# PERFORMANCE LIMITATIONS FOR UNSTABLE SISO PLANTS

**Kjetil Havre** [1] **Sigurd Skogestad** [2]

*Chemical Engineering,*
*Norwegian University of Science and Technology*
*N-7034 Trondheim, Norway.*

**Abstract:** This paper examines the fundamental limitations imposed by instability in the plant (Right Half Plane (RHP) poles) on closed-loop performance. The main limitation is that instability requires the active use of plant inputs, and we quantify this is terms of tight lower bounds on the input magnitudes required when there are disturbances or measurement noise. These new bounds involve the $\mathcal{H}_\infty$-norm, which has direct engineering significance. The output performance in terms of disturbance rejection or reference tracking is *only* limited if the plant also has RHP-zeros. It is important to stress that the derived bounds are controller independent and that they are tight, meaning that there exists controllers which achieve the lower bounds.

**Keywords:** Input usage, achievable $\mathcal{H}_\infty$ performance, linear systems, RHP zeros and poles, stabilization, bounded inputs, input output controllability.

## 1. INTRODUCTION

An unstable plant, for example an unstable chemical reactor, can only be stabilized by use of feedback control which implies active use of the plant inputs. If measurement noise and/or disturbances are present (which is always the case in practical process control), then the input usage may become unacceptable.

In this paper, the above statements are quantified by deriving tight lower bounds on the $\mathcal{H}_\infty$-norm of the closed-loop transfer functions $SV$ and $TV$, where $S$ and $T$ are the sensitivity and complementary sensitivity functions. The transfer function $V$ can be viewed as a *generalized* "weight", which for our purpose should be independent of the feedback controller $K$.

One important application is that we can *quantify* the minimum input usage for stabilization in the presence of worst case measurement noise and disturbances. It appears that *even* for SISO systems this has been a difficult task, which has not been solved analytically until now.

To give the reader some appreciation of the basis of the bounds and their usefulness, we consider as a motivating example an unstable plant with a RHP-pole $p$. We want to obtain a lower bound on the $\mathcal{H}_\infty$-norm of the closed-loop transfer function $KS$ from measurement noise $n$ to plant input $u$. We first rewrite $KS = G^{-1}T$, which is on the form $TV$ with $V = G^{-1}$. The basis of our bound is the use of the maximum modulus principle and the "interpolation constraint" $T(p) = 1$, which must apply to achieve internal stability. We obtain (see Theorem 2 for details)

$$\|KS(s)\|_\infty = \|G^{-1}T(s)\|_\infty \geq |G_{ms}^{-1}(p)|$$

where $G_{ms}$ is the "stable and minimum phase" version of $G$ (if $G(s)$ also has a RHP-zero $z$ we get the additional penalty $\frac{|z+p|}{|z-p|}$). As an example, consider the plant $G(s) = \frac{1}{s-10}$, which has an unstable pole $p = 10$. We obtain $G_{ms}(s) = \frac{1}{s+10}$. For *any* linear feedback controller $K$, we find that the lower bound

$$\|KS(s)\|_\infty \geq |G_{ms}^{-1}(p)| = 2p = 20$$

---

[1] Also affiliated with: Institute for energy technology, P.O.Box 40, N-2007 Kjeller, Norway, E-mail: kjetil@ife.no.
[2] Author to whom correspondence should be addressed, Fax: (+47) 73 59 40 80, E-mail: skoge@kjemi.unit.no.

must be satisfied. Thus, if we require that the plant inputs are bounded with $\|u\|_\infty \le 1$, then we cannot allow the magnitude of measurement noise to exceed $\|n\|_\infty = 1/20 = 0.05$.

The basis for our results is the *important* work by Zames (1981), who made use of the interpolation constraint $S(z) = 1$ and the maximum modulus theorem to derive bounds on the $\mathcal{H}_\infty$-norm of $S$ for plants with one RHP-zero. Subsequently, these results were extended to plants with one RHP-pole and then to plants with combined RHP zeros and poles, e.g. (Doyle *et al.*, 1992, pp. 93–95) and (Skogestad and Postlethwaite, 1996).

However, these generalizations to unstable plants did *not* consider the input usage which involves the closed-loop transfer function $KS$. An important contribution of this paper is therefore to use the “trick” $KS = G^{-1}T$, which enable us to derive lower bounds on input usage, by using the general lower bound on $\|TV(s)\|_\infty$ with $V = G^{-1}$.

However, when $G$ is unstable (with RHP-pole $p$), then $V = G^{-1}$ has RHP-zeros for $s = p$. A second important contribution compared to earlier work is the ability to include RHP zeros and poles in the “weight” $V$ (under the assumption that $SV$ and $TV$ are stable).

A third important contribution is that we show that the lower bounds are *tight*. That is, we give analytical expressions for stable controllers which *achieves* an $\mathcal{H}_\infty$-norm of the closed-loop transfer function which is equal to the lower bound.

The bounds on $\|S(s)\|_\infty$ for plants with RHP-zero derived by Zames (1981) are also valid for multivariable systems. It is important to note that all the results given in this paper have been generalized to multivariable systems (Havre and Skogestad, 1997*b*).

