INPUT/OUTPUT SELECTION AND PARTIAL CONTROL

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Abstract. This paper considers the process of control structure design, and in particular it addresses the issue of selecting measurements and manipulations for partial control. Partial control at a given level involves controlling a subset of the outputs with an associated control objective. The relative gain array (RGA) and singular value decomposition (SVD) are useful measures for selecting inputs and outputs.

Keywords. Input/output controllability analysis, disturbance rejection, control structure selection, relative gain array, RGA.

1. INTRODUCTION

One of the most important task in the design of a control system is the specification of the control structure. Steps in the process of *control structure design* are:

(1) Selection of controlled outputs.
(2) Selection of manipulations and measurements.
(3) Selection of control *configuration*.
(4) Selection of controller type.

One may easily recognize that the design of a control structure is more complex than the task of synthesizing a controller for given sets of measurements and actuators. This paper mainly consider steps 1, 2 and 3 and introduces controllability measures to address the input/output selection problem. With a large number of candidate measurements and/or manipulations the number of possible combinations of inputs and outputs have a combinatorial growth, so an approach consisting of performing a controllability analysis for each possible combination becomes time consuming. In this paper we therefore suggest to use measures for non-square systems such as the relative gain array (RGA) and singular value decomposition (SVD) to select inputs and outputs.

Notation. We consider linear time invariant transfer function models on the form

$$y(s) = G(s)u(s) + G_d(s)d(s) \tag{1}$$

where u is the vector of manipulated inputs, d is the vector of disturbances and y is the vector of outputs. The objective is to keep the control error $e = y - r$ small, where r is the vector of reference signals. $G(s)$ and $G_d(s)$ are transfer matrices of sizes $l \times m$ and $l \times n_d$. Throughout the paper subscripts i, j and k denotes a particular output y_i, input u_j and disturbance d_k. The notation $y_{l \neq i}$ means all outputs except output number i. $[A]_{ij} = a_{ij}$ denotes the ij'th element of the matrix A and $\tilde{A} = \mathrm{diag}\{a_{ii}\}$ contains the diagonal elements of A.

Scaling. The variables should be scaled to be within the interval -1 to 1, that is, their expected magnitudes should be normalized to be less than 1. This is done by dividing the unscaled signals by their *expected maximum allowed change* $u_{j,max}$, $d_{j,max}$, $r_{i,max}$ and $e_{i,max}$. The scaled model can then be written

$$e = y - r = Gu + G_d d - R\tilde{r} \tag{2}$$

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where $R = \text{diag}\{\frac{r_{i,max}}{e_{i,max}}\}$. At each frequency we assume $\|u\|_\infty < 1$, $\|d\|_\infty < 1$, $\|\tilde{r}\|_\infty < 1$ and we want $\|e\|_\infty < 1$.

Related and previous work. The relative gain array (RGA) was first introduced by Bristol (1966) at steady state as the ratio of the "open-loop" and "closed-loop" gains between input j and output i when all other outputs $y_{l\neq i}$ are perfectly controlled using the inputs $u_{h\neq j}$

$$\lambda_{ij}(s) = \frac{\partial y_i/\partial u_j}{\partial y_i/\partial u_j|_{y_l \neq i}} = g_{ij}\left[G^{-1}\right]_{ji} \quad (3)$$

The RGA matrix can be computed at any frequency using the formula

$$\Lambda(G(s)) = G(s) \times (G^\dagger(s))^T \quad (4)$$

where $G^\dagger$ is the pseudo-inverse of G (Chang and Yu, 1990). Interpreting the RGA in terms of perfect control at steady state is only possible when rank$(G) = l$.

