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THE USE OF RGA AND CONDITION NUMBER AS ROBUSTNESS MEASURES

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Abstract - The relative gain array (RGA) and condition number are commonly used tools in controllability analysis. In this paper we present new results that link these measures to control performance, measured in terms of the output sensitivity function with input and output uncertainty.

1 INTRODUCTION

Diagonal input and output uncertainty are always present in any real system: diagonal input uncertainty in terms of unknown characteristics in the actuators and diagonal output uncertainty in terms of imperfect measurement devices. It is therefore reasonable to consider the effect of these two types of uncertainty on performance for a given control system. In particular, ill-conditioned plants with a large condition number are often believed to be sensitive to uncertainty, and the objective of the paper is to gain insight into this by answering the question:

- How can ill-conditioning which results in poor robust performance be identified?

In the paper we consider linear time invariant transfer function models on the form $y(s) = G(s)u(s)$. For simplicity of the proofs we assume that G is stable. However, as noted in the conclusion the results are also valid for unstable plants. The results in the paper are stated in terms of the plant and controller condition numbers,

$$\gamma(G) = \frac{\bar{\sigma}(G)}{\underline{\sigma}(G)}, \qquad \gamma(K) = \frac{\bar{\sigma}(K)}{\underline{\sigma}(K)} \tag{1}$$

and the following minimized condition numbers for the plant and the controller

$$\gamma_I^*(G) = min_{D_I}\ \gamma(GD_I), \qquad \gamma_O^*(K) = min_{D_O}\ \gamma(D_O K) \tag{2}$$

where D_I and D_O are diagonal scaling matrices. These minimized condition numbers can be computed as outlined by Braatz and Morari (1994). In the paper we also make use of the relative gain array (RGA) which was introduced by Bristol (1966). The RGA matrix can be computed at any frequency using the formula

$$\Lambda(s) = G(s) \times \left(G^{-1}(s)\right)^T \tag{3}$$

where the $\times$ symbol denotes element by element multiplication (Hadamard or Schur product). An important property of the RGA is that it is scaling independent.

Previous work in the area include that of (P. Grosdidier and Holt, 1985; Nett and Manousiouthakis, 1987; Skogestad and Morari, 1987; Freudenberg, 1989*a*; Freudenberg, 1989*b*; Freudenberg and Saglik, 1991; Waller *et al.*, 1994*a*; Waller *et al.*, 1994*b*).

2 UNCERTAINTY

In practice, the true perturbed plant G' differs from that of the plant model G. This may be caused by a number of different sources, and in this paper we focus on input and output uncertainty. On multiplicative form the output and input uncertainties are (Fig. 1)

$$\text{Output uncertainty}: \quad G' = (I + E_O)G \quad \text{or} \quad E_O = (G' - G)G^{-1} \tag{4}$$

$$\text{Input uncertainty}: \quad G' = G(I + E_I) \quad \text{or} \quad E_I = G^{-1}(G' - G) \tag{5}$$

These forms of uncertainty may seem similar, but we will show that their implications for control may be very different. In particular, note that for square plants $E_O = GE_IG^{-1}$. The main reason for writing the

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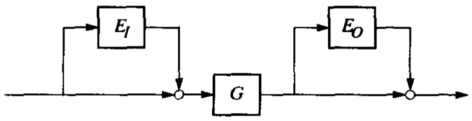

Figure 1: System with multiplicative input and output uncertainty

uncertainty in multiplicative (or relative) form is because this makes it easier to quantify the uncertainty. In most cases we assume that the magnitude of the uncertainty at each frequency can be bounded in terms of its singular value

$$\bar{\sigma}(E_I) \leq |w_I|, \qquad \bar{\sigma}(E_O) \leq |w_O| \tag{6}$$

where $w_I(s)$ and $w_O(s)$ are scalar weights. Typically the uncertainty bound, $|w_I|$ or $|w_O|$, is 0.2 at low frequencies and exceeds 1 at higher frequencies. If we allow E_I or E_O to be any uncertainty matrix satisfying the bound (6), then we have *full block* uncertainty. However, in many cases the source of uncertainty is in the individual input or output channels, and we have that E_I or E_O are diagonal matrices

$$E_I = diag\{\epsilon_{I1}, \epsilon_{I2}, \ldots\}, \qquad E_O = diag\{\epsilon_{O1}, \epsilon_{O2}, \ldots\} \tag{7}$$

