_\infty$ and $\|T(s)\|_\infty$. However, from (20) and (18) we obtain

$$\|S(s)\|_\infty \geq \max_{\text{RHP-zeros},\, z_j} |\mathcal{B}_p^{-1}(z_j)| \geq 1$$

$$\|T(s)\|_\infty \geq \max_{\text{RHP-poles},\, p_i} |\mathcal{B}_z^{-1}(p_i)| \geq 1$$

which are large if $|z_j - p_i|$ is small. If all RHP zeros and poles are different by a factor of 10 or more, then the interaction between them are small. For a plant with one RHP-zero z and one RHP-pole p we obtain

$$M_{pz} \triangleq |\mathcal{B}_p^{-1}(z)| = |\mathcal{B}_z^{-1}(p)| = \frac{|z + p|}{|z - p|}$$

Peaks in S and T less than M_{pz} are thus unavoidable. For example if $z/p = 10$ gives $M_{pz} = 1.22$, whereas $z/p = 1.5$ gives $M_{pz} = 5$. By using the lower bounds we have derived and quantified the conclusion:

- *Closely located RHP zeros and poles imply large sensitivity peaks.*

5.4 Examples

EXAMPLE 1. The intention with this example is to show the engineering application of the lower bound on $\|KSN(s)\|_\infty$ and to demonstrate the use of Theorem 3 to find the feedback controller K which minimize $\|KSN(s)\|_\infty$. We consider the unstable plant

$$G(s) = \frac{1}{s-p}, \qquad p > 0$$

with RHP-pole at p. From (40) we have the following lower bound on the $\mathcal{H}_\infty$-norm of the transfer function from normalized measurement noise $\tilde{n}$ to input u (we assume that N is minimum phase)

$$\|KSN(s)\|_\infty \geq |G_{ms}^{-1}(p)| \cdot |N(p)|$$

In our case $G^{-1} = s - p$, $G_{ms}^{-1}(s) = s + p$, $G_{ms}^{-1}(p) = 2p$, and the lower bound becomes

$$\|KSN(s)\|_\infty \geq 2p \cdot |N(p)| \tag{44}$$

The controller which minimizes $\|TV(s)\|_\infty$ and achieves the bound (44) is given in Theorem 3. Rewriting $KSN = TG^{-1}N$ and by using $V = G^{-1}N$ we obtain $V_{ms}(s) = (s+p)N(s)$, where we have assumed N to be stable minimum phase. Furthermore, $\mathcal{B}_z(s) = 1$, $\mathcal{B}_p(s) = \frac{s-p}{s+p}$. Thus, from Theorem 3 we obtain

$$P(s) = \frac{2p \cdot N(p)}{(s+p) \cdot N(s)} \quad \text{and} \quad Q(s) = \frac{s+p}{s-p} \cdot \left(1 - \frac{2p \cdot N(p)}{(s+p) \cdot N(s)}\right)$$

which gives

$$K(s) = \frac{2p \cdot N(p)\,(s-p)}{(s+p)N(s) - 2p \cdot N(p)}$$

Remark: It seems like this controller has a RHP-zero for $s = p$, but this is not the case for its minimal realization since

$$(s+p) \cdot N(s)|_{s=p} - 2p \cdot N(p) = 0$$

For the special case where $N(s)$ is a constant $N(s) = N$ we get the proportional feedback controller

$$K(s) = \frac{2p(s-p)}{s+p-2p} = 2p$$

As a numerical example, let $p = 10$, then

$$G(s) = \frac{1}{s-10}$$

and we must have for any stabilizing feedback controller K

$$\|KSN(s)\|_\infty \geq 20\,|N(p)|$$

Thus with $\|\tilde{n}\|_\infty = 1$ we will need excessive inputs ($\|u\|_\infty > 1$) if $|N(p)| \geq |G_{ms}(p)| = 0.05$. Assume that $N(s) = N(p) = 0.05$, then $K(s) = 2p = 20$. This controller gives a "flat" frequency response, i.e. $|KSN(j\omega)| = 20$, $\forall \omega$. Thus, at any frequency ω_0 the closed-loop response in u due to

$$n(t) = 0.05\sin(\omega_0 t), \quad \text{is} \quad u(t) = \sin(\omega_0 t + \varphi) \quad \forall \omega$$

So, the input $u(t)$ oscillates between ± 1. The response in u and y due to $n(t) = 0.05\sin(4t)$ is shown in Figure 3.
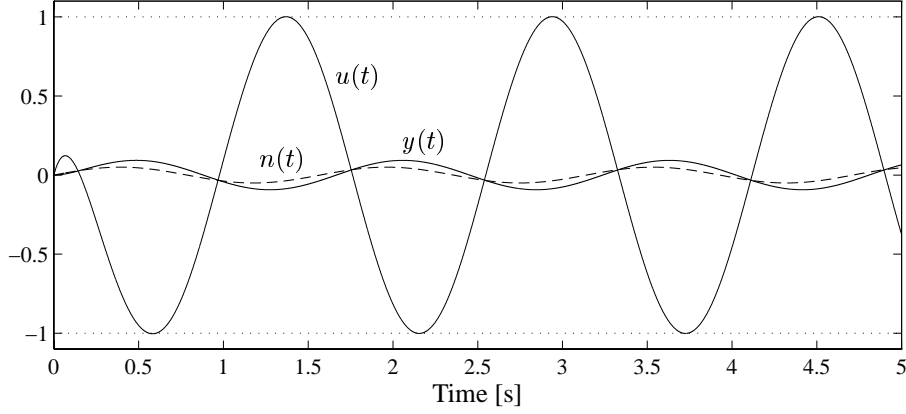
Figure 3: Closed-loop response at input u and output y of the plant G, due to $n(t) = 0.05 \sin(4t)$ (dashed), with $K = 20$

EXAMPLE 2. In this example we consider disturbance rejection for a plant with one RHP zero and pole. Let

$$G(s) = \frac{k}{z}\frac{s-z}{s-p}, \qquad G_d(s) = k_d G(s) \quad \text{with} \quad z = 2, \quad p = 1$$

We see that the disturbance is of magnitude k_d and enters at the input of the plant. Note that

$$(G_d)_{ms} = \frac{k \cdot k_d}{z}\frac{s+z}{s+p} \quad \text{and} \quad G_{ms}^{-1} = \frac{z}{k}\frac{s+p}{s+z}$$

