DIRECTIONS AND FACTORIZATIONS OF ZEROS AND POLES IN MULTIVARIABLE SYSTEMS

K. Havre [1] and **S. Skogestad** [2]

Chemical Engineering, Norwegian University of Science and Technology, N-7034 Trondheim, Norway.

INTERNAL REPORT - June 96.

Abstract. Directionality of zeros and poles in multivariable systems are examined. These directions can be computed in terms of eigenvalue problems. Furthermore, analytical factorizations of RHP-zeros and poles in *Blaschke* products, with state-space realizations dependent on the pole and zero directions are given.

1. INTRODUCTION

We consider linear time invariant systems on state space form

$$\dot{x} = Ax + Bu \tag{1}$$

$$y = Cx + Du \tag{2}$$

where $A \in \mathbb{R}^{n_x \times n_x}$, $B \in \mathbb{R}^{n_x \times m}$, $C \in \mathbb{R}^{l \times n_x}$ and $D \in \mathbb{R}^{l \times m}$ where n_x is the number of states, l is the number of outputs and m is the number of inputs. These equations may be rewritten as

$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}$$

This gives rise to the short-hand notation

$$G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \tag{3}$$

which is frequently used to describe a state-space model of a system G. The transfer function of G defined by (3) can be evaluated as a function of the complex variable $s \in \mathbb{C}$, $G(s) = C(sI - A)^{-1}B + D$. We often omit to show the dependence on the complex variable s for transfer functions.

The second section present definitions of zeros, zero directions, poles and pole directions in multivariable systems. We use the letter u for input directions and the letter y for output directions. The subscripts p or z is used to distinguish the pole direction from the zero direction. If there are more than one zero or one pole we use an additional subscript to denote the direction of that particular zero or pole. For the state directions the letter x is used with subscript z or p as above. To distinguish input state direction from the output state direction we use an additional subscript I or O. The results regarding the zero and pole directions are published in (Havre and Skogestad, 1996).

The main objective with this paper is to derive and write down analytical state-space expressions for factorizations of RHP-zeros and poles. Section three contains these input and output factorizations for systems with RHP-zeros and poles. Factorizations of zeros has been known for a period (Wall *et al.*, 1980; Zhou *et al.*, 1996). The main reason for writing these factorizations down is that they are used extensively in the work (Havre and Skogestad, 1996).

All the proofs are given in appendix A.

[1] Present address: Institute of energy technology, P.O.Box 40, N-2007 Kjeller, Norway, E-mail: kjetil@ife.no.
[2] Author to whom correspondence should be addressed. Fax: (+47) 73 59 40 80, E-mail: skoge@kjemi.unit.no.

2. ZEROS AND POLES OF MULTIVARIABLE SYSTEMS

The concept of normal rank is essential in the definition of zeros below. The normal rank is defined as follows

DEFINITION 1. (NORMAL RANK). *The normal rank of $G(s)$, denoted n_r, is the rank of $G(s)$ at all values of s except at a finite number of singularities. The system $G(s)$ has* full normal rank *if $n_r = min\{m, l\}$,* full row rank *if $n_r = l$ and* full column rank *if $n_r = m$. Where m is the number of inputs and l is the number of outputs.*

2.1 Zeros

Zeros of a system may arise when competing effects internal to the system are such that the output is zero even when the inputs (and the states) are not themselves identically zero. For a SISO system $G(s)$ the zeros are the solutions $s = z_i$ to $G(s) = 0$, and thus it could be argued that they are values of s at which $G(s)$ looses rank (from rank 1 to rank 0). This is the basis for the following definition for zeros for the multivariable system (MacFarlane and Karcanias, 1976).

DEFINITION 2. (ZEROS). *$z_i \in \mathbb{C}$ is a zero of $G(s)$ if the rank of $G(z_i)$ is less than the normal rank of $G(s)$. The zero polynomial is defined as $z(s) = \prod_{i=1}^{N_z}(s - z_i)$ where N_z is the number of finite zeros of $G(s)$.*

This definition of zeros is based on the transfer function matrix, corresponding to a minimal realization of a system. These zeros are sometimes called "transmission zeros", but we shall simply call them "zeros". We may sometimes use the term "multivariable zeros" to distinguish them from the zeros of the elements of the transfer function matrix.