This is denoted *diagonal input uncertainty* and *diagonal output uncertainty*. We will assume that in each input channel j and in each output channel i the uncertainty is bounded as follows

$$|\epsilon_{Ij}| \leq |w_{Ij}|, \qquad |\epsilon_{Oi}| \leq |w_{Oi}| \tag{8}$$

It is important to stress that diagonal input uncertainty is *always* present in real systems (whereas full block input uncertainty is present only in some cases).

3 EFFECT OF UNCERTAINTY ON FEEDFORWARD CONTROL

For the nominal model with no disturbances we have $y = Gu$. The control error can be expressed as $e = y - r = Gu - r$. Consider "perfect" feedforward control, $e = 0$, assuming an invertible plant G and solving for u gives the manipulated inputs $u = G^{-1}r$. However, for the actual plant G' we have $y' = G'u$ and the control error becomes $e' = y' - r = G'G^{-1}r - r$. We get for the two sources of uncertainty

$$\text{Output uncertainty}: \quad e' = E_O r \tag{9}$$

$$\text{Input uncertainty}: \quad e' = GE_I G^{-1} r \tag{10}$$

From (9) we see that with output uncertainty the relative error $\frac{\|e'\|_2}{\|r\|_2}$ is equal to the relative input uncertainty $\|E_O\|_2$. However, for input uncertainty the sensitivity may be much larger because the elements in the matrix GE_IG^{-1} can be much larger than the elements in E_I. In particular for diagonal input uncertainty the elements of GE_IG^{-1} are directly related to the RGA of G, Skogestad and Morari (1987)

$$\text{Diagonal uncertainty}: \quad [GE_IG^{-1}]_{ii} = \sum_{j=1}^{n} \lambda_{ij}(G)\epsilon_j \tag{11}$$

Since diagonal input uncertainty is *always* present we can conclude

- If the plant has large RGA elements within the frequency range where effective control is desired, then it is *not* possible to achieve good reference tracking with feedforward control because of strong sensitivity to diagonal input uncertainty.

4 EFFECT OF UNCERTAINTY ON FEEDBACK CONTROL

One of the main reasons for applying feedback control rather than feedforward control is to reduce the effect of uncertainty. In particular, with integral action in the controller we can achieve zero steady-state control error even with quit large model errors. Nevertheless, uncertainty poses limitations on the achievable feedback control performance, and the objective of this section is to show how the condition number and RGA can be used as tools to detect potential problems. We will base our arguments on the singular values of the perturbed sensitivity function

$$S' = (I + G'K)^{-1} \tag{12}$$

which is directly related to performance measured at the output of the plant. For example, we have that

$$e' = -S'r, \qquad \text{and} \qquad \max_r \frac{\|e'\|_2}{\|r\|_2} = \bar{\sigma}(S') \tag{13}$$

We will derive upper bounds on $\bar{\sigma}(S')$ which involves the condition number and a lower bound on $\bar{\sigma}(S')$ which involves the RGA. The lower bound is useful for identifying plants which are difficult to control. Proofs of some of the results in this section are given in appendix A.

4.1 Factorizations of the sensitivity function

The upper bounds are based on the following factorizations of the sensitivity function

$$\text{Output uncertainty:} \qquad S' = S(I + E_O T)^{-1} \tag{14}$$

$$\text{Input uncertainty :} \qquad S' = S(I + GE_I G^{-1}T)^{-1} = SG(I + E_I T_I)^{-1}G^{-1} \tag{15}$$

$$S' = (I + TK^{-1}E_I K)^{-1}S = K^{-1}(I + T_I E_I)^{-1}KS \tag{16}$$

We assume that the plants, G and G', are stable. We also assume closed loop stability, so that both S and S' are stable. We then get that $(I + E_O T)^{-1}$ and $(I + E_I T_I)^{-1}$ are stable (equivalently $(I + TE_O)^{-1}$ and $(I + T_I E_I)^{-1}$ are stable). When deriving bounds we make use of properties like