The factors involving the interactions between the RHP zero z and pole p become

$$|\mathcal{B}_p^{-1}(z)| = |\mathcal{B}_z^{-1}(p)| = \frac{|z+p|}{|z-p|} = 3$$

and we find that peaks in the sensitivity S and the complementary sensitivity T less than 3 are unavoidable since

$$\|S(s)\|_\infty \geq \frac{|z+p|}{|z-p|} = 3 \quad \text{and} \quad \|T(s)\|_\infty \geq \frac{|z+p|}{|z-p|} = 3$$

Since G has a RHP-zero, we have a bound on the $\mathcal{H}_\infty$-norm of the closed-loop transfer function from disturbance d to output $e = y - r$

$$\|SG_d(s)\|_\infty \geq |\mathcal{B}_p^{-1}(z)| \cdot |(G_d)_{ms}|_{s=z} = \frac{|z+p|}{|z-p|} \cdot \frac{2\,|k \cdot k_d|}{|z+p|} = 2\,|k \cdot k_d|$$

and for $\|d\|_\infty = 1$, the output e will be unacceptable ($\|e\|_\infty > 1$) for $|k \cdot k_d| > 0.5$.
Similarly, since G has a RHP-pole p we have a bound on the $\mathcal{H}_\infty$-norm of the closed-loop transfer function from disturbance d to input u

$$\|KSG_d(s)\|_\infty \geq |\mathcal{B}_z^{-1}(p)| \cdot |G_{ms}^{-1}(p)| \cdot |(G_d)_{ms}|_{s=p} = \frac{|z+p|}{|z-p|} \cdot |k_d| = 3\,|k_d|$$

and for $\|d\|_\infty = 1$ the input usage will be unacceptable ($\|u\|_\infty > 1$) when $|k_d| > 1/3$.

EXAMPLE 3. In this example we look at the effect of a RHP zero and pole in G_d. Let the plant be

$$G(s) = \frac{5}{(10s+1)(s-1)}$$

where $\mathcal{B}_z(s) = 1$ since there is no RHP-zeros in G. We consider the three disturbances

$$G_{d1}(s) = \frac{k_d}{(s-1)(0.2s+1)}, \qquad G_{d2}(s) = \frac{k_d}{(s+1)(0.2s+1)}$$

13

$$\text{and} \quad G_{d3}(s) = \frac{k_d(s \Leftrightarrow 2)}{(s+1)(0.2s+1)(s+2)}$$

For disturbance d_1 we must assume that the unstable pole at $p = 1$ is the same as the one in the plant G, such that it can be stabilized using feedback control. There is no RHP-zero in G, so we have no lower bound on $\|SG_{dk}(s)\|_\infty$. However, since G has a RHP-pole p there is a bound on $\|KSG_{dk}(s)\|_\infty$, and we find that the same lower bound applies to all three disturbances ($k \in \{1, 2, 3\}$), since

$$(G_{d1})_{ms} = (G_{d2})_{ms} = (G_{d3})_{ms} = \frac{k_d}{(s+1)(0.2s+1)}$$

We obtain

$$\|KSG_{dk}(s)\|_\infty \geq |G_{ms}^{-1}(G_{dk})_{ms}|_{s=p} = \left| \frac{(10s+1)(s+1)}{5} \frac{k_d}{(s+1)(0.2s+1)} \right|_{s=1} = \frac{11}{6} \cdot |k_d|$$

Thus, for $\|d\|_\infty = 1$ and if we require $\|u\|_\infty \leq 1$ we need to have $|k_d| \leq \frac{6}{11} \approx 0.55$. In other words, we may encounter excessive plant inputs (for all controllers) if $|k_d| > \frac{6}{11} \approx 0.55$.

6   Stabilization with input saturation

Our results provide tight lower bounds for the required input signals for an unstable plant. Can these bounds be used to say anything about the possibility of stabilizing a plant with constrained inputs (e.g. $|u(t)| \leq 1$, $\forall t$)? Assume that we have found, from one of these bounds, that we need $\|u\|_\infty > 1$. That is, at some frequency ω_0 we need $u(t) = u_{\max} \sin(\omega_0 t)$, with $u_{\max} > 1$. Will the system become unstable in the case where input is constrained such that $|u(t)| \leq 1$ ($\forall t$)?

Unfortunately, all our results are for linear systems, and we have not derived any results for this nonlinear effect of input saturation. Nevertheless, for simple low order systems we find as expected very good agreement between our lower bounds and the actual stability limit in systems with input saturation. Intuitively, this agreement should be good if the input remains saturated for a time which is longer than about $1/p$, where p is the RHP-pole.

6.1   Examples

EXAMPLE 1 CONTINUED. Consider again the plant

$$G(s) = \frac{1}{s \Leftrightarrow 10}$$

with the controller $K = 20$ which minimizes $\|KSN(s)\|_\infty$ when N is constant. With this controller we get $|KS(j\omega)| = 20$, $\forall \omega$, from which we know that sinusoidal measurement noise

$$n(t) = n_0 \sin(\omega_0 t)$$

cause the input to become

$$u(t) = 20n_0 \sin(\omega_0 t + \varphi)$$

for any frequency ω_0. Thus, for $n_0 = f \cdot 0.05$ we have that $u(t) = f \sin(\omega_0 t + \varphi)$, and for $f > 1$ the plant input will exceed ± 1 in magnitude. The question is: what happens if the inputs are constrained to be within ± 1? Will the stability be maintained? We will investigate this numerically by considering three frequencies; $\omega_0 = 1$ [rad/s], $\omega_0 = 10$ [rad/s] and $\omega_0 = 100$ [rad/s].