Stanley *et al.* (1985) introduced the Relative Disturbance Gain (RDG) and Skogestad and Morari (1987) found that it may be evaluated at any frequency using

$$\beta_{jk} = \frac{\partial u_j/\partial d_k|_{y_i=0,\ \forall i}}{\partial u_j/\partial d_k|_{y_i=0,\ (i=j)}} = \frac{[\tilde{G}G^{-1}G_d]_{jk}}{[G_d]_{jk}} \quad (5)$$

Skogestad and Wolff (1992) introduced the Partial Disturbance Gain (PDG) as the effect of disturbance d_k on the uncontrolled output y_i when a set of outputs $y_{l\neq i}$ are controlled by a set of inputs $u_{h\neq j}$. For square G they presented

$$PDG = \left.\frac{\partial y_i}{\partial d_k}\right|_{u_j, y_{l\neq i}} = \frac{[G^{-1}G_d]_{jk}}{[G^{-1}]_{ji}} \quad (6)$$

The Partial Disturbance Gain has been applied to a continuos bioreactor (Zhao and Skogestad, 1994) and to a FCC process (Wolff *et al.*, 1992).

Also the partial disturbance gain can be generalized to a non-square and singular G by the use of the pseudo-inverse. However, the gains at steady state can only be interpreted in terms of perfect control when rank$(G) = l$ ($G^\dagger$ is then a right-inverse of G). Otherwise, PDG can be interpreted in terms of *least square control*.

2. DEFINITION OF PARTIAL CONTROL

Partial control at a given level involves controlling only a subset of the outputs for which there is a performance objective. Divide the outputs and inputs into the sets:

- y_1 uncontrolled outputs at the present control layer.
- y_2 controlled outputs at the present control layer.
- u_1 inputs not used at the present control layer.
- u_2 inputs used to control y_2.

To analyze the feasibility of partial control, one may consider the effect of the disturbances and reference changes on the *uncontrolled* outputs. Let **1** and **2** denote the sets of indicies for the uncontrolled and controlled outputs. The partial disturbance gain defined in (6) can be used to quantify the effect of disturbances on the uncontrolled outputs. To be more precise, the following notation is used for the effect of disturbance d_k on uncontrolled output y_i when y_2 are controlled using u_2

$$P^{y_2,u_2}_{y_i,d_k} = \left.\frac{\partial y_i}{\partial d_k}\right|_{y_2=0 \text{ using } u_2,\ i\in\mathbf{1}} \quad (7)$$

Note that the partial disturbance gain depends not only on the disturbance d_k and the output y_i, but also on the choice of controlled outputs y_2 and manipulated inputs u_2. This is the reason to the notation $P^{y_2,u_2}_{y_i,d_k}$.

Similarly, the "partial reference gain" is defined as the effect of a reference change in the controlled output r_f on uncontrolled output y_i when the outputs y_2 are controlled using the inputs u_2

$$P^{y_2,u_2}_{y_i,r_f} = \left.\frac{\partial y_i}{\partial r_f}\right|_{y_2=0 \text{ using } u_2,\ i\in\mathbf{1},\ f\in\mathbf{2}} \quad (8)$$

P_d and P_r are used to denote the matrices of all disturbance and reference gains when it is clear from the context what y_2 and u_2 are.

3. TRANSFER FUNCTIONS FOR PARTIALLY CONTROLLED SYSTEMS

By rearranging and partitioning the inputs and outputs as given above, the overall model $y = Gu + G_d d$ and the error $e = y - R\tilde{r}$ can be written

$$y_1 = G_{11}u_1 + G_{12}u_2 + G_{d1}d, \quad e_1 = y_1 - R_1\tilde{r}_1 \quad (9)$$

$$y_2 = G_{21}u_1 + G_{22}u_2 + G_{d2}d, \quad e_2 = y_2 - R_2\tilde{r}_2 \quad (10)$$

With feedback control of y_2 using u_2, $u_2 = K_2(r_2 - y_2)$, the partially controlled system becomes

$$\begin{aligned} y_1 = {} & (G_{11} - G_{12}K_2(I + G_{22}K_2)^{-1}G_{21})u_1 + \\ & (G_{d1} - G_{12}K_2(I + G_{22}K_2)^{-1}G_{d2})d + \\ & G_{12}K_2(I + G_{22}K_2)^{-1}r_2 \end{aligned} \quad (11)$$