$$\bar{\sigma}((I + E_I T_I)^{-1}) = \frac{1}{\underline{\sigma}(I+E_I T_I)} \leq \frac{1}{1-\bar{\sigma}(E_I T_I)} \leq \frac{1}{1-\bar{\sigma}(E_I)\bar{\sigma}(T_I)} \leq \frac{1}{1-|w_I|\bar{\sigma}(T_I)}$$

where we have made use of $\bar{\sigma}(E_I) \leq |w_I|$. Of course these inequalities only apply if we assume $\bar{\sigma}(E_I T_I) < 1$, $\bar{\sigma}(E_I)\bar{\sigma}(T_I) < 1$ and $|w_I|\bar{\sigma}(T_I) < 1$. For simplicity, we will not state these assumptions each time.

4.2 Upper bounds on the sensitivity function

4.2.1 Output uncertainty

$$\bar{\sigma}(S') \leq \bar{\sigma}(S)\bar{\sigma}((I + E_O T)^{-1}) \leq \frac{\bar{\sigma}(S)}{1-\bar{\sigma}(E_O)\bar{\sigma}(T)} \leq \frac{\bar{\sigma}(S)}{1-|w_O|\bar{\sigma}(T)} \tag{17}$$

From (17) we see that output uncertainty, be it diagonal or full block, poses no particular problem when performance is measured at the plant output. That is, if we have a reasonable margin to stability ($\|(I + E_O T)^{-1}\|_\infty$ is not to much larger than 1) then the nominal and perturbed sensitivity do not differ very much.

4.2.2 Input uncertainty

The sensitivity function can be much more sensitive to input uncertainty than output uncertainty.

1. General case (full block or diagonal input uncertainty and any controller).

$$\bar{\sigma}(S') \leq \gamma(G)\bar{\sigma}(S)\bar{\sigma}((I + E_I T_I)^{-1}) \leq \gamma(G)\frac{\bar{\sigma}(S)}{1-\bar{\sigma}(E_I)\bar{\sigma}(T_I)} \leq \gamma(G)\frac{\bar{\sigma}(S)}{1-|w_I|\bar{\sigma}(T_I)} \tag{18}$$

$$\bar{\sigma}(S') \leq \gamma(K)\bar{\sigma}(S)\bar{\sigma}((I + T_I E_I)^{-1}) \leq \gamma(K)\frac{\bar{\sigma}(S)}{1-\bar{\sigma}(T_I)\bar{\sigma}(E_I)} \leq \gamma(K)\frac{\bar{\sigma}(S)}{1-|w_I|\bar{\sigma}(T_I)} \tag{19}$$

From (19) we have the important result that if we use a "round" controller with $\gamma(K)$ close to 1, then the sensitivity function is *not* sensitive to input uncertainty. In many cases the bounds (18) and (19) are not very useful because they yield unnecessary large upper bounds. To improve on this, we present below bounds for some special cases, where we either restrict the uncertainty to be diagonal or restrict the controller to be of a particular form.

2. Diagonal uncertainty *and* diagonal control. In this case we have $K^{-1}E_I K = E_I$ and we get

$$\bar{\sigma}(S') \leq \bar{\sigma}(S)\bar{\sigma}((I + TE_I)^{-1}) \leq \frac{\bar{\sigma}(S)}{1-\bar{\sigma}(T)\bar{\sigma}(E_I)} \leq \frac{\bar{\sigma}(S)}{1-|w_I|\bar{\sigma}(T)} \tag{20}$$

Thus, in this important case S' is not sensitive to input uncertainty.