First, Figure 4 shows the response to $n(t) = 1.01 \cdot 0.05 \sin(t)$ ($\omega_0 = 1$ [rad/s], $f = 1.01$). We see that the plant becomes unstable due to the input saturation. Next, we consider $\omega_0 = 10$ [rad/s]. In this case we do *not*
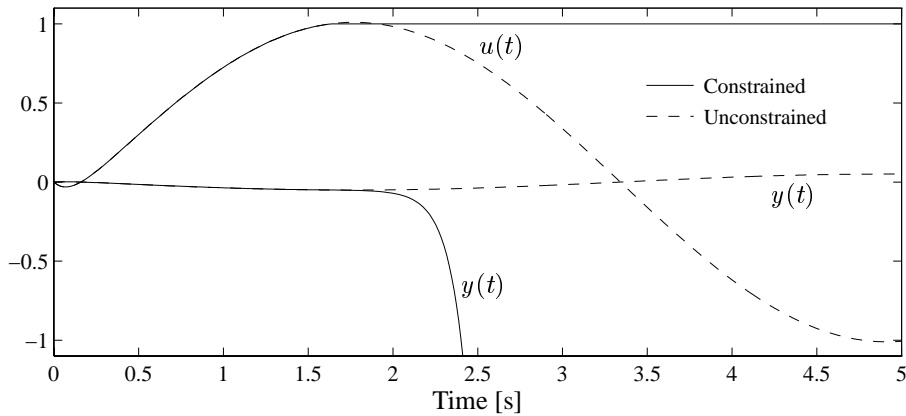
Figure 4: Closed-loop response at input u and output y of the plant G, due $n(t) = 1.01 \cdot 0.05 \sin(t)$
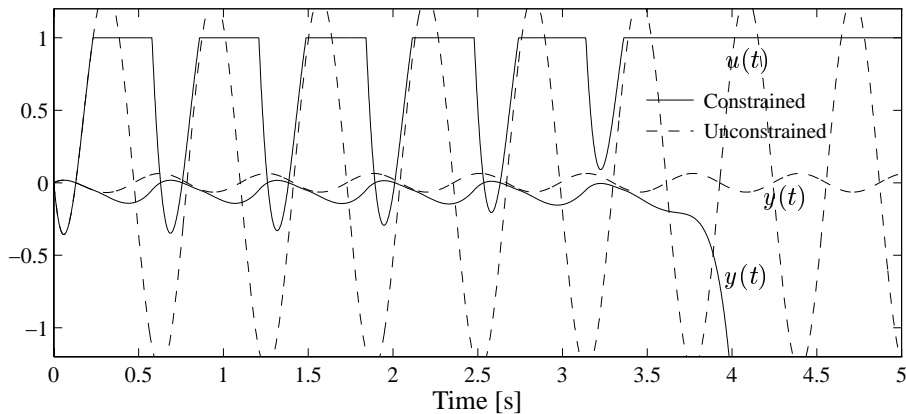

Figure 5: Closed-loop response at input u and output y of the plant G, due $n(t) = 1.29 \cdot 0.05 \sin(10t)$

get instability with $f = 1.01$ and $\omega_0 = 10$ [rad/s]. We find numerically that we need to increase the magnitude of the sinusoidal noise to about $f = 1.29$ to get instability for this particular frequency. Figure 5 shows the response to $n(t) = 1.29 \cdot 0.05 \sin(10t)$ ($\omega = 10$ [rad/s] and $f = 1.29$). Finally, as shown in Figure 6 we get instability with $n(t) = 1.6 \cdot 0.05 \sin(100t)$ ($\omega = 100$ [rad/s] and $f = 1.6$).

We experience that we have to increase the magnitude of the noise somewhat to get instability for sinusoidal measurement noise with frequency around the bandwidth and higher. However, we are still within a factor of two for a large frequency range for this particular plant. Measurement noise usually contain a large range of frequencies, which makes it even more probable that one loose stability of the plant if the lower bounds exceeds the allowable input range.

Note that the control system designer seldom wants the input to saturate when stabilizing an unstable plant due to the possibility of loosing stability. So our "engineering bounds" are really applicable in practical controller design.

As a final simulation, Figure 7 shows the closed-loop response due to a step of size $1.01 \cdot 0.05$ in n. (1% increase relative to the limit which cause u to exceed ± 1). This input signal can be viewed as consisting of infinite number of frequencies with decreasing magnitude, where the steady-state effect is the most important and can be viewed as a slowly varying sinusoid with $\omega_0 = 0$ [rad/s] and amplitude $1.01 \cdot 0.05$. As can be seen from the figure, the unconstrained input exceeds 1 slightly. When the input is constrained to be within ± 1, stability of the plant is lost.

EXAMPLE 3 CONTINUED. Consider again the plant
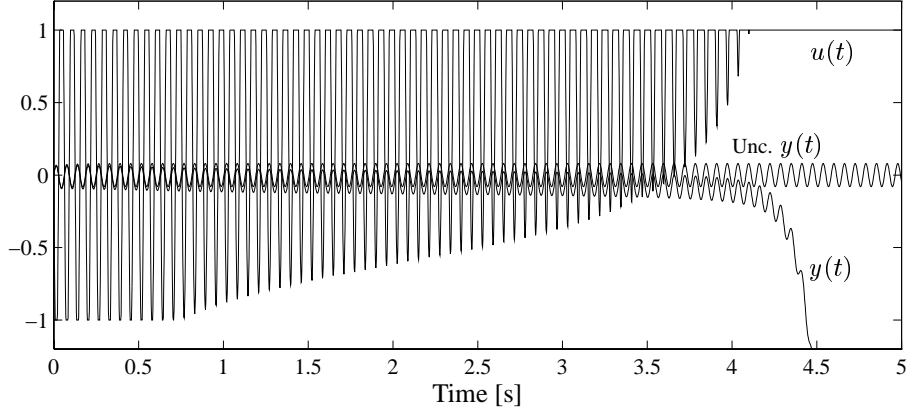
$$G(s) = \frac{5}{(10s + 1)(s \Leftrightarrow 1)}$$

15

Figure 6: Closed-loop response at input u and output y of the plant G, due $n(t) = 1.6 \cdot 0.05 \sin(100t)$, (unconstrained input not shown)
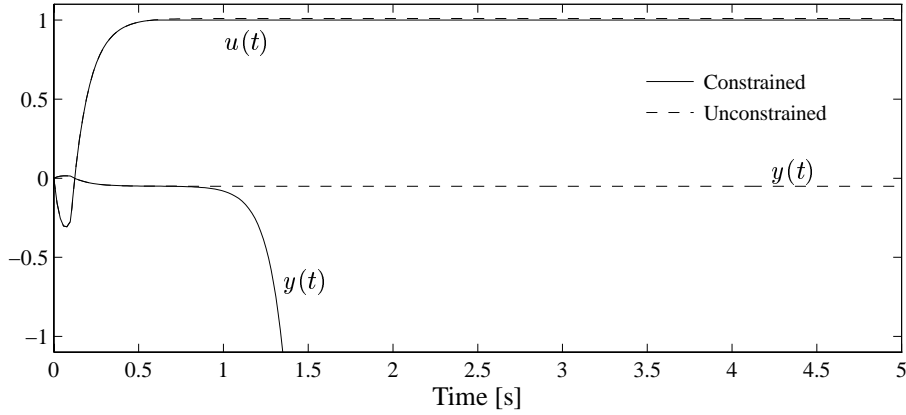

Figure 7: Closed-loop response at input u and output y of the plant G, due to step in measurement noise, $n(s) = 1.01 \cdot \frac{0.05}{s}$

In the simulations shown in this example, we have used the disturbance plant $G_d = G_{d3}$

$$G_d(s) = \frac{k_d(s \Leftrightarrow 2)}{(s+1)(0.2s+1)(s+2)}$$

However, it does not really matter which G_{dk} one uses, except that the initial responses may be different.