DEFINITION 3. (INPUT AND OUTPUT ZERO DIRECTIONS). *If $G(s)$ has a zero for $s = z \in \mathbb{C}$ then there exist non-zero vectors labeled the output zero direction $y_z \in \mathbb{C}^l$ and the input zero direction $u_z \in \mathbb{C}^m$, such that $y_z^H y_z = 1$, $u_z^H u_z = 1$ and*

$$y_z^H G(z) = 0; \quad G(z)u_z = 0 \tag{4}$$

The input zero direction is a basis vector for a part of the null-space of $G(z)$ and the output direction is a basis vector for a part of the left null-space of $G(z)$. The parts of the two null-spaces corresponds to the singular directions at the input and the output resulting from the singularity occurring when evaluating $G(s)$ for $s = z$. For square G with full normal rank and z of multiplicity one the dimensions of the null-spaces are both one. In the general case where z is of multiplicity m there are m input and output directions associated with the zero $s = z$. The definitions of input and output zero directions can further be extended with the state input and output zero directions through the use of generalized eigenvalues for computation of zeros. For a system $G(s)$, the zeros z of the system, the zero input directions u_z and the zero input state directions $x_{z,\mathrm{I}} \in \mathbb{C}^{n_x}$ can all be computed from the generalized eigenvalue problem

$$\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} \begin{bmatrix} x_{z,\mathrm{I}} \\ u_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{5}$$

In this setup we normalize the length of u_z, so that $u_z^H u_z = 1$. This imply that the length of $x_{z,\mathrm{I}}$ most likely is different from one.

Similarly one can compute the zeros z, the output zero direction y_z and the output zero state direction $x_{z,\mathrm{O}} \in \mathbb{C}^{n_x}$ through the generalized eigenvalue problem

$$\begin{bmatrix} x_{z,\mathrm{O}}^H & y_z^H \end{bmatrix} \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix} \tag{6}$$

Where the length of y_z is normalized, so that $y_z^H y_z = 1$. By taking the transpose of (6) one obtains

$$\begin{bmatrix} A^T - sI & C^T \\ B^T & D^T \end{bmatrix} \begin{bmatrix} \bar{x}_{z,\mathrm{O}} \\ \bar{y}_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{7}$$

From this we see that the input directions of the transposed system G^T is equal to the conjugate of the output directions of G. In MATLAB the generalized eigenvalue problem (6) can be solved via the transposed problem.

Another possibility is to calculate the zero directions from the singular value decomposition of $G(z)$

$$\begin{aligned} G(z) = U_z \Sigma_z V_z^H &= \sum_{i=1}^{n_r} u_i \sigma_i v_i^H \\ &= u_1 \sigma_1 v_1^H + u_2 \sigma_2 v_2^H + \cdots + u_{n_r} \sigma_{n_r} v_{n_r}^H \end{aligned}$$

If one assumes that the system has rank n_r and the zero is of multiplicity one, then the zero directions are given in the columns of V and U corresponding to the singular value which becomes zero due to $s = z$. Under normal circumstances this is column n_r, giving the input zero direction $u_z = v_{n_r}$ and output zero direction $y_z = u_{n_r}$.

REMARK 1. If $G(s)$ does not have full normal rank, or zero is of multiplicity greater than one, it may be difficult to pick out the zero directions since more than one singular value are equal to zero.

REMARK 2. The calculation of zero directions with SVD requires knowledge of the zeros. However, the generalized eigenvalues can be used to both compute the zeros and the directions in one operation.

REMARK 3. Sometimes it is also necessary to have the associated state direction, the generalized eigenvalue method is the only way to compute the state direction.