3. Diagonal uncertainty *and* decoupling control. Consider a decoupling controller on the form $K(s) = D(s)G^{-1}(s)$ where $D(s)$ is a diagonal matrix. In this case KG is diagonal so $T_I = KG(I+KG)^{-1}$ is diagonal (and we have that $E_I T_I = T_I E_I$). With diagonal uncertainty we get

$$\bar{\sigma}(S') \leq \gamma_I^*(G)\bar{\sigma}(S)\bar{\sigma}((I + E_I T_I)^{-1}) \leq \gamma_I^*(G)\frac{\bar{\sigma}(S)}{1-\bar{\sigma}(E_I)\bar{\sigma}(T_I)} \leq \gamma_I^*(G)\frac{\bar{\sigma}(S)}{1-|w_I|\bar{\sigma}(T_I)} \tag{21}$$

$$\bar{\sigma}(S') \leq \gamma_O^*(K)\bar{\sigma}(S)\bar{\sigma}((I + T_I E_I)^{-1}) \leq \gamma_O^*(K)\frac{\bar{\sigma}(S)}{1-\bar{\sigma}(T_I)\bar{\sigma}(E_I)} \leq \gamma_O^*(K)\frac{\bar{\sigma}(S)}{1-|w_I|\bar{\sigma}(T_I)} \tag{22}$$

The bounds (21) and (22) apply to any decoupling controller on the form $K = DG^{-1}$. In particular, they apply to inverse based control, $K = l(s)G^{-1}(s)$ which yields input-output decoupling with $T_I = T = t \cdot I$ where $t = \frac{l}{1+l}$. A diagonal controller has $\gamma_O^*(K) = 1$, so from (20) we see that (22) applies to both a diagonal and decoupling controller. Nevertheless, it does not seem like (22) applies generally for any controller. However, another bound which applies to any controller is given in (24).

4. Diagonal uncertainty (Any controller).

$$\bar{\sigma}(S') \leq \frac{\bar{\sigma}(S)}{1-\gamma_I^*(G)\bar{\sigma}(E_I)\bar{\sigma}(T)} \leq \frac{\bar{\sigma}(S)}{1-|w_I|\gamma_I^*(G)\bar{\sigma}(T)} \tag{23}$$

$$\bar{\sigma}(S') \leq \frac{\bar{\sigma}(S)}{1-\gamma_O^*(K)\bar{\sigma}(E_I)\bar{\sigma}(T)} \leq \frac{\bar{\sigma}(S)}{1-|w_I|\gamma_O^*(K)\bar{\sigma}(T)} \tag{24}$$

Again note that $\gamma_O^*(K) = 1$ for a diagonal controller so (24) confirms that diagonal uncertainty poses little problems when we use a diagonal controller.

4.3 Lower bound on the sensitivity function

Consider the special case of diagonal input uncertainty and inverse based control, $K(s) = l(s)G^{-1}(s)$ (which is a special case of decoupling control which yields $T = T_I = t \cdot I$ and $S = S_I = s \cdot I$). In this case we can generalize the *lower* bound on the sensitivity function for the 2×2 case given in Gjøsæter (1995).

Theorem 1 Lower bound with input uncertainty and decoupling control. *Consider a decoupling controller $K(s) = l(s)G^{-1}(s)$ which results in a nominally decoupled response with sensitivity $S = s \cdot I$ and complementary sensitivity $T = t \cdot I$ where $t(s) = 1 - s(s)$. Suppose the plant has diagonal input uncertainty of relative magnitude $|w_I(j\omega)|$ in each input channel. Then there exists a combination of input uncertainties such that at each frequency*

$$\bar{\sigma}(S') \geq \bar{\sigma}(S)\left(1 + \frac{|w_I t|}{1+|w_I t|}\|\Lambda(G)\|_{i\infty}\right) \tag{25}$$

where $\|\Lambda(G)\|_{i\infty}$ is the maximum row sum of the RGA and $\bar{\sigma}(S) = |s|$.

It is important to notice that (25) provides a *lower* bound on $\bar{\sigma}(S')$, whereas our previous results discussed in Sec. 4.2 gave *upper* bounds. *A lower bound is more useful because it allows us to make definite conclusions about when the plant is not controllable.* Specifically, from (25) we see that with an inverse based controller the worst case sensitivity will be much larger than the nominal at frequencies where the plant has large RGA-elements. At frequencies where control is effective ($|s|$ is small and $|t| \approx 1$) this implies that control is not as good as expected, but it may still be acceptable. However, at crossover frequencies where $|s|$ and $|t| = |1-s|$ are both close to 1, we find that $\bar{\sigma}(S')$ in (25) may become much larger than 1 if the plant has large RGA-elements at these frequencies.