By using Theorem 3 with $V = G^{-1}G_d$, we obtain:

$$\mathcal{B}_z(s) = 1, \quad \mathcal{B}_p(s) = \frac{s \Leftrightarrow 1}{s+1}, \quad G_{ms}(s) = \frac{5}{(s+1)(10s+1)}, \quad (G_d)_{ms}(s) = \frac{k_d}{(s+1)(0.2s+1)},$$

$$V_{ms}(s) = \frac{k_d}{5}\frac{10s+1}{0.2s+1}, \quad V_{ms}(p) = \frac{11}{6}\cdot k_d, \quad P(s) = \frac{55}{6}\frac{0.2s+1}{10s+1} \quad \text{and} \quad Q(s) = \frac{49}{6}\frac{s+1}{10s+1}$$

The $\mathcal{H}_\infty$-optimal controller minimizing $\|KSG_d(s)\|_\infty$ becomes

$$K_\infty(s) = \frac{49}{11}(0.2s+1)(10s+1)$$

which is not proper. For $k_d = \frac{6}{11}$ the controller K_∞ results in $\|K_\infty SG_d(s)\|_\infty = 1$, and when $k_d = 0.55 > \frac{6}{11}$ (0.55 is the value of k_d used in the simulations) $\|K_\infty SG_d(s)\|_\infty = 1.008$. We note that the specter of $K_\infty SG_d(j\omega)$ is flat (constant). To get a realizable (proper) controller we add second order dynamics at high frequency to obtain the $\mathcal{H}_\infty$-suboptimal controller

$$\widetilde{K}_\infty(s) = \frac{49}{11}\frac{(0.2s+1)(10s+1)}{(0.01s+1)^2} \tag{45}$$

16

The $\mathcal{H}_\infty$-norm of the closed-loop transfer function $\widetilde{K}_\infty SG_d$ with $k_d = 0.55$ is

$$\|\widetilde{K}_\infty SG_d(s)\|_\infty = 1.027, \quad \text{for } \omega = 1.35 \text{ [rad/s]}.$$

To compare with a more traditional controller, which emphasize tight control at low frequencies, we also consider controlling the plant G using the feedback controller

$$K(s) = \frac{0.4 \cdot (10s + 1)^2}{s(0.1s + 1)^2} \tag{46}$$

With this K the $\mathcal{H}_\infty$-norm of the closed-loop transfer function KSG_d for $k_d = 0.55$ becomes

$$\|KSG_d(s)\|_\infty = 2.845, \quad \text{for } \omega = 2.056 \text{ [rad/s]}.$$

The magnitude of the closed-loop transfer functions $\widetilde{K}_\infty SG_d$ for $\widetilde{K}_\infty$ given by (45) is shown in Figure 8 together with the magnitude of KSG_d for K given in (46). From the figure we see that forcing $|KSG_d(j\omega)|$
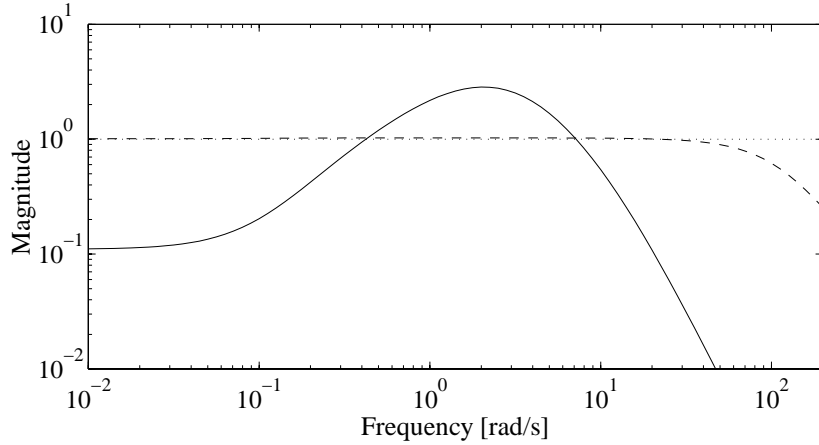

Figure 8: Closed-loop transfer functions KSG_d (dashed) and $\widetilde{K}_\infty SG_d$ (solid)

to be small at low frequencies, results in a peak in the medium frequency range (compare $|KSG_d(j\omega)|$ with $|\widetilde{K}_\infty SG_d(j\omega)|$ in Figure 8).

The non-linear constrained and the linear unconstrained responses to the unit step in disturbance d using the suboptimal $\mathcal{H}_\infty$-controller $\widetilde{K}_\infty$ given by (45) and the controller K given by (46), are shown in Figures 9 and 10. From the simulations we see that the input saturates (it may be difficult to separate the unconstrained input from the constrained input in Figure 9, since the unconstrained input only slightly exceeds $\Leftrightarrow$1), with the consequence that we loose stability of the plant for both controllers.