## 2. BASICS FROM LINEAR CONTROL THEORY

We consider linear time invariant transfer function models on the form

$$y(s) = G(s)u(s) + G_d(s)d(s) \tag{1}$$

where $u$ is the manipulated input, $d$ is the disturbance, $y$ is the output, $G$ is the SISO plant model and $G_d$ is the SISO disturbance plant model. The measured output is $y_m = y + n$ where $n$ is the measurement noise.

The $\mathcal{H}_\infty$-norm of a stable rational transfer function $M(s)$ is defined as the peak value in the magnitude $|M(j\omega)|$ over all frequencies.

$$\|M(s)\|_\infty \triangleq \sup_\omega |M(j\omega)| \tag{2}$$

### 2.1 *Factorizations of* RHP *zeros and poles*

A rational transfer function $M(s)$ with zeros and poles in the open RHP, $\{z_j, p_i\} \in \mathbb{C}_+$, can be factorized in *Blaschke products* as follows [3]

$$M(s) = \mathcal{B}_z(M)\, M_m(s) \tag{3}$$

$$M(s) = \mathcal{B}_p^{-1}(M)\, M_s(s) \tag{4}$$

$$M(s) = \mathcal{B}_z(M)\, \mathcal{B}_p^{-1}(M)\, M_{ms}(s) \tag{5}$$

where

- $M_m$ – Minimum phase (subscript $m$) version of $M$ with the RHP-zeros mirrored across the imaginary axis.
- $M_s$ – Stable (subscript $s$) version of $M$ with the RHP-poles mirrored across the imaginary axis.
- $M_{ms}$ – Minimum phase, stable (subscript $ms$) version of $M$ with the RHP zeros and poles mirrored across the imaginary axis.
- $\mathcal{B}_z(M)$ – Stable all-pass rational transfer function ($|\mathcal{B}_z(j\omega)| = 1,\ \forall\omega$) containing the RHP-zeros (subscript $z$) of $M$.
- $\mathcal{B}_p(M)$ – Stable all-pass rational transfer function ($|\mathcal{B}_p(j\omega)| = 1,\ \forall\omega$) containing the RHP-poles (subscript $p$) of $M$ as RHP-zeros.

The all-pass filters are

$$\mathcal{B}_z(M(s)) = \prod_{j=1}^{N_z} \frac{s - z_j}{s + \bar{z}_j} \tag{6}$$

$$\mathcal{B}_p(M(s)) = \prod_{i=1}^{N_p} \frac{s - p_i}{s + \bar{p}_i} \tag{7}$$

where $N_z$ is the number of RHP-zeros $z_j \in \mathbb{C}_+$ and $N_p$ is the number of RHP-poles $p_i \in \mathbb{C}_+$ in $M$.

In most cases $M = G$ and to simplify the notation we often omit to show that the all-pass filters are dependent on $G$, i.e. we write $\mathcal{B}_p(s)$ and $\mathcal{B}_z(s)$ in the meaning of $\mathcal{B}_p(G(s))$ and $\mathcal{B}_z(G(s))$.

The order of the two operations $(\cdot)_m$ and $(\cdot)_s$ in the combined operator $(\cdot)_{ms}$ is arbitrary. It also follows that

$$(G^{-1})_{ms} = (G_{ms})^{-1} = G_{ms}^{-1} \tag{8}$$

And we note that

$$\|M(s)\|_\infty = \|M_m(s)\|_\infty = \|M_{ms}(s)\|_\infty \tag{9}$$

The first identity follows since $|\mathcal{B}_z(M(j\omega))| = 1,\ \forall\omega$, and the latter identity follows since $M$ is stable, i.e. $M_{ms} = M_m$ and $\mathcal{B}_p(M_m) = \mathcal{B}_p(M) = 1$.

[3] Note that the notation on the all-pass factorizations of RHP zeros and poles used in this paper is reversed compared to the notation used in (Skogestad and Postlethwaite, 1996; Havre and Skogestad, 1997*a*). The reason to this change of notation is to get consistent with what the literature generally defines as an all-pass filter.

To prove the main results in this paper we make use of the following Lemma.

LEMMA 1. *Consider a stable* SISO *transfer function $AB$ which can be expressed by the product of the* SISO *transfer functions $A$ and $B$, where both $A$ and $B$ may be unstable. Then*

$$\|AB\|_\infty = \|(AB)_m\|_\infty = \|A_{ms}B_{ms}\|_\infty \quad (10)$$

### 2.2 *Closing the loop*

A typical control problem is shown in Figure 1. In

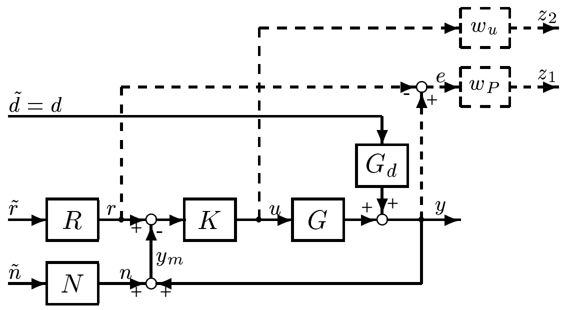


Fig. 1. One degree-of-freedom control configuration

the figure possible performance weights are given in dashed lines. Without loss of generality the performance weights $w_P$ and $w_u$ may be assumed to be stable and minimum phase. We have included both the reference $r$ and the measurement noise $n$, in addition to disturbances $d$ as external inputs. The transfer functions, $G_d$, $R$ and $N$ can be viewed as weights on the inputs, and the inputs: $\tilde{d}$, $\tilde{r}$ and $\tilde{n}$ are normalized in magnitude. Normally, $N$ is the inverse of signal to noise ratio.