4.3.1 Relationship to structured singular value, μ

The appropriate measure to analyze exactly the worst-case sensitivity under influence of input uncertainty $|w_I|$ is skewed-μ (μ^s). This involves computing $\mu_{\tilde{\Delta}}(N)$ with $\tilde{\Delta} = diag(\Delta_I, \Delta_p)$ where $N = \begin{bmatrix} w_I T_I & w_I KS \\ \frac{1}{\mu^s}SG & \frac{1}{\mu^s}S \end{bmatrix}$ and varying μ^s until $\mu(N) = 1$, where Δ_p is full block. The worst-case performance at a given frequency is then $\bar{\sigma}(S') = \mu^s$.

5 EXAMPLES

Example 1 Distillation column, LV configuration. In this example we consider the following model of an ill-conditioned distillation column, taken from Skogestad *et al.* (1988).

$$G = G_{LV}(s) = \frac{1}{\tau s+1}\begin{bmatrix} 87.8 & -86.4 \\ 108.2 & -109.6 \end{bmatrix}, \qquad \Lambda(G) = \begin{bmatrix} 35.1 & -34.1 \\ -34.1 & 35.1 \end{bmatrix} \tag{26}$$

We consider diagonal input uncertainty of magnitude $|w_I| = 0.2$ at all frequencies. We have that $\|\Lambda(G(j\omega))\|_{i\infty} = 69.14$, $\gamma^*(G) \approx \gamma_O^*(G) \approx 138.3$ and $\gamma(G) \approx \gamma_I^*(G) \approx 141.7$ at all frequencies. So, we may expect problems with input uncertainty for both feedforward and feedback control.

1. Inverse based feedback controller:

$$K_{inv}(s) = \frac{k_1}{s}G^{-1}(s) = \frac{k_1(\tau s+1)}{s}\begin{bmatrix} 0.3994 & -0.3149 \\ 0.3943 & -.3200 \end{bmatrix}, \qquad k_1 = 0.7\ [min^{-1}]$$

The peak value for the lower bound in (25) is 6.81 for $w = 0.79$. As a comparison, the actual peak value with the inverse-based controller with 20% gain uncertainty is (Skogestad *et al.*, 1988)

$$\|S'\|_\infty = \left\| \left(I + \frac{0.7}{s}G\begin{bmatrix} 1.2 & \\ & 0.8 \end{bmatrix}G^{-1}\right)^{-1} \right\|_\infty = 14.21$$

and occurs for $w = 0.69$. The difference between 6.76 and 14.2 illustrates that the bound in terms of the RGA is not generally tight, but it is nevertheless very useful.

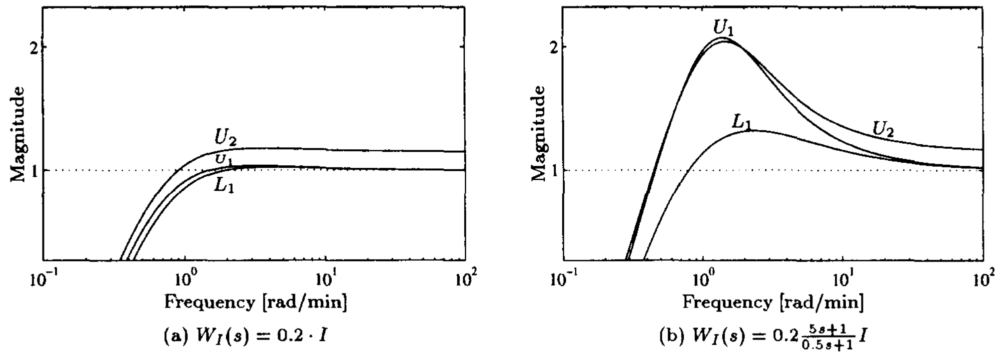

Figure 2: Bounds on sensitivity function for distillation column with DV configuration, lower bound L_1 from (25), upper bounds U_1 from (23) and U_2 from (21).