Perfect control of y_2. At some frequencies it may be reasonable to assume y_2 perfectly controlled. We can then set $e_2 = 0$ and eliminate u_2 in (9) and (10) to get

$$\begin{aligned} y_1 = {} & \overbrace{(G_{11} - G_{12}G_{22}^{-1}G_{21})}^{P_u} u_1 + \\ & \underbrace{(G_{d1} - G_{12}G_{22}^{-1}G_{d2})}_{P_d} d + \underbrace{G_{12}G_{22}^{-1}R_2}_{P_r} \tilde{r}_2 \end{aligned} \quad (12)$$

where P_d is the *partial disturbance gain*, P_r is the partial reference gain, P_u is the gain for the unused inputs

u_1 for a system under partial control and $r_2 = R_2\tilde{r}_2$. The advantage with the model (12) is that the model is independent of K_2, but we stress that it only applies at frequencies where y_2 is tightly controlled.

Remark 1. (6) and (12) reflect two different ways of computing the PDG's which yield the same results. This follows from the definition. With one uncontrolled output and one unused input, (6) gives an efficient way of computing all combinations of PDG's for disturbance d_k. With more than one uncontrolled output and one unused input, it is easier to use (12).

Remark 2. One advantage with (6) is that it provides direct insight into which uncontrolled output and unused input to select. We have: Select u_j such that row j in $G^{-1}G_d$ has small elements (keep the input constant for which the desired change is small), and select y_i corresponding to a large element in row j of G^{-1} (keep an output uncontrolled which is insensitive to u_j).

4. USES OF PARTIAL CONTROL

Three applications of partial control are:

(1) *Inner cascade loops* with extra measurements y_2.
(2) *Indirect control* of y_1 by controlling y_2.
(3) *Sequential design* of decentralized controllers.

In the cases 2 and 3 there are performance objectives associated with the outputs y_2 (so $\tilde{r}_2$ is given). The three problems are closely related, and in all cases we want the effect of the disturbances on y_1 to be small when y_2 is controlled. In particular we want $\|P_d\| < \|G_{d1}\|$. An additional desirable property for all three cases is to achieve fast and acceptable control of y_2. A common controllability requirement is

- $\underline{\sigma}(G_{22}) > 1, \ \forall \omega < \omega_{B2}$ (ω_{B2} denotes the desired bandwidth of the secondary loop).

One justification of this requirement is to avoid input constraints in u_2.

1. Inner cascade loops. In this case, y_2 are additional "secondary" measurements with *no associated performance objectives.* The control objective is to achieve satisfactory performance for the "primary" outputs y_1. One way to improve the control of y_1 may be to control y_2. In particular, this may reduce the effect of the disturbances d on y_1, when d enters between the input of the plant and y_2. Note that with the inner loop closed, $\tilde{r}_2$ is a degree of freedom for controlling y_1.

Define the frequencies $\omega_{G_{d1}}$ and ω_{P_d} where $\|G_{d1}\| = 1$ and $\|P_d\| = 1$ (the upper frequencies). In the cases where $\omega_{P_d} < \omega_{G_{d1}}$, disturbance rejection in y_1 is improved for frequencies $\omega > \omega_{P_d}$. Performance in y_1 can further be improved by using u_1 and $\tilde{r}_2$ at low frequencies. A common controllability imposed on P_r and P_u is then

- $\underline{\sigma}([\,P_r \quad P_u\,]) > 1, \ \forall \omega < \omega_{G_{d1}}$.

This is to guarantee that u_1 and $\tilde{r}_2$ stay within the desired limits and applies irrespective of the controller type, provided that the secondary loops $y_2 \leftrightarrow u_2$ are intact.

2. Indirect control ("true" partial control). In some cases, the outputs are correlated such that controlling the outputs y_2 indirectly gives acceptable control of some other outputs y_1. Two examples of "true" partial control are given in section 6 for a binary distillation column and a FCC process.

In the following ω_B denotes the desired bandwidth for the control loop. For rejection of combined disturbances in y_1, the following requirement must be satisfied

- A set of outputs y_1 may be considered kept uncontrolled if $\|P_d\|_{i\infty} < 1, \ \forall \omega < \omega_B$.