We apply negative feedback control

$$u = K(r - y_m) = K(r - y - n) \quad (11)$$

The closed-loop transfer function $F$ from

$$v = \begin{bmatrix} \tilde{r} \\ \tilde{d} \\ \tilde{n} \end{bmatrix} \quad \text{to} \quad z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} w_P(y - r) \\ w_u u \end{bmatrix}$$

is

$$F(s) = \begin{bmatrix} -w_P SR & w_P SG_d & -w_P TN \\ w_u SKR & -w_u KSG_d & -w_u KSN \end{bmatrix} \quad (12)$$

where the sensitivity $S$ and the complementary sensitivity $T$ are defined as

$$S \triangleq (1 + GK)^{-1} = \frac{1}{1 + GK} \quad (13)$$

$$T \triangleq 1 - S = \frac{GK}{1 + GK} \quad (14)$$

To have good control performance (keep $z_1$ small) with a small input usage (keep $z_2$ small) we would like to have $\|F(s)\|_\infty$ small. That is we want all the SISO transfer functions in (12) small. In addition, there are robustness issues. For example, we wish to have $\|w_{\text{unc}}T(s)\|_\infty$ small, where $w_{\text{unc}}$ is the magnitude of the relative plant uncertainty.

### 2.3 *Interpolation constraints*

If $G$ has a RHP-zero $z$ or a RHP-pole $p$ then for internal stability of the feedback system the following interpolation constraints must apply (e.g. Skogestad and Postlethwaite, 1996):

$$T(z) = 0; \qquad S(z) = 1 \quad (15)$$

$$S(p) = 0; \qquad T(p) = 1 \quad (16)$$

## 3. LOWER BOUNDS ON CLOSED-LOOP TRANSFER FUNCTIONS

In this section we will give the main results, which are lower bounds on the $\mathcal{H}_\infty$-norm of closed-loop transfer functions which can be written on the form $SV$ or $TV$. The generalized "weight" $V$ is assumed to be independent of the feedback controller $K$. $V$ may be unstable but $SV$ and $TV$ must be stable. That is, it must be possible to stabilize all transfer functions by controlling the output $y$ using the input $u$ (this implies that all unstable modes of $N$, $R$ and $G_d$ also are modes of $G$).

*Some examples.* Consider the six transfer functions in (12). The first two can be written on the form $SV$ by selecting $V_{11} = w_P R$ and $V_{12} = w_P G_d$. The remaining four can be written on the form $TV$ by selecting $V_{13} = w_P N$, $V_{21} = w_u G^{-1} R$, $V_{22} = G^{-1} G_d$ and $V_{23} = w_u G^{-1} N$. From this we see that the "weight" $V$ may be unstable (if one or both of $G_d$ and $G^{-1}$ are unstable) and may contain RHP-zeros (if one or both of $G_d$ and $G^{-1}$ contain RHP-poles).

We now present the two main results.

THEOREM 1. (LOWER BOUND ON $\|SV(s)\|_\infty$). *Consider the* SISO *plant $G$ with $N_z$* RHP*-zeros $z_j$ and $N_p$* RHP*-poles $p_i \in \mathbb{C}_+$. Let $V$ be a rational transfer function, and assume that $SV$ is (internally) stable. Then the following lower bound on $\|SV(s)\|_\infty$ applies:*

$$\|SV(s)\|_\infty \geq \max_{\text{RHP-zeros, } z_j} |\mathcal{B}_p^{-1}(z_j)| \cdot |V_{ms}(z_j)| \quad (17)$$

REMARK 1. With $|\mathcal{B}_p(z_j)|$ we mean $|\mathcal{B}_p(G(s))|_{\text{evaluated at } s=z_j}$.
REMARK 2. The assumption that $SV$ is internally stable, means that it must be possible to stabilize the system using the feedback controller $K$, without having any RHP zero/pole cancellations between $G$ and $K$.

The lower bound (17) is independent of the controller $K$, if the weight $V$ is independent of $K$. The factor $|\mathcal{B}_p^{-1}(z_j)|$ takes into account the interactions between all RHP-poles $p_i \in \mathbb{C}_+$ and the single RHP-zero $z_j$ of $G$. As we shall see, this factor can be quite large if $G$ contains one or more RHP-poles close to the RHP-zero $z_j$.

*Proof of Theorem 1.*

1) **Factor out RHP zeros and poles in $S$ and $V$.** Lemma 1 gives

$$\|SV(s)\|_\infty = \|S_{ms}V_{ms}(s)\|_\infty = \|S_m V_{ms}(s)\|_\infty$$

where the last equality holds since $S$ is stable.