Next we look at the upper bounds. Unfortunately, in this case $\gamma_I^*(G) = \gamma_O^*(K) \approx 141.7$, so the upper bound in (21) and (22) are not very useful (they are of magnitude 141.7, at high frequencies).

2. Diagonal feedback controller:

$$K_{dia}(s) = \frac{k_2(\tau s+1)}{s}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}, \qquad k_2 = 2.4 \cdot 10^{-2}\ [min^{-1}]$$

We have $\gamma(K_{dia}) = \gamma_O^*(K_{dia}) = 1$ since both loops are tuned equally. Tight upper bounds on perturbed sensitivity functions are therefore provided by (19), (20) and (24). We find that the actual peak in the perturbed sensitivity function is $\|S'\|_\infty = 1.05$ for $\omega = 1.30$ $[rad/min]$ when $E_I = diag\{0.2, -0.2\}$, whereas the peaks in the upper bounds (19), (20) and (24) are all 1.26 for $\omega = 0.56$ $[rad/min]$.

Example 2 Distillation column, DV configuration. In this example we consider the following model of a distillation column with DV configuration, also taken from Skogestad *et al.* (1988)

$$G = G_{DV}(s) = \frac{1}{\tau s+1}\begin{bmatrix}-87.8 & 1.4\\ -108.2 & -1.4\end{bmatrix}, \qquad \Lambda(G) = \begin{bmatrix}0.448 & 0.552\\ 0.552 & 0.448\end{bmatrix} \tag{27}$$

We have that $\|\Lambda(G(j\omega))\|_{i\infty} = 1$, $\gamma^*(G) \approx 1.00$ and $\gamma_I^*(G) \approx 1.11$ and $\gamma(G) \approx 70.76$ and $\gamma_O^*(G) \approx 69.24$ at all frequencies. We do not expect problems with input uncertainty and therefore design an inverse based controller, similar to the one considered by Skogestad *et al.* (1988). The controller is $K_{inv}(s) = \frac{k_1}{s}G^{-1}(s)$, $k_1 = 0.7$ $[min^{-1}]$. Since we use an inverse based controller we have $\gamma(K) = \gamma(G)$, and $\gamma_O^*(K) = \gamma_I^*(G)$. Also since $\gamma(G)$ is much larger than $\gamma_I^*(G)$ we find that the bounds in (18) and (19) are more conservative than the bounds in (21), (22), (23) and (24). In Fig. 2 we show the lower bound given by (25) and the two upper bounds given by (21) and (23) for two different uncertainty weights. From these curves we see that the upper bounds (denoted U_1 and U_2) can be close in some cases, and conclude that the system is robust against input uncertainty.

6 CONCLUSION PERFORMANCE ROBUSTNESS WITH INPUT UNCERTAINTY

Our conclusions on input minimized condition number, condition number and RGA are summarized below. The statements apply to the frequency-range around crossover. By "small', we mean about 2 or smaller. By "large" we mean about 10 or larger.

1. Condition number's $\gamma(G)$ or $\gamma(K)$ small: Robust to both diagonal and full-block input uncertainty.
2. Minimized condition number's $\gamma_I^*(G)$ or $\gamma_O^*(K)$ small: Robust to diagonal input uncertainty. Note that a diagonal controller has $\gamma_O^*(K) = 1$.
3. RGA(G) has large elements: Inverse-based controller is *not* robust to diagonal input uncertainty and should therefore *not* be used (since diagonal input uncertainty is unavoidable). Furthermore, a diagonal controller will most likely yield poor nominal performance for a plant with large RGA-elements, so we conclude that plants with large RGA-elements are fundamentally difficult to control.
4. $\gamma_I^*(G)$ is large while at the same time the RGA has small elements: Cannot make any definite conclusion about the sensitivity to input uncertainty based on the bounds in this paper.

The results also applies to unstable plants G, however, the proofs are then somewhat more complicated than shown in this paper.