Reference changes in $\tilde{r}_2$ may also be regarded as disturbances for the uncontrolled outputs y_1.

- For combinations of reference changes, a set of outputs y_1 may be considered kept uncontrolled if $\|P_r\|_{i\infty} < 1, \ \forall \omega < \omega_B$.
- For combined reference changes and disturbances, a set of outputs y_1 may be considered kept uncontrolled if $\|\,[\,P_d \quad P_r\,]\,\|_{i\infty} < 1, \ \forall \omega < \omega_B$.

The induced infinity norm $\|\cdot\|_{i\infty}$ computes the maximum row sum (sum of element magnitudes) and reflects the effect of d and $\tilde{r}_2$ for the worst case output.

3. Sequential controller design. One common way to implement a hierarchical control system is to first implement a *lower-level* control system for controlling the outputs y_2. With this lower-level control system in place, one designs a controller K_1 for control of y_1. Some criteria for selecting u_2 and y_2 in this case are given in (Hovd and Skogestad, 1993).

5. PARTITIONING TOOLS

The subsets y_1, y_2, u_1 and u_2 can be expressed as:

$$y_1 = N_{y^\perp}^T y, \quad y_2 = N_y^T y, \quad u_1 = N_{u^\perp}^T u, \quad u_2 = N_u^T u$$

where N is a selection (projection) matrix. For example to select the two first outputs of a plant set $N_y = [\,e_1 \quad e_2\,]$ where e_i is a vector of size l with zeros in all elements except in position i which contains 1. With this notation the uncontrolled and controlled outputs can be written in terms of d, u_1 and u_2

$$y_1 = \overbrace{N_{y^\perp}^T G N_{u^\perp}}^{G_{11}} u_1 + \overbrace{N_{y^\perp}^T G N_u}^{G_{12}} u_2 + \overbrace{N_{y^\perp}^T G_d}^{G_{d1}} d \quad (13)$$

$$y_2 = \underbrace{N_y^T G N_{u^\perp}}_{G_{21}} u_1 + \underbrace{N_y^T G N_u}_{G_{22}} u_2 + \underbrace{N_y^T G_d}_{G_{d2}} d \quad (14)$$

5.1 *RGA and the selection problem*

Several authors have used the relative gain array (4) as a selection tool for control structure design and in particular for the pairing problem for decentralized control. Results which are connected to the row sums and column sums have also been suggested. Chang and Yu (1990) recognized that the row sums of the RGA stayed between zero and one for non-square plants with full column rank (more outputs than inputs). They used this to rank candidate outputs corresponding to the row sums of the RGA. Recently Cao (1995) presented a similar suggestion for the input selection problem, involving the column sums of the RGA. Cao (1995) also derived the relation between input singular vectors and the column sums of the RGA. In Theorem 1 we generalize the result in (Cao, 1995) to the outputs (row sums of RGA) and also provide a simpler proof.

In the following, consider the model $y = Gu$ and write the singular value decomposition of G as

$$G = U\Sigma V^H = U_r \Sigma_r V_r^H \tag{15}$$

where Σ_r consists only of the $r = \text{rank}(G)$ nonzero singular values, U_r consists of the r first columns of U, and V_r consists of the r first columns of V.

Let e_j and e_i be defined as above ($u_j = e_j^T u$, $y_i = e_i^T y$). Then $e_j^T V_r$ yields the projection of a unit input u_j onto the non-zero input space of G and $e_i^T U_r$ yields the projection of a unit output y_i onto the non-zero output space of G. We follow (Cao, 1995) and define

$$\text{Effectiveness for input } j\text{:} \quad \eta_{\text{I},j} = \|e_j^T V_r\|_2 \tag{16}$$

$$\text{Effectiveness for output } i\text{:} \quad \eta_{\text{O},i} = \|e_i^T U_r\|_2 \tag{17}$$

The following theorem links the SVD to the RGA.