REMARK 4. If a zero and a pole is present at the same location but with different directions so that they do not cancel, the SVD of $G(z)$ may not give the correct directions due to the numerical problems when evaluating $G(z)$. The numerical problems occur due to the singularity of $zI - A$ in $G(z) = C(zI - A)^{-1}B + D$.

2.2 Poles

DEFINITION 4. (POLES). *The poles $p_i \in \mathbb{C}$ of a system with state-space description (3) are the eigenvalues $\lambda_i(A)$, $i = 1, \ldots, n_x$ of the matrix A. The pole or characteristic polynomial $\phi(s)$ is defined as*

$$\phi(s) = \det(sI - A) = \prod_{i=1}^{n_x}(s - p_i) \qquad (8)$$

Thus the poles are the roots of the characteristic equation

$$\phi(s) = \det(sI - A) = 0 \qquad (9)$$

The gain of the system G evaluated at $s = p$, $G(p)$, is infinite in some directions at the input and the output. This is the basis for the following definition of input and output pole directions.

DEFINITION 5. (INPUT AND OUTPUT POLE DIRECTIONS). *If $s = p \in \mathbb{C}$ is a pole of $G(s)$ then there exist an output direction $y_p \in \mathbb{C}^l$ and an input direction $u_p \in \mathbb{C}^m$ with infinite gain for $s = p$.*

From the singular value decomposition of $G(p)$ we have

$$\begin{aligned} G(p) &= U_p \Sigma_p V_p^H = \sum_{i=1}^{n_r} u_i \sigma_i v_i^H \\ &= u_1\sigma_1 v_1^H + u_2\sigma_2 v_2^H + \cdots + u_{n_r}\sigma_{n_r} v_{n_r}^H \end{aligned}$$

The directions with largest gain are associated with σ_1, the input direction u_p is v_1 and the output direction y_p is u_1. Since $G(p)u_p = \infty$ and $y_p^H G(p) = \infty$ we can not evaluate $G(p)$. Instead we can consider $G(p + \epsilon)$ when $\epsilon \to 0$.

For a square system, G, with state space realization (3) the inverse is given by (Zhou *et al.*, 1996, p. 67)

$$G^{-1} = \left[\begin{array}{c|c} A - BD^{-1}C & -BD^{-1} \\ \hline D^{-1}C & D^{-1} \end{array}\right] \qquad (10)$$

provided that D^{-1} exists. The pole output direction is then given by $G^{-1}(p)y_p = 0$, similarly the pole input direction is given by $u_p^H G^{-1}(p) = 0$. The pole directions can therefore be found as the the zero directions of $G^{-1}(p)$, $G^{-1}(p) = U\Sigma V^H$, y_p as the zero direction in V and u_p as the zero direction in U.

To calculate the pole directions from SVD of $G(p)$ or $G^{-1}(p)$ has rather poor numerical properties. The following result shows how to compute the pole directions for a general system with state space realization (3).

LEMMA 1. (POLE DIRECTIONS). *For a system G with state space description (3) the pole directions associated with the pole $p \in \mathbb{C}$ can be computed from*

$$y_p = Cx_R; \quad u_p = B^H x_L \qquad (11)$$

where $x_R \in \mathbb{C}^{n_x}$ and $x_L \in \mathbb{C}^{n_x}$ are the eigenvectors corresponding to the two eigenvalue problems $Ax_R = px_R$ and $x_L^H A = x_L^H p$.

REMARK 1. The right and left eigenvectors are the singular input and output directions of $pI - A$. So, the eigenvectors can be computed from the SVD of $pI - A$.

REMARK 2. The pole directions are independent of the matrix D in the state space description of G.

REMARK 3. In this setup the length of the state directions x_R and x_L are normalized. However, the relations in (11) can be multiplied by any non-zero constant, so that y_p and u_p can be normalized instead of x_R and x_L.