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A PROOFS

Proof of (14): $I + G'K = I + (I + E_O)GK = (I + E_O GK(I + GK)^{-1})(I + GK) = (I + E_O T)(I + GK)$. Eq. (15) and (16) are proved in similar ways. *Proof of (17)*: Apply singular value inequalities to (14). *Proofs of (18) and (19)*: Apply singular value inequalities to last identity in (15) and (16). *Proof of (20)*: Set $KE_I K^{-1}$ in (16) first identity to obtain $S' = (I + TE_I)^{-1}S$ and apply singular value inequalities. *Proofs of (21) and (22)*: Since E_I and T_I are diagonal, we have $E_I = DE_I D^{-1} = D^{-1}E_I D$ and $T_I = DT_I D^{-1} = D^{-1}T_I D$ for any diagonal matrix D, then (15) first identity can be written

$$S' = S(I + GE_I G^{-1}T)^{-1} = S(I + (GD_I)E_I(GD_I)^{-1}T)^{-1} = S(GD_I)(I + E_I T_I)^{-1}(GD_I)^{-1} \tag{28}$$

Since (28) applies to any diagonal D_I, (21) follows by applying singular value inequalities to (28). Similarly (22) follows from (16). *Proofs of (23) and (24)*: Apply singular value inequalities to second identity in (28) similar for (24) form equivalent equation with controller.

Proof of Theorem 1. Write the sensitivity function as $S' = (I + G'K)^{-1} = SG(I + E_I T_I)^{-1}G^{-1} = SGDG^{-1}$, $E_I = diag\{\epsilon_k\}$. Since D is a diagonal matrix, we have from (11) that the diagonal elements of S' are given in terms of the RGA of the plant G as

$$s'_{ii} = s\sum_{k=1}^{n}\lambda_{ik}d_k; \qquad d_k(s) = \frac{1}{1 + t(s)\epsilon_k(s)}; \qquad \Lambda = G \times (G^{-1})^T \tag{29}$$

The singular value of a matrix is larger than any of its elements, so $\bar{\sigma}(S') \geq \max_i |s'_{ii}|$, and the objective in the following is to choose a combination of input errors ϵ_k such that the worst-case $|s'_{ii}|$ is as large (poor) as possible. Consider a given i and write each term in the sum in (29) as

$$\lambda_{ik}d_k = \frac{\lambda_{ik}}{1 + t\epsilon_k} = \lambda_{ik} - \frac{\lambda_{ik}t\epsilon_k}{1 + t\epsilon_k} \tag{30}$$

We choose all ϵ_k to have the same magnitude $|w_I(j\omega)|$, so we have $\epsilon_k(j\omega) = |w_I|e^{j\angle\epsilon_k}$. We also assume that $|t\epsilon_k| < 1$ at all frequencies, such that the phase of $1 + t\epsilon_k$ lies between -90° and 90°.[1] It is then always possible to select $\angle\epsilon_k$ (the phase of ϵ_k) such that the last term in (30) is real and negative, and we have at each frequency with these choices for ϵ_k

$$\frac{s'_{ii}}{s} = \sum_{k=1}^{n}\lambda_{ik}d_k = 1 + \sum_{k=1}^{n}\frac{|\lambda_{ik}|\cdot|t\epsilon_k|}{|1 + t\epsilon_k|} \geq 1 + \sum_{k=1}^{n}\frac{|\lambda_{ik}|\cdot|w_I t|}{1 + |w_I t|} = 1 + \frac{|w_I t|}{1 + |w_I t|}\sum_{k=1}^{n}|\lambda_{ik}| \tag{31}$$

where the first equality makes use of the fact that the row-elements of the RGA sum to 1, ($\sum_{k=1}^{n}\lambda_{ik} = 1$) and the inequality follows since $|\epsilon_k| = |w_I|$ and $|1 + t\epsilon_k| \leq 1 + |t\epsilon_k| = 1 + |w_I t|$. This derivation holds for any i (but only for one at a time), and (25) follows by selecting i to maximize $\sum_{k=1}^{n}|\lambda_{ik}|$ (the maximum row-sum of the RGA of G).

[1] The assumption $|t\epsilon_k| < 1$ is not included in the theorem since it is actually needed for robust stability, so if it does not hold we may have $\bar{\sigma}(S')$ infinite for some allowed uncertainty, and (25) clearly holds.