Theorem 1 RGA and SVD.

$$\sum_{j=1}^{m} \lambda_{ij} = \|e_i^T U_r\|_2^2; \quad \sum_{i=1}^{l} \lambda_{ij} = \|e_j^T V_r\|_2^2 \tag{18}$$

Proof. See Appendix A.

Note that $\|e_i^T U_r\|_2$ is simply the 2-norm of row i in U_r. Essentially, for the case of extra measurements (outputs) one may consider eliminating measurements corresponding to rows in the RGA where the sum of the elements is much smaller than 1. Similarly, for the case of extra manipulations (inputs) one may consider eliminating manipulations corresponding to columns in the RGA where the sum of the elements is much smaller than 1. The RGA used in this way can be a useful tool for screening because it may be computed only once by including all the alternative measurements and/or manipulations and thus avoids the combinatorial problem.

When G is square and non-singular the input and output effectivenesses are equal to one (column and rows of RGA for a square non-singular G sums to one), and the RGA provides no ranking of inputs and outputs. We may then obtain more information by directly considering the SVD. We have

Result 1 SVD for input/output selection. *A ranking of potential inputs and outputs used in a control structure of dimension $k_y \times k_u$ ($k_y < r$ and $k_u < r$, where $r = \text{rank}(G)$) can be obtained by considering the 2-norm of the rows in the matrices U_{ky} and V_{ku} where U_{ky} consists of the first k_y columns of U_r and V_{ku} consists of the first k_u columns V_r.*

This can be justified from the fact that the first k_y columns of U_r and the first k_u columns of V_r correspond to the most significant directions and the 2-norms of the rows correspond to the input and output effectivenesses of the subsystem of dim $k_y \times k_u$. The singular value σ_{k+1} is a measure of the information disregarded in the partial control structure, i.e. $\|G - N_y G_{22} N_u^T\| \geq \sigma_{k+1}$, where $k = \max\{k_y,\ k_u\}$. See also example 2.

Remark 1. The criteria for selecting inputs and outputs through the use of RGA, considering all non-singular directions or only the k first non-singular directions, can be viewed in terms of maximizing the information contained in those directions on the selected inputs/outputs.
Remark 2. It is not clear what this selection procedure implies in terms of measures like $\underline{\sigma}(G_{22})$, $\bar{\sigma}(G_{22})$, $\gamma(G_{22})$ and $\|G - N_y G_{22} N_u^T\|$. However, since there is a finite number of combinations, it is possible to find the projection matrices N_y and N_u which maximize $\underline{\sigma}(G_{22})$ or minimize $\|G - N_y G_{22} N_u^T\|$ for a given dimension $k_y \times k_u$ of G_{22} simply by testing all possibilities.

5.2 *Least square and "true" partial control*

By considering the least square solution to the full (optimal) and the partial control problems, one can quantify the imposed performance loss by partial control for a particular choice of N_y and N_u. Due to space limitations we simply state the results in terms of control errors (ρ) for reference changes in $\tilde{r}_2$ and disturbance rejection for the full (ρ_f) and the partial (ρ_p) control problems

$$\rho_f = \left[\, (GG^\dagger - I)N_y R_2 \quad -(GG^\dagger - I)G_d \,\right] \begin{bmatrix} \tilde{r}_2 \\ d \end{bmatrix} \tag{19}$$

$$\rho_p = \begin{bmatrix} G_{12}G_{22}^\dagger R_2 & G_{d1} - G_{12}G_{22}^\dagger G_{d2} \\ (G_{22}G_{22}^\dagger - I)R_2 & -(G_{22}G_{22}^\dagger - I)G_{d2} \end{bmatrix} \begin{bmatrix} \tilde{r}_2 \\ d \end{bmatrix} \tag{20}$$

For the performance loss to be small we want $\|\rho_p\|$ to be close to $\|\rho_f\|$.