2) **Introduce the stable scalar function $f(s) = S_m V_{ms}(s)$.**

3) **Apply the maximum modulus theorem to $f(s)$ at the RHP-zeros $z_j$ of $G$.**

$$\|f(s)\|_\infty \geq |f(z_j)|$$

4) **Resubstitute the factorization of RHP-zeros in $S$**, i.e. use $S_m(z_j) = S(z_j)\mathcal{B}_p^{-1}(z_j)$ to get

$$f(z_j) = S_m(z_j)V_{ms}(z_j) = S(z_j)\mathcal{B}_p^{-1}(z_j)V_{ms}(z_j)$$

5) **Use the interpolation constraint (15) for RHP-zeros $z_j$ in $G$**, i.e. use $S(z_j) = 1$.

6) **Evaluate the lower bound.**

$$|f(z_j)| = |\mathcal{B}_p^{-1}(z_j)| \cdot |V_{ms}(z_j)| \tag{18}$$

Note, at that $f(z_j)$ is independent of the controller $K$ if $V$ is independent of $K$.

Since these steps holds for all RHP-zeros $z_j$, Theorem 1 follows. □

THEOREM 2. (LOWER BOUND ON $\|TV(s)\|_\infty$). *Consider the* SISO *plant $G$ with $N_z$* RHP*-zeros $z_j \in \mathbb{C}_+$ and $N_p$* RHP*-poles $p_i$. Let $V$ be a rational transfer function, and assume that $TV$ is (internally) stable. Then the following lower bound on $\|TV(s)\|_\infty$ applies:*

$$\|TV(s)\|_\infty \geq \max_{\text{RHP-poles, } p_i} |\mathcal{B}_z^{-1}(p_i)| \cdot |V_{ms}(p_i)| \tag{19}$$

The lower bound (19) is independent of the controller $K$, if the weight $V$ is independent of $K$. The factor $|\mathcal{B}_z^{-1}(p_i)|$ takes into account the interactions between all the RHP-zeros $z_j \in \mathbb{C}_+$ and the single RHP-pole $p_i$ of $G$. As we shall see this factor can be quite large if $G$ contains one or more RHP-zeros close to the RHP-pole $p_i$.

Remarks on Theorems 1 and 2:

1) The bound on $\|SV(s)\|_\infty$ is caused by the RHP-zeros $z_j$ in $G$, and the term $|\mathcal{B}_p^{-1}(z_j)| \geq 1$ gives an additional penalty for plants which also have RHP-poles. For the case when $G$ has *no* RHP-poles, then $\mathcal{B}_p^{-1}(z_j) = 1$.
2) The bound on $\|TV(s)\|_\infty$ is caused by the RHP-poles $p_i$ in $G$, and the term $|\mathcal{B}_z^{-1}(p_i)| \geq 1$ gives an additional penalty for plants which also have RHP-zeros. For the case when $G$ has *no* RHP-zeros, then $\mathcal{B}_z^{-1}(p_i) = 1$.
3) For a plant with a single RHP-zero $z$ and a single RHP-pole $p$ the additional penalty is given by the term

$$|\mathcal{B}_p(z)| = |\mathcal{B}_z(p)| = \frac{|z+p|}{|z-p|}$$

## 4. TIGHTNESS OF LOWER BOUNDS

Theorems 1 and 2 provide lower bounds on $\|SV(s)\|_\infty$ and $\|TV(s)\|_\infty$. The question is whether these bounds are tight, meaning that there actually exists controllers which achieves the bounds?

The answer is "yes" if there is only one RHP-zero or one RHP-pole. We prove tightness of the lower bounds by constructing controllers which achieves the bounds. In this short version of the paper we only give the controller minimizing $\|TV(s)\|_\infty$.

THEOREM 3. ($K$ WHICH MINIMIZE $\|TV(s)\|_\infty$). *Consider the* SISO *plant $G$ with one* RHP*-pole $p$ and $N_p$* RHP*-zeros $z_j \in \mathbb{C}_+$. Then the feedback controller $K$ which minimize $\|TV(s)\|_\infty$ is given by*

$$K(s) = G_{ms}^{-1}K_o(s), \quad K_o(s) = PQ^{-1}(s) \tag{20}$$

*where*

$$P(s) = \mathcal{B}_z^{-1}(p)\, V_{ms}(p)\, V_{ms}^{-1}(s) \tag{21}$$

$$Q(s) = \mathcal{B}_p^{-1}(s)\ (1 - \mathcal{B}_z(s)\, P(s)) \tag{22}$$

*With this controller we have*

$$\|TV(s)\|_\infty = |\mathcal{B}_z^{-1}(p)| \cdot |V_{ms}(p)| \tag{23}$$

The controller in Theorem 3 gives constant $|TV(j\omega)|$ for all $\omega$. We note that no properness restriction has been imposed on the controller, so the controller given in Theorem 3 may be improper. Also note that the controller $K(s)$ in Theorem 3 is always stable and minimum phase. This may seem surprising since it is known that some plants with RHP zeros and poles require an unstable controller (Youla *et al.*, 1974) to achieve closed-loop stability. However, this assumes that the controller is strictly proper, and does therefore not apply in our case.