REMARK 4. We have pole directions x_L and x_R for G given as $(A - pI)x_R = 0$ and $x_L^H(A - pI) = 0$. For the transposed system $G^T = \left[\begin{array}{c|c} A^T & C^T \\ \hline B^T & D^T \end{array}\right]$, we have $(A^T - pI)x'_R = 0$ and $x'^H_L(A^T - pI) = 0$ which implies $x'^T_R(A - pI) = 0$ and $(A - pI)\bar{x}'_L = 0$. Relations between pole directions for G and G^T are: $\bar{x}'_R = x_L$, $\bar{x}'_L = x_R$, $y'_p = B^T x'_R = B^T \bar{x}_L = \bar{u}_p$ and $u'_p = Cx'_L = C\bar{x}_R = \bar{y}_p$. Note that the input and output pole directions for G, $G(p)u_p = \infty$, $y_p^H G(p) = \infty$, and for G^T, $G^T(p)u'_p = \infty$ and $y'^H_p G^T(p) = \infty$, also gives the relations $u'_p = \bar{y}_p$ and $y'_p = \bar{u}_p$. This follows from $u'^T_p G(p) = \infty$ and $G(p)\bar{y}'_p = \infty$.

REMARK 5. To find a relationship between output pole directions y_p and $x_p = x_R$ and the input zero directions u_z and x_z for G^{-1} assume that G is a square system with a non-singular D matrix. The input zero directions of G^{-1} are defined through the use of (10).

$$\begin{bmatrix} A - BD^{-1}C - pI & -BD^{-1} \\ D^{-1}C & D^{-1} \end{bmatrix} \begin{bmatrix} x_z \\ u_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (12)$$

From (12) we have

$$(A - pI)x_z - BD^{-1}(Cx_z + u_z) = 0 \qquad (13)$$

$$D^{-1}(Cx_z + u_z) = 0 \qquad (14)$$

Clearly, $x_z = -x_p$ and $u_z = y_p$ is a solution.

3. FACTORIZATIONS OF RHP-ZEROS AND POLES

Right half plane zeros and poles (zeros/poles in the open right half plane, $\mathbb{C}^+$), $G(s)$ can be factorized in either of the two *Blaschke* products labeled "input factorization" and "output factorization" as follows

$$G(s) = G_{\mathrm{I}}(s)B_{\mathrm{I}}(s); \quad G(s) = B_{\mathrm{O}}(s)G_{\mathrm{O}}(s) \quad (15)$$

where $B_{\mathrm{I}}(s)$ and B_{O} are transfer matrices containing the RHP-zeros/poles. When factorizing RHP-zeros/poles, the filters $B_{\mathrm{I}}(s)$ and $B_{\mathrm{O}}(s)$ consist of N_z/N_p series connected first order filters $B_i(s)$ of size $k \times k$, each factorizing one RHP-zero/pole, z_i/p_i. If an output factorization is considered then $k = l$ and if an input factorization is considered then $k = m$. The general filter $B(s)$ describing both $B_{\mathrm{I}}(s)$ and $B_{\mathrm{O}}(s)$ for RHP-zeros and some of it's properties are summarized in Lemma 2. The filters $B_{\mathrm{I}}(s)$ and $B_{\mathrm{O}}(s)$ for factorizations RHP-poles are the inverse of $B(s)$ with N_z replaced by N_p and z_i replaced by p_i in Lemma 2.

LEMMA 2. *Let the filter $B(s)$ be defined as*

$$B(s) = B_{N_z}(s)B_{N_z-1}\cdots B_1(s) = \prod_{i=0}^{N_z-1} B_i(s) \quad (16)$$

$$B_i(s) = I - \frac{2\mathrm{Re}(z_i)}{s+\bar{z}_i} v_i v_i^H \quad (17)$$

where $z_i \in \mathbb{C}$, $v_i \in \mathbb{C}^k$. Consider the factor $B_i(s)$ in $B(s)$, $B_i(s)$ has $k-1$ singular values and eigenvalues equal to one. The last eigenvalue and the last singular value are given by

$$\lambda_k(B_i(s)) = \frac{s-z_i}{s+\bar{z}_i}$$
$$\sigma_k(B_i(s)) = |\lambda_k(B_i(s))| = \frac{|s-z_i|}{|s+\bar{z}_i|}$$