Example 1 Partial control of distillation column. This example discusses partial control of a 2×2 distillation column. We use the reduced 5-state model of the distillation column given in (Hovd and Skogestad, 1992). The full 82-state model consists of 40 theoretical trays plus a total condenser and includes both liquid flow dynamics and composition dynamics. Disturbances in feed flow rate F (d_1) and feed composition z_F (d_2) are included. The LV configuration is used, that is, the manipulated inputs are reflux L (u_1) and boilup V (u_2). Outputs are product compositions y_D (y_1) and x_B (y_2). The disturbances and outputs have been scaled such that a magnitude of 1 corresponds to a change in F of 20%, a change in z_F of 20% and a change in x_B and y_D of 0.01 mole fraction units. One and two point control of binary distillation columns has also been studied by (Waller *et al.*, 1988).

At steady-state the model and the RGA are

$$G=\begin{bmatrix} 88.2 & -86.8 \\ 108.8 & -110.1 \end{bmatrix} \; G_d=\begin{bmatrix} 7.9 & 8.9 \\ 11.7 & 11.3 \end{bmatrix} \; \Lambda=\begin{bmatrix} 36.1 & -35.1 \\ -35.1 & 36.1 \end{bmatrix}$$

The RGA-elements are much larger than 1 which indicates that the plant is fundamentally difficult to control. It also indicates that the two outputs are closely related. Consider the SVD at steady state, $G(0) = USV^H$

$$U=\begin{bmatrix} 0.62 & -0.78 \\ 0.78 & 0.62 \end{bmatrix} \; S=\begin{bmatrix} 198.2 & 0.0 \\ 0.0 & 1.36 \end{bmatrix} \; V=\begin{bmatrix} 0.71 & -0.71 \\ -0.71 & -0.71 \end{bmatrix}$$

From U we see that the gain to the bottom composition is slightly larger than the top composition. This may indicate that it is best to control bottom composition and leave top composition uncontrolled.

The partial disturbance gain for the two disturbances for the four alternative partial control schemes are

$$P_{1,d}^{2,2} = [\,-1.32 \quad 0.013\,] \quad P_{1,d}^{2,1} = [\,-1.59 \quad -0.24\,]$$
$$P_{2,d}^{1,2} = [\,1.68 \quad -0.016\,] \quad P_{2,d}^{1,1} = [\,1.95 \quad 0.297\,]$$

In all four cases we see that control of one output significantly reduces the effect of the disturbances on the uncontrolled output. In particular, this is the case for disturbance 2, for which the gain is reduced from about 10 to 0.30 and less. The best partial scheme is seen to be scheme 1 where the effect of disturbance 1 is -1.32, which is only slightly above one in magnitude. This scheme corresponds to controlling output y_2 (the bottom composition) with u_2 (the boilup V) while letting y_1 (the top composition) being uncontrolled, which also from a physical point of view, is a reasonable control scheme. Frequency-dependent plots for scheme 1 show that the same conclusion applies also at higher frequencies. This is seen in Fig. 1 where we show for disturbance 1 both the open-loop disturbance gain (G_{d11}, Curve 1) and the partial disturbance gain ($P_{1,d1}^{2,2}$, Curve 2) as function of frequency.

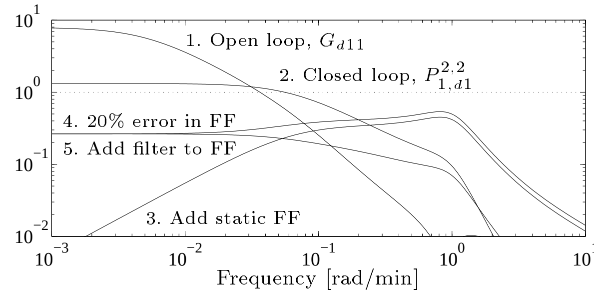

Fig. 1. Effect of d_1 on y_1 for distillation column example.