## 5. APPLICATIONS OF LOWER BOUNDS

### 5.1 *Bounds on important closed-loop transfer functions*

Consider again the six transfer functions in (12), and the weighted complementary sensitivity function $w_{\text{unc}}T$. For simplicity we assume that $w_P$, $w_u$, $w_{\text{unc}}$, $R$ and $N$ are all stable minimum phase (or have been replaced by the stable minimum phase counterparts with same magnitude). From Theorems 1 and 2 we obtain:

*Output performance, reference tracking*:

$$\|w_P SR(s)\|_\infty \geq \max_{\text{RHP-zeros, } z_j} |w_P(z_j)| \cdot |\mathcal{B}_p^{-1}(z_j)| \cdot |R(z_j)| \tag{24}$$

*Output performance, disturbance rejection*:

$$\|w_P S G_d(s)\|_\infty \geq \max_{\text{RHP-zeros}, z_j} |w_P(z_j)| \cdot |\mathcal{B}_p^{-1}(z_j)| \cdot |(G_d)_{ms}|_{s=z_j} \quad (25)$$

*Output performance, measurement noise rejection*:

$$\|w_P S N(s)\|_\infty \geq \max_{\text{RHP-zeros}, z_j} |w_P(z_j)| \cdot |\mathcal{B}_p^{-1}(z_j)| \cdot |N(z_j)| \quad (26)$$

*Input usage, reference tracking*:

$$\|w_u K S R(s)\|_\infty = \|w_u T G^{-1} R(s)\|_\infty \geq \max_{\text{RHP-poles}, p_i} |w_u(p_i)| \cdot |\mathcal{B}_z^{-1}(p_i)| \cdot |G_{ms}^{-1} R(p_i)| \quad (27)$$

*Input usage, disturbance rejection*:

$$\|w_u K S G_d(s)\|_\infty = \|w_u T G^{-1} G_d(s)\|_\infty \geq \max_{\text{RHP-poles}, p_i} |w_u(p_i)| \cdot |\mathcal{B}_z^{-1}(p_i)| \cdot |G_{ms}^{-1} (G_d)_{ms}|_{s=p_i} \quad (28)$$

*Input usage, measurement noise rejection*:

$$\|w_u K S N(s)\|_\infty = \|w_u T G^{-1} N(s)\|_\infty \geq \max_{\text{RHP-poles}, p_i} |w_u(p_i)| \cdot |\mathcal{B}_z^{-1}(p_i)| \cdot |G_{ms}^{-1} N(p_i)| \quad (29)$$

*Closed-loop sensitivity to plant uncertainty*:

$$\|w_{\text{unc}} T(s)\|_\infty \geq \max_{\text{RHP-poles}, p_i} |w_{\text{unc}}(p_i)| \cdot |\mathcal{B}_z^{-1}(p_i)| \quad (30)$$

Note that we mainly have inherent limitations on (output) performance when the plant has RHP-zeros. The exception is for measurement noise, where the requirement of stabilizing an unstable pole may give poor performance.

All the bounds on input usage are caused by the presence of RHP-poles. This is reasonable since we need active use of the input in order to stabilize the plant. This is considered in more detail in the next section.

## 5.2 *Implications for stabilization with bounded inputs*

Our bounds involve the $\mathcal{H}_\infty$-norm, and their large engineering usefulness may not be immediate. In the following we will concentrate on the bounds involving input usage and we will use the lower bounds to derive and *quantify* the conclusion:

- *Bounded inputs combined with disturbances and noise may make stabilization impossible.*

*Measurement noise.* The transfer function from normalized measurement noise $\tilde{n}$ to the input $u$ is $KSN$. Then from (29) with $w_u = 1$

$$\|u\|_\infty = \|KSN(s)\|_\infty \geq \max_{\text{RHP-poles}, p_i} |\mathcal{B}_z^{-1}(p_i)| \cdot |G_{ms}^{-1}(p_i) N(p_i)| \quad (31)$$

Thus, to have $\|u\|_\infty \leq 1$ for $\|\tilde{n}\|_\infty = 1$, we must require

$$|G_{ms}(p_i)| \geq |\mathcal{B}_z^{-1}(p_i)| \cdot |N(p_i)| \quad (32)$$

for the worst case $p_i$ (we have here assumed that $N$ is minimum phase).

*Disturbances.* Similar results as those for measurement noise apply to disturbances if we replace $N$ by $G_d$. From (28) with $w_u = 1$ we obtain

$$\|u\|_\infty = \|KSG_d(s)\|_\infty \geq \max_{\text{RHP-poles}, p_i} |\mathcal{B}_z^{-1}(p_i)| \cdot |G_{ms}^{-1} (G_d)_{ms}|_{s=p_i} \quad (33)$$

To have $\|u\|_\infty \leq 1$ for $\|d\|_\infty = 1$ we must require

$$|G_{ms}(p_i)| \geq \left|\mathcal{B}_z^{-1}(p_i)\right| \cdot |(G_d)_{ms}|_{s=p_i} \quad (34)$$

for the worst case pole $p_i$.

*References.* For reference changes with $\|\tilde{r}\|_\infty = 1$, we find the same bound (33), but with $G_d$ replaced by $R$. However, the implications are less severe since we may choose *not* to follow the references (e.g. set $R = 0$).