7   Two degrees-of-freedom control

In this section we consider the 2-DOF controller where

$$u = K_1 r - K_2(y + n) \tag{47}$$

(the 1-DOF considered above follows by setting $K_1 = K_2 = K$). For a 2-DOF controller the closed-loop transfer function from references $\tilde{r}$ to outputs $z_1 = w_P(y - r)$ becomes

$$w_P(SGK_1 - 1)R \tag{48}$$

We then have the following "special" lower bound on this transfer function.
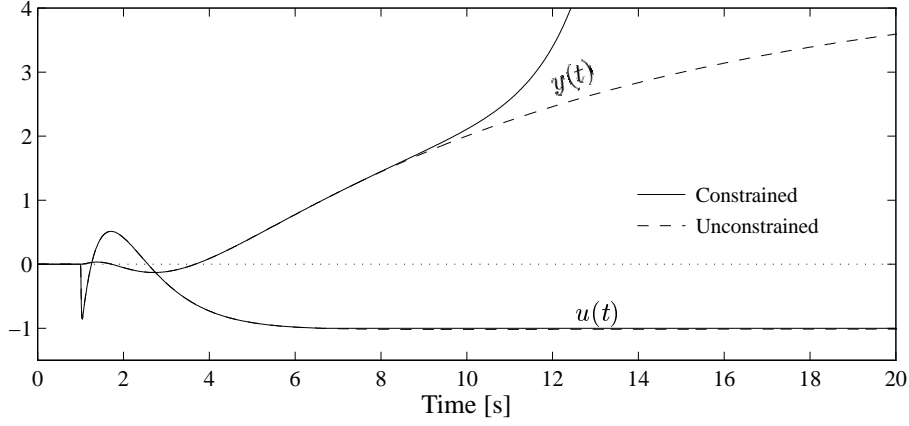
Figure 9: Responses in y and u due to unit step in disturbance d for constrained ($|u| \leq 1$) and unconstrained input with $\widetilde{K}_\infty$ given by (45)

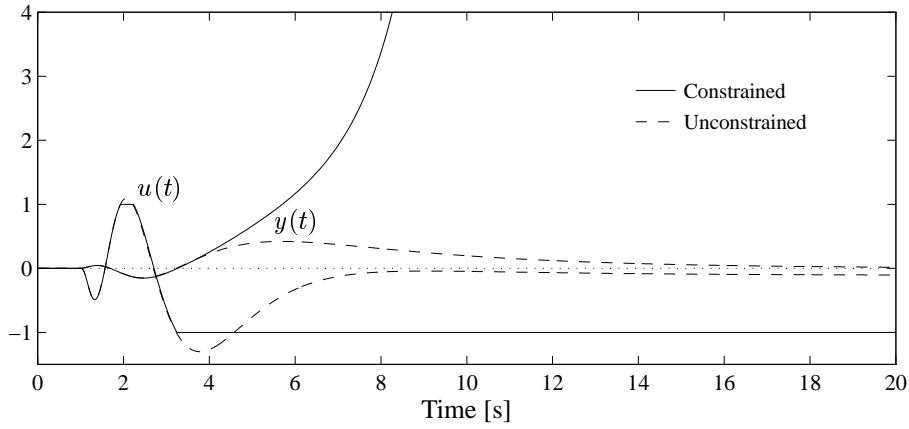Figure 10: Responses in y and u due to unit step in disturbance d for constrained ($|u| \leq 1$) and unconstrained input with K given by (46)

THEOREM 5. *Consider the* SISO *plant G with $N_z \geq 1$* RHP-*zeros z_j and $N_p \geq 0$* RHP-*poles $p_i \in \mathbb{C}_+$. Let the performance weight w_P be stable and minimum phase, and let the closed-loop transfer function $w_P(SGK_1 - 1)R$ be stable. Then for a two degrees-of-freedom controller the following lower bound applies*

$$\|w_P(SGK_1 - 1)R\|_\infty \geq \max_{\text{RHP-zeros, } z_j} |w_P(z_j)| \cdot |R_{ms}(z_j)| \tag{49}$$

Furthermore, the bound (49) is tight if the plant has one RHP-*zero z, and the controllers K_1 and K_2 which achieve the lower bound (49) are given by*

$$K_1 = \mathcal{B}_p(z)\, G_{ms}^{-1}(z) \cdot \left(1 - w_P^{-1}(s) R_{ms}^{-1}(s) w_P(z) R_{ms}(z)\right)_m \tag{50}$$

$$K_2 = \textit{The controller given in Theorem 4, minimizing } \|SG(s)\|_\infty. \tag{51}$$

REMARK 1. The bound (49) is clearly a lower bound (both for 1-DOF and 2-DOF controllers). The important fact is that (49) provides a tight lower bound for a plant with one RHP-zero and with the 2-DOF controller given in Theorem 5.

REMARK 2. It follows that K_1 is stable since w_P^{-1} is stable and R_{ms}^{-1} is stable. From Theorem 4 it follows that K_2 is stable.

The bound in (49) should be compared to the corresponding bound for 1-DOF controller (32):

$$\|w_P SR(s)\|_\infty \geq \max_{\text{RHP-zeros, } z_j} |w_P(z_j)| \cdot |\mathcal{B}_p^{-1}(z_j)| \cdot |R_{ms}(z_j)| \tag{52}$$

18

The fact that the lower bound (49) is tight when the plant has one RHP-zero and 2-DOF is applied makes it possible to conclude that *only* the RHP-zero pose limitations in this case. Thus, with a 2-DOF controller there is no additional penalty for having RHP-poles in G when performance is measured as $z_1 = w_P(y - r)$. However, from (52) we see that the penalty for having both a RHP-zero z_j and RHP-poles is $|\mathcal{B}_p^{-1}(z_j)| \geq 1$ for a 1-DOF controller.

8   Discussion

Form the lower bounds on input usage (see Section 5.2) we can easily *quantify* how much measurement noise and the magnitude of disturbance we can tolerate to avoid that the input exceeds some prespecified limits. We find this quantification appealing, and it should be useful for control engineers doing practical control design. We therefore used the term "engineering bounds" for this application of the lower bounds in the second part of Example 1. Here we will only stress that these bounds are of fundamental theoretical importance, and they are (in many cases) tight for the best possible controller. So the bounds are exact, i.e. these bounds are not rules of thumb.

In the $\mathcal{H}_\infty$-controller design procedure, the $\mathcal{H}_\infty$-norm of some weighted closed-loop transfer function is minimized. It has been shown that the resulting minimization problem is a convex problem, which can be solved numerically for example by introducing Linear Matrix Inequalities (LMI) or using γ-iteration.