Let us next consider how we may reduce the effect of disturbance 1 (the feed flow rate F) on y_1 (which is $P_d(0) = -1.32$ at steady-state) to be less than 1 by using a feed-forward controller based on measuring d_1 (the feed flow rate F) and adjusting u_1 (the reflux L). In practice, this is easily implemented as a ratio controller which keeps L/F constant. This eliminates the steady-state effect of d_1 on y_1 (provided the other control loop is closed). With $P_u(0) = g_{11} - g_{12}g_{22}^{-1}g_{21} = -2.45$ we get $u_1 = -P_u(0)^{-1}P_d(0)d_1 = -1.32/2.45d_1 = -0.54d_1$. The resulting disturbance effect is shown in Fig. 1 as curve 3. However, due to measurement error we cannot achieve perfect feed-forward control, so let us assume the error is 20% and use $u_1 = -1.2 \cdot 0.54d_1$. The steady-state effect of the disturbance is then $P_d(0)(1-1.2) = 0.265$, which is still acceptable. However, as seen from the frequency-dependent plot (curve 4), the effect is above 0.5 at higher frequencies, which may not be desirable. The reason for the peak is that the feed-forward controller, which is purely static, reacts too fast and in fact makes the response worse at higher frequencies (as seen when comparing curves 3 and 4 with curve 2). To avoid this we filter the feed-forward action with a time constant of 3 min resulting in the following feed-forward controller: $u_1 = -\frac{0.54}{3s+1}d_1$. To be realistic we again assume 20% error, and the resulting effect of the disturbance on the uncontrolled output is shown by curve 5, and we see that the effect is now less than 0.265 at all frequencies.

Example 2 Input/output selection for FCC process. For the linear model of the FCC process considered by (Hovd and Skogestad, 1993; Wolff *et al.*, 1992) we have at steady state

$$G = \begin{bmatrix} 10.16 & 5.59 & 1.43 \\ 15.52 & -8.37 & -0.71 \\ 18.05 & 0.42 & 1.80 \end{bmatrix} \quad G_d = \begin{bmatrix} 1.66 & 0.36 & -13.61 \\ 0.47 & 0.23 & -3.89 \\ 1.86 & 0.56 & -15.30 \end{bmatrix}$$

From a SVD of $G(j\omega)$ we find $\underline{\sigma}(G(j\omega)) < 1 \; \forall\omega$. Hence, it is likely to encounter input constraints for certain combinations of disturbances and reference changes. We therefore consider 2×2 control of the FCC process. With the two strongest input and output directions i.e. $U_1 =$

$[u_1 \quad u_2]$, $V_1 = [v_1 \quad v_2]$, the 2-norms of the rows becomes

$$\eta_{\rm I} = \begin{bmatrix} 0.997 & 0.982 & 0.201 \end{bmatrix}^T \quad \eta_{\rm O} = \begin{bmatrix} 0.774 & 0.927 & 0.736 \end{bmatrix}^T$$

We clearly see that the input u_3 has little effect on G and that this input may be considered unused. For the outputs the situation is not so obvious, y_1 and y_3 seems to be of equal importance. However, for higher frequencies (not shown) $\eta_{{\rm O},1}$ and $\eta_{{\rm O},2}$ approaches 1, whereas $\eta_{{\rm O},3}$ approaches 0, so we select y_1 and y_2 together with u_1 and u_2 in a partial control structure of size 2×2. It is worth noting that this control structure (denoted Hicks) was considered by (Hovd and Skogestad, 1993) as the best one. They argued from a controllability point of view taking into account RHP-zeros, constraints and different operating modes.

The following example shows that although the RGA is an efficient screening tool, it must be used with some caution.

Example 3 Consider a plant with 2 inputs and 4 candidate outputs of which we want to select 2. We have

$$G = \begin{bmatrix} 10 & 10 \\ 10 & 9 \\ 2 & 1 \\ 2 & 1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} -2.57 & 3.27 \\ 1.96 & -1.43 \\ 0.80 & -0.42 \\ 0.80 & -0.42 \end{bmatrix}$$

The four row sums of RGA are 0.70, 0.53, 0.38 and 0.38. To maximize the output effectiveness we would select outputs 1 and 2. However, this yields a plant $G_1 = \begin{bmatrix} 10 & 10 \\ 10 & 9 \end{bmatrix}$ which is ill-conditioned with large RGA-elements, and most likely difficult to control. Selecting output 1 and 3 yields $G_2 = \begin{bmatrix} 10 & 10 \\ 2 & 1 \end{bmatrix}$ which is well-conditioned. For comparison, the minimum singular values are $\underline{\sigma}(G) = 1.05$, $\underline{\sigma}(G_1) = 0.51$, $\underline{\sigma}(G_2) = 0.70$. The minimized condition numbers ($\gamma^*(A) = min_{D_1,D_2}\, \gamma(D_2 A D_1)$, where D_1 and D_1 are diagonal matrices) are $\gamma^*(G) = 11.69$, $\gamma^*(G_1) = 37.97$, $\gamma^*(G_2) = 5.83$.