## 5.3 *Examples*

EXAMPLE 1. Consider the unstable plant

$$G(s) = \frac{1}{s-p}, \quad p > 0$$

with the RHP-pole at $p$. From (31) we have the following lower bound on the $\mathcal{H}_\infty$-norm of the transfer function from normalized measurement noise $\tilde{n}$ to input $u$ (we assume that $N$ is minimum phase)

$$\|KSN(s)\|_\infty \geq |G_{ms}^{-1}(p)| \cdot |N(p)|$$

In our case $G^{-1} = s - p$, $G_{ms}^{-1}(s) = s + p$, $G_{ms}^{-1}(p) = 2p$, and the lower bound becomes

$$\|KSN(s)\|_\infty \geq 2p \cdot |N(p)| \quad (35)$$

The controller which minimizes $\|TV(s)\|_\infty$ and achieves the bound (35) is given in Theorem 3. For the special case where $N(s)$ is a constant $N(s) = N$ we get the proportional feedback controller $K(s) = 2p$.

As a numerical example, let $p = 10$, then we must have for any stabilizing feedback controller $K$

$$\|KSN(s)\|_\infty \geq 20 \cdot |N(p)|$$

Thus with $\|\tilde{n}\|_\infty = 1$ we will need excessive inputs ($\|u\|_\infty > 1$) if $|N(p)| \geq |G_{ms}(p)| = 0.05$. Assume that $N(s) = N(p) = 0.05$, then $K(s) = 2p = 20$. This controller gives a “flat” frequency response, i.e. $|KSN(j\omega)| = 20, \forall\omega$. Thus, at any frequency $\omega_0$ the closed-loop response in $u$ due to

$$n(t) = 0.05\sin(\omega_0 t), \quad \text{is} \quad u(t) = \sin(\omega_0 t + \varphi) \quad \forall\omega$$

So, the input $u(t)$ oscillates between $\pm 1$. The response in $u$ and $y$ due to $n(t) = 0.05\sin(4t)$ is shown in Figure 2.

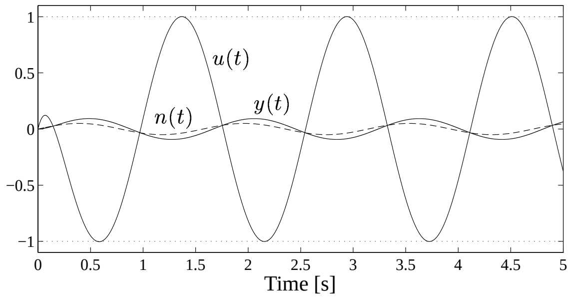


Fig. 2. Closed-loop response at input $u$ and output $y$ of the plant $G$, due to $n(t) = 0.05\sin(4t)$ (dotted), with $K = 20$

EXAMPLE 2. In this example we look at the effect of a RHP zero and pole in $G_d$. Let the plant be

$$G(s) = \frac{5}{(10s+1)(s-1)}$$

where $\mathcal{B}_z(s) = 1$ since there is no RHP-zeros in $G$. We consider the three disturbances

$$G_{d1}(s) = \frac{k_d}{(s-1)(0.2s+1)}, \quad G_{d2}(s) = \frac{k_d}{(s+1)(0.2s+1)}$$
$$\text{and} \quad G_{d3}(s) = \frac{k_d(s-2)}{(s+1)(0.2s+1)(s+2)}$$

For disturbance $d_1$ we must assume that the unstable pole at $p = 1$ is the same as the one in the plant $G$, such that it can be stabilized using feedback control. There is no RHP-zero in $G$, so we have no lower bound on $\|SG_{dk}(s)\|_\infty$. However, since $G$ has a RHP-pole $p$ there is a bound on $\|KSG_{dk}(s)\|_\infty$, and we find that the same lower bound applies to all three disturbances ($k \in \{1, 2, 3\}$), since

$$(G_{d1})_{ms} = (G_{d2})_{ms} = (G_{d3})_{ms} = \frac{k_d}{(s+1)(0.2s+1)}$$

We obtain

$$\|KSG_{dk}(s)\|_\infty \geq |G_{ms}^{-1}(G_{dk})_{ms}(p)|$$
$$= \left|\frac{(10s+1)(s+1)}{5}\frac{k_d}{(s+1)(0.2s+1)}\right|_{s=1} = \frac{11}{6}\cdot|k_d|$$

Thus, for $\|d\|_\infty = 1$ and if we require $\|u\|_\infty \leq 1$ we need to have $|k_d| \leq \frac{6}{11} \approx 0.55$. In other words, we may encounter excessive plant inputs (for all controllers) if $|k_d| > \frac{6}{11} \approx 0.55$.

## 6. STABILIZATION WITH INPUT SATURATION

Our results provide tight lower bounds for the required input signals for an unstable plant. Assume that we have found, from one of these bounds, that we need $\|u\|_\infty > 1$. That is, at some frequency $\omega_0$ we need $u(t) = u_{\max}\sin(\omega_0 t)$, with $u_{\max} > 1$. Will the system become unstable in the case where input is constrained such that $|u(t)| \leq 1$ $(\forall t)$?

Unfortunately, all our results are for linear systems, and we have not derived any results for this nonlinear effect of input saturation.

Nevertheless, for simple low order systems we find as expected very good agreement between our lower bounds and the actual stability limit in systems with input saturation.

Intuitively, this agreement should be good if the input remains saturated for a time which is longer than about $1/p$, where $p$ is the RHP-pole.