In this paper we have looked at single closed-loop transfer functions which can be written as TV or SV. Practical $\mathcal{H}_\infty$-controller designs are usually set up as a stacked transfer function consisting of several closed-loop transfer functions. Usually the sensitivity appears as a factor in one or more of the closed-loop transfer functions, which is the origin to the name "mixed sensitivity". The controller designed will then reflect a trade-off between the different requirements expressed in each of the closed-loop transfer functions. For example, it is common to put weight on both the output performance and input usage. This can be expressed as in the mixed S/KS $\mathcal{H}_\infty$ controller design where the problem is to find the controller K such that the $\mathcal{H}_\infty$-norm of $\begin{bmatrix} w_P S \\ w_u K S \end{bmatrix}$ is minimized, i.e.

$$\min_K \left\| \begin{bmatrix} w_P S(s) \\ w_u K S(s) \end{bmatrix} \right\|_\infty$$

Lower and upper bounds on the $\mathcal{H}_\infty$-norm of the mixed S/KS sensitivity are

$$\max\{\|w_P S(s)\|_\infty , \|w_u K S(s)\|_\infty\} \leq \left\| \begin{bmatrix} w_P S(s) \\ w_u K S(s) \end{bmatrix} \right\|_\infty \leq \sqrt{2} \max\{\|w_P S(s)\|_\infty , \|w_u K S(s)\|_\infty\}$$

which shows that our individual lower bounds on $\|w_P S(s)\|_\infty$ and $\|w_u K S(s)\|_\infty$ provide useful information also for practical $\mathcal{H}_\infty$-controller designs.

In the γ-iteration the $\mathcal{H}_\infty$-minimization over the controller K is transformed to a convex minimization problem in the free variable γ, which is the $\mathcal{H}_\infty$-norm of the closed-loop transfer function[2]. Most packages[3] perform the γ-iteration using the bisection method. That is, given a high and a low value of γ (upper and lower bound) and a stabilizing controller, the bisection method is used to iterate on the value of γ. This "modern" controller synthesis shows one application of lower and upper bounds on the $\mathcal{H}_\infty$-norm of general closed-loop transfer functions. The lower bounds derived in this

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[2] In MATLAB Robust Control Toolbox γ is the inverse of the $\mathcal{H}_\infty$-norm of the closed-loop transfer function.

[3] See MATLAB, μ-tools or Robust Control Toolbox.

paper can be used as the low value of γ supplied to the γ-iteration. This follows since the largest singular value of matrix is larger than the largest element in the matrix. So, the largest lower bound on the $\mathcal{H}_\infty$-norm of a SISO transfer function in a larger multivariable stacked transfer function matrix still is a lower bound on the $\mathcal{H}_\infty$-norm of the stacked closed-loop transfer function in question.

9 Conclusion

- We have derived tight lower bounds on closed-loop transfer functions. The bounds are independent of the controller, and therefore reflects the controllability of the plant.
- The bounds extend and generalizes the SISO results by Zames (1981), Doyle et al. (1992) and Skogestad and Postlethwaite (1996) to also handle non-minimum phase and unstable weights. This allow us to derive *new* lower bounds on input usage due to disturbances, measurement noise and reference changes.
- The new lower bounds on input usage make it possible to *quantify* the minimum input usage for stabilization of unstable plants in the presence of worst case disturbances, measurement noise and reference changes.
- It is proved that the lower bounds are *tight*, by deriving analytical expressions for stable controllers which achieves an $\mathcal{H}_\infty$-norm of the closed-loop transfer functions equal to the lower bound for large classes of systems.
- Theorem 5 express the benefit of applying a 2-DOF controller compared to a 1-DOF controller when the plant is unstable and has a RHP-zero.
- The application of the lower bounds have been illustrated and the implications have studied in several examples. Nonlinear simulations have been used to find the amount of noise and disturbances which in combination with input constraints, cause loss of stability for unstable plants. The results show good agreement between this amount of noise and disturbances and the corresponding values predicted by the lower bounds, in the examples studied.

References

Doyle, J. C., Francis, B. and Tannenbaum, A. (1992). *Feedback Control Theory*, Macmillan Publishing Company.

Freudenberg, J. S. and Looze, D. P. (1988). *Frequency Domain Properties of Scalar and Multivariable Feedback Systems*, Vol. 104 of *Lecture Notes in Control and Information Sciences*, Springer-Verlag, Berlin.

Havre, K. and Skogestad, S. (1997a). Effect of RHP zeros and poles on performance in multivariable systems, *Accepted for publication in Journal of Process Control* .

Havre, K. and Skogestad, S. (1997b). Limitations imposed by RHP zeros/poles in multivariable systems, *Proc. from ECC97*, Brussels, Belgium.

Kwakernaak, H. (1995). Symmetries in control system design, *Trends in Control - A European Perspective, Alberto Isidori (ed.)* pp. 17–51.

Skogestad, S. and Postlethwaite, I. (1996). *Multivariable Feedback Control, Analysis and Design*, John Wiley & Sons, Chichester.

Youla, D. C., Bongiorno, J. J. and Lu, C. N. (1974). Single-loop feedback stabilization of linear multivariable dynamical plants, *Automatica* **10**: 159–173.

Zames, G. (1981). Feedback and optimal sensitivity: model reference transformations, multiplicative semi-norms, and approximate inverses, *IEEE Transactions on Automatic Control* **AC-26**(2): 301–320.

A Proofs of the results

Proof of Lemma 1. The first identity in (11) follows since extracting RHP-zeros in the product AB in terms of the all-pass filter $\mathcal{B}_z(AB)$, does not change the $\mathcal{H}_\infty$-norm. The reason is of course that $\mathcal{B}_z(AB)$ is all-pass for $s = j\omega$. To prove the latter identity, assume A has RHP-zeros which does not appear in the product AB, then B has RHP-poles for those RHP-zeros, and these RHP-poles can be factorized as $\mathcal{B}_p^{-1}(B)$. Similarly, if B has RHP-zeros which does not appear in the product AB, then A has RHP-poles for those RHP-zeros, and these RHP-poles can be factorized as $\mathcal{B}_p^{-1}(A)$. We obtain

$$AB = \mathcal{B}_p^{-1}(A)\,\mathcal{B}_z(A)\,A_{ms}\,\mathcal{B}_p^{-1}(B)\,\mathcal{B}_z(B)\,B_{ms} = \underbrace{\mathcal{B}_p^{-1}(A)\,\mathcal{B}_z(A)\,\mathcal{B}_p^{-1}(B)\,\mathcal{B}_z(B)}_{=\mathcal{B}_z(AB)}\,\underbrace{A_{ms}\,B_{ms}}_{=(AB)_m}$$

Since, AB is stable then $(AB)_m = A_{ms}B_{ms}$, and it follows that

$$\mathcal{B}_p^{-1}(A)\,\mathcal{B}_z(A)\,\mathcal{B}_p^{-1}(B)\,\mathcal{B}_z(B) = \mathcal{B}_z(AB)$$

Note that, $\mathcal{B}_p^{-1}(A)\,\mathcal{B}_z(B)$ are the RHP-zeros of B which are *not* RHP-poles in A and $\mathcal{B}_z(A)\,\mathcal{B}_p^{-1}(B)$ are the RHP-zeros of A which are *not* RHP-poles in B. □

Proof of Theorem 2.