7. CONCLUSION

We have stated relationships between the RGA and SVD which generalizes the results of (Cao, 1995). These tools can be used to obtain a ranking of the possible inputs and outputs. However, they should be used with care since there may be many other factors which determine controllability.

8. REFERENCES

Bristol, E. H. (1966). On a new measure of interactions for multivariable process control. *IEEE Trans. Auto. Cont.*, **AC-11**, 133–134.

Cao, Y. (1995). Control Structure Selection for Chemical Processes Using Input-output Controllability Analysis. PhD thesis. University of Exeter.

Chang, J.W. and C.C. Yu (1990). The relative gain for non-square multivariable systems. *Chem. Eng. Sci.,* **45**(5), 1309–1323.

Hovd, M. and S. Skogestad (1992). Simple frequency-dependent tools for control system analysis, structure selection and design. *Automatica,* **28**(5), 989–996.

Hovd, M. and S. Skogestad (1993). Procedure for regulatory control structure selection with application to the FCC process. *AIChE J,* **39**(12), 1938–1953.

Skogestad, S. and E. A. Wolff (1992). Controllability measures for disturbance rejection. In: *IFAC Workshop on Interactions between Process Design and Process Control*. London, England. pp. 23–29.

Skogestad, S. and M. Morari (1987). The effect of disturbance directions on closed loop performance. *Ind. Eng. Chem. Res.,* **26**, 2029–2035.

Stanley, G., M. Marino-Galarraga and T. J. McAvoy (1985). Shortcut operability analysis. 1, the relative disturbance gain. *Ind. Eng. Chem. Proc. Des. Dev.,* **24**, 1181–1188.

Waller, K. V., K. E. Häggblom, P. M. Sandelin and D. H. Finnerman (1988). Disturbance sensitivity of distillation control structures. *AIChE J,* **34**(5), 853–858.

Wolff, E. A., S. Skogestad, M. Hovd and K.W. Mathisen (1992). A procedure for controllability analysis. In: *IFAC Workshop on Interactions between Process Design and Process Control*. London, England. pp. 127–132.

Zhao, Y. and S. Skogestad (1994). A comparison of various control schemes for continuous bioreator. In: *Proc. IFAC-Symposium ADCHEM'94*. Kyoto, Japan. pp. 325–330.

Appendix A. PROOF OF (18)

The proof of the identities in (18) are given for the general case. Write the SVD of G as $G = U_r \Sigma_r V_r^H$ where Σ_r is invertible. We have that $g_{ij} = e_i^T U_r \Sigma_r V_r^H e_j$, $[G^\dagger]_{ji} = e_j^T V_r \Sigma_r^{-1} U_r^H e_i$, $U_r^H U_r = I_r$ and $V_r^H V_r = I_r$ where I_r denotes identity matrix of dim $r \times r$. The row sum becomes

$$\sum_{j=1}^{m} \lambda_{ij} = \sum_{j=1}^{m} e_i^T U_r \Sigma_r V_r^H e_j e_j^T V_r \Sigma_r^{-1} U_r^H e_i$$

$$= e_i^T U_r \Sigma_r V_r^H \underbrace{\sum_{j=1}^{m} e_j e_j^T}_{I_m} V_r \Sigma_r^{-1} U_r^H e_i = \|e_i^T U_r\|_2^2$$

The proof of the column sum is similar when changing the order of the two scalar terms g_{ij} and $[G^\dagger]_{ji}$ and summing over i.