When s and z_i are both in $\mathbb{C}^+$ or both in $\mathbb{C}^-$ then

$$\sigma_k(B_i(s)) = |\lambda_k(B_i(s))| = \underline{\sigma}(B_i(s)) = \frac{|s-z_i|}{|s+\bar{z}_i|} < 1$$

otherwise

$$\sigma_k(B_i(s)) = |\lambda_k(B_i(s))| = \bar{\sigma}(B_i(s)) = \frac{|s-z_i|}{|s+\bar{z}_i|} \geq 1$$

For $s = j\omega$, all eigenvalues and all singular values are equal to one.

The inverse of $B(s)$ is given by

$$B^{-1}(s) = B_1^{-1}(s)B_2^{-1}(s)\cdots B_{N_z}^{-1}(s) = \prod_{i=1}^{N_z} B_i^{-1}(s) \quad (18)$$

$$B_i^{-1}(s) = I + \frac{2\mathrm{Re}(z_i)}{s-z_i} v_i v_i^H \quad (19)$$

$B_i^{-1}(s)$ has $k-1$ singular values and eigenvalues equal to one. The last eigenvalue and the last singular value are given by

$$\lambda_k(B_i^{-1}(s)) = \frac{s-z_i}{s+\bar{z}_i}$$
$$\sigma_k(B_i^{-1}(s)) = |\lambda_k(B_i^{-1}(s))| = \frac{|s+\bar{z}_i|}{|s-z_i|}$$

When s and z_i are both in $\mathbb{C}^+$ or both in $\mathbb{C}^-$ then

$$\sigma_k(B_i^{-1}(s)) = |\lambda_k(B_i(s))| = \underline{\sigma}(B_i(s)) = \frac{|s+\bar{z}_i|}{|s-z_i|} \geq 1$$

otherwise

$$\sigma_k(B_i^{-1}(s)) = |\lambda_k(B_i(s))|\bar{\sigma}(B_i(s)) = \frac{|s+\bar{z}_i|}{|s-z_i|} \leq 1$$

For $s = j\omega$, all eigenvalues and all singular values are equal to one.

REMARK 1. The eigenvectors of $B_i(s)$ equals the singular input vectors which again equals the singular output vectors. Also note that these vectors are independent of frequency. Since the input and output singular directions of $B_i(s)$ are equal it follows that there is no rotation from input of $B_i(s)$ to the output of $B_i(s)$.

REMARK 2. $B_i(s)$ has a zero for $s = z_i$, the zero input direction equals the zero output direction v_i. Furthermore, $B_i(s)$ has a pole for $s = -\bar{z}_i$ with input and output direction v_i.

3.1 Right half plane zeros

The input factorization of RHP-zeros into B_{I} is given in the following theorem.

THEOREM 1. (INPUT FACTORIZATION OF RHP-ZEROS). *A system $G(s)$ containing N_z RHP-zeros z_i, with input directions $\hat{u}_{zi}$ and $\hat{x}_{zi}$ defined by*

$$\begin{bmatrix} A - z_i I & B_{i-1} \\ C & D \end{bmatrix} \begin{bmatrix} \hat{x}_{zi} \\ \hat{u}_{zi} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (20)$$

can be factorized in a minimum phase system $G_{\mathrm{I}}(s)$ and an all pass filter $B_{\mathrm{I}}(s)$, $G(s) = G_{\mathrm{I}}(s)B_{\mathrm{I}}(s)$ where

$$G_{\mathrm{I}}(s) = \left[\begin{array}{c|c} A & B' \\ \hline C & D \end{array}\right] \quad (21)$$