1) **Factor out RHP zeros and poles in S and V.** Lemma 1 gives

$$\|SV(s)\|_\infty = \|S_{ms}V_{ms}(s)\|_\infty = \|S_m V_{ms}(s)\|_\infty$$

where the last equality holds since S is stable, i.e. $S_{ms} = S_m$.

2) **Introduce the stable scalar function $f(s) = S_m V_{ms}(s)$.**

3) **Apply the maximum modulus theorem to $f(s)$ at the RHP-zeros z_j of G.**

$$\|f(s)\|_\infty \geq |f(z_j)|$$

4) **Resubstitute the factorization of RHP-zeros in S**, i.e. use $S_m(z_j) = S(z_j)\mathcal{B}_p^{-1}(z_j)$ to get

$$f(z_j) = S_m(z_j)V_{ms}(z_j) = S(z_j)\mathcal{B}_p^{-1}(z_j)V_{ms}(z_j)$$

5) **Use the interpolation constraint (16) for RHP-zeros z_j in G**, i.e. use $S(z_j) = 1$.

6) **Evaluate the lower bound.**
$$|f(z_j)| = |\mathcal{B}_p^{-1}(z_j)| \cdot |V_{ms}(z_j)| \tag{53}$$

Note that $f(z_j)$ is independent of the controller K if V is independent of K.

Since these steps holds for all RHP-zeros z_j, Theorem 2 follows. □

Proof of Theorem 3. The transfer function P is stable, since $V_{ms}^{-1}(s)$ is stable and the remaining matrices $\mathcal{B}_z^{-1}(p)$ and $V_{ms}(p)$ are finite constant matrices. Consider $(1 \Leftrightarrow \mathcal{B}_z(s)P(s))$ which has a RHP-zero for $s = p$

$$(1 \Leftrightarrow \mathcal{B}_z(s)P(s)) = \mathcal{B}_p(s)\,(1 \Leftrightarrow \mathcal{B}_z(s)P(s))_m \quad \text{or} \quad (1 \Leftrightarrow \mathcal{B}_z(s)P(s))_m = \mathcal{B}_p^{-1}(s)\,(1 \Leftrightarrow \mathcal{B}_z(s)P(s))$$

We obtain

$$L(s) = GK(s) = \mathcal{B}_p^{-1}(s)\,\mathcal{B}_z(s)\,K_o = \frac{M(s)}{1 \Leftrightarrow M(s)} = \Leftrightarrow 1 + \frac{1}{1 \Leftrightarrow M(s)}$$

where

$$M(s) = \mathcal{B}_z(s)\,\mathcal{B}_z^{-1}(p)\,V_{ms}(p)\,V_{ms}^{-1}(s)$$

Since, $S^{-1} = 1 + L = \frac{1}{1-M}$ we have

$$S(s) \;=\; 1 \Leftrightarrow M(s) = 1 \Leftrightarrow \mathcal{B}_z(s)\,\underbrace{\mathcal{B}_z^{-1}(p)\,V_{ms}(p)\,V_{ms}^{-1}(s)}_{=P(s)} = 1 \Leftrightarrow \mathcal{B}_z(s)\,P(s)$$

Consider S at the complex value p

$$S(p) = 1 \Leftrightarrow \mathcal{B}_z(p)\,\mathcal{B}_z^{-1}(p)\,V_{ms}(p)\,V_{ms}^{-1}(p) = 0$$

We can therefore write

$$S(s) = \mathcal{B}_p(s)\,Q(s) \;\Leftrightarrow\; Q(s) = \mathcal{B}_p^{-1}(s)(1 \Leftrightarrow \mathcal{B}_z(s)\,P(s))$$

where Q is stable. Since, P, Q and G_{ms}^{-1} are all stable we have that K is stable. Furthermore,

$$T = \mathcal{B}_p^{-1}(s)\,\mathcal{B}_z(s)\,P(s)Q^{-1}(s)\mathcal{B}_p(s)Q(s) = \mathcal{B}_z(s)P(s)$$

and we get

$$\begin{aligned}
TV(s) &= \mathcal{B}_z(s)\,\mathcal{B}_z^{-1}(p)\,V_{ms}(p)\,V_{ms}^{-1}(s)\,\overbrace{\mathcal{B}_z(V(s))\,\mathcal{B}_p^{-1}(V(s))\,V_{ms}(s)}^{=V(s)} \\
&= \mathcal{B}_z^{-1}(p)\,V_{ms}(p)\,\mathcal{B}_z(s)\,\mathcal{B}_z(V(s))\,\mathcal{B}_p^{-1}(V(s))
\end{aligned}$$

The $\mathcal{H}_\infty$-norm of TV is

$$\|TV(s)\|_\infty = |\mathcal{B}_z^{-1}(p)| \cdot |V_{ms}(p)| \tag{54}$$

since $\mathcal{B}_z(s)\,\mathcal{B}_z(V(s))\,\mathcal{B}_p^{-1}(V(s))$ is all-pass for $s = j\omega$. Since the value of $\|TV(s)\|_\infty$ in (27) is the same as the lower bound (18), this is the controller which minimize $\|TV(s)\|_\infty$. $\qquad\square$

Proof of Theorem 4. The transfer function Q is stable, since $V_{ms}^{-1}(s)$ is stable and the remaining matrices $\mathcal{B}_p^{-1}(z)$ and $V_{ms}(z)$ are finite constant matrices. Consider $(1 \Leftrightarrow \mathcal{B}_p(s)Q(s))$ which has a RHP-zero for $s = z$

$$(1 \Leftrightarrow \mathcal{B}_p(s)Q(s)) = \mathcal{B}_z(s)\,(1 \Leftrightarrow \mathcal{B}_p(s)Q(s))_m \quad \text{or} \quad (1 \Leftrightarrow \mathcal{B}_p(s)Q(s))_m = \mathcal{B}_z^{-1}(s)\,(1 \Leftrightarrow \mathcal{B}_p(s)Q(s))$$