### 6.1 *Examples*

EXAMPLE 1 CONTINUED. Consider again the plant

$$G(s) = \frac{1}{s-10}$$

with the controller $K = 20$ which minimizes $\|KSN(s)\|_\infty$ when $N$ is constant. With this controller we get $|KS(j\omega)| = 20, \forall\omega$, from which we know that sinusoidal measurement noise

$$n(t) = n_0\sin(\omega_0 t)$$

cause the input to become

$$u(t) = 20n_0\sin(\omega_0 t + \varphi)$$

for any frequency $\omega_0$. Thus, for $n_0 = f \cdot 0.05$ we have that $u(t) = f\sin(\omega_0 t + \varphi)$, and for $f > 1$ the plant input will exceed $\pm 1$ in magnitude. The question is: what happens if the inputs are constrained to be within $\pm 1$? Will the stability be maintained?

We will investigate this numerically by considering two frequencies; $\omega_0 = 1$ [rad/s] and $\omega_0 = 10$ [rad/s].

First, Figure 3 shows the response to $n(t) = 1.01 \cdot 0.05\sin(t)$ ($\omega_0 = 1$ [rad/s], $f = 1.01$). We see that the plant becomes unstable due to the input saturation.

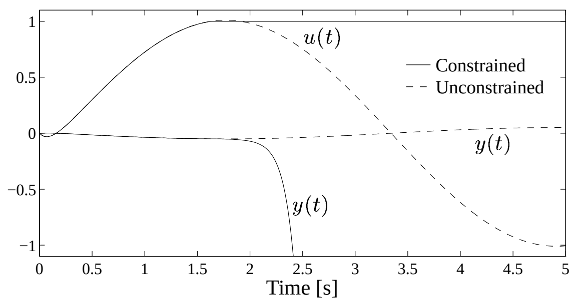


Fig. 3. Closed-loop response at input $u$ and output $y$ of the plant $G$, due $n(t) = 1.01 \cdot 0.05\sin(t)$

Next, we consider $\omega_0 = 10$ [rad/s]. In this case we do *not* get instability with $f = 1.01$ and $\omega_0 = 10$ [rad/s]. We find numerically that we need to increase the magnitude of the sinusoidal noise above $f = 1.29$ to get instability for this frequency. Figure 4 shows the response to $n(t) = 1.29 \cdot 0.05\sin(10t)$ ($\omega = 10$ [rad/s] and $f = 1.29$).

EXAMPLE 2 CONTINUED. Consider again the plant

$$G(s) = \frac{5}{(10s+1)(s-1)}$$

In the simulations shown in this example, we have used the disturbance plant $G_d = G_{d3}$

$$G_d(s) = \frac{k_d(s-2)}{(s+1)(0.2s+1)(s+2)}$$

However, it does not really matter which $G_{dk}$ one uses, except that the initial responses may be different.

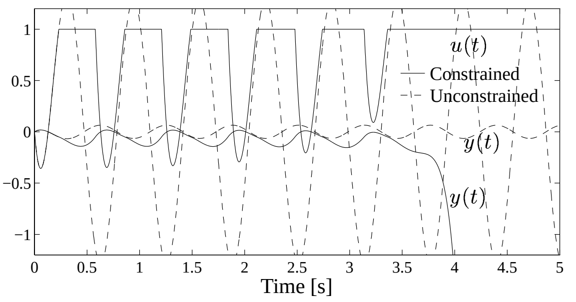


Fig. 4. Closed-loop response at input $u$ and output $y$ of the plant $G$, due $n(t) = 1.29 \cdot 0.05 \sin(10t)$

By using Theorem 3 with $V = G^{-1}G_d$, we obtain:

$$\mathcal{B}_z(s) = 1, \quad \mathcal{B}_p(s) = \frac{s-1}{s+1}, \quad G_{ms}(s) = \frac{5}{(s+1)(10s+1)},$$

$$(G_d)_{ms}(s) = \frac{k_d}{(s+1)(0.2s+1)},$$

$$V_{ms}(s) = \frac{k_d}{5}\frac{10s+1}{0.2s+1}, \quad V_{ms}(p) = \frac{11}{6} \cdot k_d,$$

$$P(s) = \frac{55}{6}\frac{0.2s+1}{10s+1} \quad \text{and} \quad Q(s) = \frac{49}{6}\frac{s+1}{10s+1}$$

The $\mathcal{H}_\infty$-optimal controller minimizing $\|KSG_d(s)\|_\infty$ becomes

$$K_\infty(s) = \frac{49}{11}(0.2s+1)(10s+1)$$

which is not proper. For $k_d = \frac{6}{11}$ the controller $K_\infty$ results in $\|K_\infty SG_d(s)\|_\infty = 1$, and when $k_d = 0.55 > \frac{6}{11}$ (0.55 is the value of $k_d$ used in the simulations) $\|K_\infty SG_d(s)\|_\infty = 1.008$. We note that the specter of $K_\infty SG_d(j\omega)$ is flat (constant). To get a realizable (proper) controller we add second order dynamics at high frequency to obtain the $\mathcal{H}_\infty$-suboptimal controller

$$\widetilde{K}_\infty(s) = \frac{49}{11}\frac{(0.2s+1)(10s+1)}{(0.01s+1)^2} \tag{36}$$