The modified input matrix B' can be calculated by applying the following formula repeatedly for $i = 1, \ldots, N_z$

$$B_i = B_{i-1} - 2\mathrm{Re}(z_i)\hat{x}_{zi}\hat{u}_{zi}^H \quad (22)$$

with $B_0 = B$ and $B' = B_{N_z}$. The (all pass) filter $B_{\mathrm{I}}(s)$ has all singular values $\sigma_i(s)$ and absolute value of all eigenvalues $\lambda_i(s)$ equal to one for $s = j\omega$. The all pass filter $B_{\mathrm{I}}(s)$ is given by

$$\begin{aligned} B_{\mathrm{I}}(s) &= B_{N_z}(s)B_{N_z-1}(s)\cdots B_1(s) \\ &= \prod_{i=0}^{N_z-1} B_{N_z-i}(s) \end{aligned} \quad (23)$$

where

$$B_i(s) = I - \frac{2\mathrm{Re}(z_i)}{s+\bar{z}_i}\hat{u}_{zi}\hat{u}_{zi}^H \quad (24)$$

REMARK 1. When one RHP-zero z_1 has been factorized the directions of the remaining zeros are modified, this is so because the input matrix B_{i-1} in (20) has been modified. It then follows that the input directions $\hat{u}_{zi}$ and $\hat{x}_{zi}$ are not the same as the zero input directions u_{zi} and $x_{z,\mathrm{I}}$ for zero z_i (except for the first zero factorized).

REMARK 2. The expressions above are valid for $z \in \mathbb{C}$. However, for the case with $\mathrm{Im}(z) \neq 0$ the factorization yield complex realizations of G_I and B_I.

REMARK 3. When $G(s)$ contains more than one RHP-zero, different sequences of factorizations yield the same overall $B_\mathrm{I}(s)$ and $G_\mathrm{I}(s)$ however, the individual filters $B_i(s)$ are different. Take as an example a system $G(s)$ with two RHP-zeros z_1 and z_2. Factorizing first z_1 and then z_2 yields $B_\mathrm{I}(s) = B_1(s)B_2(s)$ and $G_\mathrm{I}(s)$. Factorizing in the opposite sequence $\{z_2, z_1\}$ gives $\tilde{B}_\mathrm{I}(s) = \tilde{B}_2(s)\tilde{B}_1(s)$ and $\tilde{G}_\mathrm{I}(s)$ it then turns out that $G_\mathrm{I}(s) = \tilde{G}_\mathrm{I}(s)$ and $B_\mathrm{I}(s) = \tilde{B}_\mathrm{I}(s)$. However $B_1(s) \neq \tilde{B}_1(s)$ and $B_2(s) \neq \tilde{B}_2(s)$.

The output factorization of a system $G(s)$ with N_z RHP-zeros, can be expressed in a similar theorem.

THEOREM 2. (OUTPUT FACTORIZATION OF RHP-ZEROS). *A system $G(s)$ containing N_z RHP-zeros z_i, with output directions $\hat{y}_{zi}$ and $\hat{x}_{zi}$ defined by*

$$\begin{bmatrix} \hat{x}_{zi}^H & \hat{y}_{zi}^H \end{bmatrix} \begin{bmatrix} A - z_i I & B \\ C_{i-1} & D \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix} \tag{25}$$

can be factorized in a minimum phase system $G_\mathrm{O}(s)$ and an all pass filter $B_\mathrm{O}(s)$, $G(s) = B_\mathrm{O}(s)G_\mathrm{O}(s)$ where

$$G_\mathrm{O}(s) = \left[\begin{array}{c|c} A & B \\ \hline C' & D \end{array}\right] \tag{26}$$

The modified output matrix C' can be calculated by applying the following formula repeatedly for $i = 1, \dots, N_z$

$$C_i = C_{i-1} - 2\mathrm{Re}(z_i)\hat{y}_{zi}\hat{x}_{zi}^H \tag{27}$$

with $C_0 = C$ and $C' = C_{N_z}$. The (all pass) filter $B_\mathrm{O}(s)$ has all singular values $\sigma_i(s)$ and absolute value of all eigenvalues $\lambda_i(s)$ equal to one for $s = j\omega$. The all pass filter $B_\mathrm{O}(s)$ is given by

$$B_\mathrm{O}(s) = B_1(s)B_2(s)\cdots B_{N_z}(s) = \prod_{i=1}^{N_z} B_i(s) \tag{28}$$

where

$$B_i(s) = I - \frac{2\mathrm{Re}(z_i)}{s + \bar{z}_i}\hat{y}_{zi}\hat{y}_{zi}^H \tag{29}$$

REMARK 1. The output directions $\hat{y}_{zi}$ and $\hat{x}_{zi}$ are not the same as the output directions y_{zi} and x_{zi} for zero z_i (except for the first zero factorized) since the output matrix C_{i-1} in (25) is modified for each zero factorized.