We obtain

$$\begin{aligned}
1 + L(s) &= 1 + GK(s) = 1 + \mathcal{B}_p^{-1}(s)\,\mathcal{B}_z(s)\,G_{ms}G_{ms}^{-1}K_o = 1 + \mathcal{B}_p^{-1}(s)\,\mathcal{B}_z(s)\,K_o \\
&= 1 + \mathcal{B}_p^{-1}(s)\,(1 \Leftrightarrow \mathcal{B}_p(s)\,Q(s))\,Q^{-1}(s) = 1 + \big(\mathcal{B}_p^{-1}(s)\,Q^{-1}(s) \Leftrightarrow 1\big) \\
&= \mathcal{B}_p^{-1}(s)\,Q^{-1}(s)
\end{aligned}$$

Which implies that

$$S(s) = \mathcal{B}_p(s)\,Q(s)$$

Since both $Q(s)$ and $\mathcal{B}_p(s)$ are stable, it follows that $S(s)$ is stable. At the complex value z, we have

$$Q(z) = \mathcal{B}_p^{-1}(z) \quad \text{and} \quad S(z) = \mathcal{B}_p(z)Q(z) = 1$$

It then follows that $T = 1 \Leftrightarrow S$ is stable and has a RHP-zero for $s = z$

$$T(z) = 1 \Leftrightarrow S(z) = 1 \Leftrightarrow 1 = 0$$

Since $T(s) = 1 \Leftrightarrow S(s) = 1 \Leftrightarrow \mathcal{B}_p(s)Q(s)$, we obtain

$$T(s) = \mathcal{B}_z(s)\,P(s) \;\Leftrightarrow\; P(s) = \mathcal{B}_z^{-1}(s)\,(1 \Leftrightarrow \mathcal{B}_p(s)\,Q(s))$$

where P is stable. Since, Q, P and G_{ms}^{-1} are all stable we have that K is stable. We get

$$
\begin{aligned}
SV(s) &= \mathcal{B}_p(s)\,\mathcal{B}_p^{-1}(z)\,V_{ms}(z)\,V_{ms}^{-1}(s)\overbrace{\mathcal{B}_z(V(s))\,\mathcal{B}_p^{-1}(V(s))\,V_{ms}(s)}^{=V(s)} \\
&= \mathcal{B}_p^{-1}(z)\,V_{ms}(z)\,\mathcal{B}_p(s)\,\mathcal{B}_z(V(s))\,\mathcal{B}_p^{-1}(V(s))
\end{aligned}
\tag{55}
$$

The $\mathcal{H}_\infty$-norm of SV is

$$
\|SV(s)\|_\infty = |\mathcal{B}_p^{-1}(z)| \cdot |V_{ms}(z)|
\tag{56}
$$

since $\mathcal{B}_p(s)\,\mathcal{B}_z(V(s))\,\mathcal{B}_p^{-1}(V(s))$ is all-pass for $s = j\omega$. Since the value of $\|SV(s)\|_\infty$ in (31) is the same as the lower bound (20), this is the controller which minimize $\|SV(s)\|_\infty$. □

Proof of Theorem 5. We first prove the lower bound (49). From Lemma 1 we have

$$
\|w_P(SGK_1 - 1)R(s)\|_\infty = \|w_P(SGK_1 - 1)_{ms}R_{ms}(s)\|_\infty
$$

since w_P is stable and minimum phase. Consider the scalar function $f(s) = w_P(SGK_1 - 1)_{ms}R_{ms}$ which is analytic (stable) in RHP since the closed-loop system is stable. By applying the maximum modulus theorem to $f(s)$ we get

$$
\|w_P(SGK_1 - 1)_{ms}R_{ms}(s)\|_\infty = \|f(s)\|_\infty \geq |f(z_j)|
$$

We get

$$
|f(z_j)| = |w_P(SGK_1 - 1)_{ms}R_{ms}|_{s=z_j} = |w_P(z_j)(-1)R_{ms}(z_j)| = |w_P(z_j)| \cdot |R_{ms}(z_j)|
$$

The second equality follows since SGK_1 must have RHP-zeros for $s = z_j$, since G has RHP-zeros for $s = z_j$, and S and K_1 must be stable (no RHP-poles in S or K_1 to cancel the RHP-zeros in G). It then follows that $(SGK_1 - 1)$ has no RHP-zeros for $s = z_j$.

We next prove that the controllers K_1 and K_2 given in Theorem 5, achieves this lower bound for the case when the plant has one RHP-zero z. From equation (55) in the proof of Theorem 4 we find that SG with $K = K_2$ (minimizing $\|SG(s)\|_\infty$) and $V = G$ becomes

$$
SG(s) = \mathcal{B}_p^{-1}(z)G_{ms}(z)\mathcal{B}_z(s)
$$

We obtain

$$
\begin{aligned}
SGK_1(s) - 1 &= \mathcal{B}_z(s)\overbrace{(1 - w_P^{-1}(s)R_{ms}^{-1}(s)w_P(z)R_{ms}(z))_m}^{\text{RHP-zero for } s = z} - 1 \\
&= \mathcal{B}_z(s)\mathcal{B}_z^{-1}(s)\left(1 - w_P^{-1}(s)R_{ms}^{-1}(s)w_P(z)R_{ms}(z)\right) - 1 \\
&= -w_P^{-1}(s)R_{ms}^{-1}(s)w_P(z)R_{ms}(z)
\end{aligned}
$$

which gives

$$
w_P(SGK_1(s) - 1)R(s) = -\mathcal{B}_z(R)\,\mathcal{B}_p^{-1}(R)\,w_P(z)R_{ms}(z)
$$

Since $w_P(SGK_1(s) - 1)R(s)$ is stable, so is $\mathcal{B}_z(R)\,\mathcal{B}_p^{-1}(R)\,w_P(z)R_{ms}(z)$ and RHP-poles in R may only cancel against RHP-zeros in $SGK_1 - 1$. It follows that

$$
\|w_P(SGK_1(s) - 1)R(s)\|_\infty = |w_P(z)| \cdot |R_{ms}(z)|
$$

and the controllers K_1 and K_2 given in (50) and (51) minimizes the $\mathcal{H}_\infty$-norm of $w_p(SGK_1 - 1)R(s)$. □