The $\mathcal{H}_\infty$-norm of the closed-loop transfer function $\widetilde{K}_\infty SG_d$ with $k_d = 0.55$ is

$$\|\widetilde{K}_\infty SG_d(s)\|_\infty = 1.027, \quad \text{for } \omega = 1.35 \text{ [rad/s]}.$$

To compare with a more traditional controller, which emphasize tighter control at low frequencies we also consider controlling the plant $G$ using the feedback controller

$$K(s) = \frac{0.4 \cdot (10s+1)^2}{s(0.1s+1)^2} \tag{37}$$

With this $K$ the $\mathcal{H}_\infty$-norm of the closed-loop transfer function $KSG_d$ for $k_d = 0.55$ becomes

$$\|KSG_d(s)\|_\infty = 2.845, \quad \text{for } \omega = 2.056 \text{ [rad/s]}.$$

The magnitude of the closed-loop transfer functions $\widetilde{K}_\infty SG_d$ for $\widetilde{K}_\infty$ given by (36) is shown in Figure 5 together with the magnitude of $KSG_d$ for $K$ given in (37). From the figure we see that forcing $|KSG_d(j\omega)|$ to be small at low frequencies, results in a peak in the medium frequency range (compare $|KSG_d(j\omega)|$ with $|\widetilde{K}_\infty SG_d(j\omega)|$ in Figure 5).

The non-linear constrained and the linear unconstrained responses to the unit step in disturbance $d$ using the suboptimal $\mathcal{H}_\infty$-controller $\widetilde{K}_\infty$ given by (36) and the controller $K$ given by (37), are shown in Figures 6 and 7. From the simulations we see that the input saturates (it may be difficult to separate the unconstrained input from the constrained input in Figure 6, since the unconstrained input only slightly exceeds $-1$), with the consequence that we loose stability of the plant for both controllers.

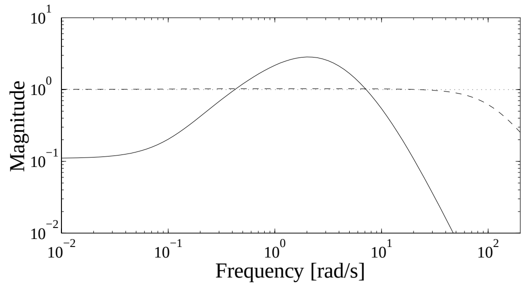


Fig. 5. Closed-loop transfer functions $KSG_d$ (dashed) and $\widetilde{K}_\infty SG_d$ (solid)

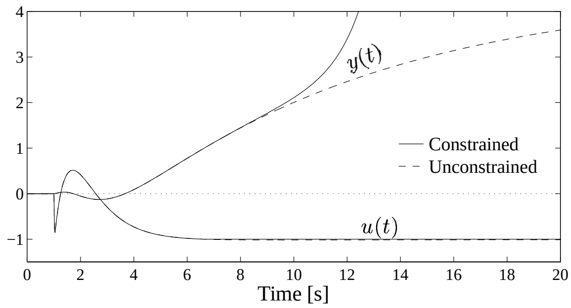


Fig. 6. Responses in $y$ and $u$ due to unit step in disturbance $d$ for constrained ($|u| \leq 1$) and unconstrained input with $\widetilde{K}_\infty$ given by (36)

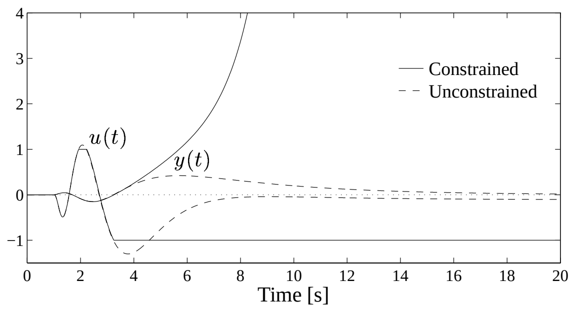


Fig. 7. Responses in $y$ and $u$ due to unit step in disturbance $d$ for constrained ($|u| \leq 1$) and unconstrained input with $K$ given by (37)

## 7. REFERENCES


Doyle, J. C., B. Francis and A. Tannenbaum (1992). *Feedback Control Theory*. Macmillan Publishing Company.

Havre, K. and S. Skogestad (1997*a*). Effect of RHP zeros and poles on performance in multivariable systems. *Accepted for publication in Journal of Process Control.*

Havre, K. and S. Skogestad (1997*b*). Limitations imposed by RHP zeros/poles in multivariable systems. In: *Proc. from ECC97*. Brussels, Belgium.

Skogestad, S. and I. Postlethwaite (1996). *Multivariable Feedback Control, Analysis and Design*. John Wiley & Sons. Chichester.

Youla, D. C., H. A. Jabr and C. N. Lu (1974). Single-loop feedback stabilization of linear multivariable dynamical plants. *Automatica* **10**, 159–173.

Zames, G. (1981). Feedback and optimal sensitivity: model reference transformations, multiplicative seminorms, and approximate inverses. *IEEE Transactions on Automatic Control* **AC-26**(2), 301–320.