REMARK 2. The expressions above are valid for $z \in \mathbb{C}$. However, for the case with $\mathrm{Im}(z) \neq 0$ the factorization yield complex realizations of G_O and B_O.

3.2 Right half plane poles

Right half plane poles can also be factorized in "input" and "output" factorizations in similar ways as RHP-zeros.

THEOREM 3. (OUTPUT FACTORIZATION OF RHP-POLES). *A system $G(s)$ containing N_p RHP-poles p_i, with output directions $\hat{y}_{pi}$ and $\hat{x}_{pi}$ defined by*

$$(A_{i-1} - p_i I)\hat{x}_{pi} = 0; \quad \hat{y}_{pi} = C\hat{x}_{pi} \tag{30}$$

can be factorized in a stabel system G_O and an all pass filter B_O containing the RHP-poles p_i, $G(s) = B_\mathrm{O}(s)G_\mathrm{O}(s)$ where

$$G_\mathrm{O}(s) = \left[\begin{array}{c|c} A' & B' \\ \hline C & D \end{array}\right] \tag{31}$$

The modified state-space matrices A' and B' can be calculated by applying the following formula repeatedly for $i = 1, \dots, N_p$

$$A_i = A_{i-1} - 2\mathrm{Re}(p_i)\hat{x}_{pi}\hat{y}_{pi}^H C \tag{32}$$

$$B_i = B_{i-1} - 2\mathrm{Re}(p_i)\hat{x}_{pi}\hat{y}_{pi}^H D \tag{33}$$

with $A_0 = A$, $B_0 = B$, $A' = A_{N_p}$ and $B' = B_{N_p}$. The (all pass) filter $B_\mathrm{O}(s)$ has all singular values $\sigma_i(s)$ and absolute value of all eigenvalues $\lambda_i(s)$ equal to one for $s = j\omega$. The all pass filter B_O is given by

$$B_\mathrm{O} = B_1(s)B_2(s)\cdots B_{N_p}(s) = \prod_{i=1}^{N_p} B_i(s) \tag{34}$$

where

$$B_i(s) = I + \frac{2\mathrm{Re}(p_i)}{s - p_i}\hat{y}_{pi}\hat{y}_{pi}^H \tag{35}$$

The input pole factorization follows.

THEOREM 4. (INPUT FACTORIZATION OF RHP-POLES). *A system $G(s)$ containing N_p RHP-poles p_i, with input directions $\hat{u}_{pi}$ and $\hat{x}_{pi}$ defined by*

$$\hat{x}_{pi}^H(A_{i-1} - p_i I) = 0; \quad \hat{u}_{pi} = B^T\hat{x}_{pi} \tag{36}$$

can be factorized in a stabel system G_I and an all pass filter B_I containing the RHP-poles p_i, $G(s) = G_\mathrm{I}(s)B_\mathrm{I}(s)$ where

$$G_\mathrm{I}(s) = \left[\begin{array}{c|c} A' & B \\ \hline C' & D \end{array}\right] \tag{37}$$

The modified state-space matrices A' and C' can be calculated by applying the following formula repeatedly for $i = 1, \dots, N_p$

$$A_i = A_{i-1} - 2\mathrm{Re}(p_i)B\hat{u}_{pi}\hat{x}_{pi}^H \tag{38}$$

$$C_i = C_{i-1} - 2\mathrm{Re}(p_i)D\hat{u}_{pi}\hat{x}_{pi}^H \tag{39}$$

with $A_0 = A$, $C_0 = C$, $A' = A_{N_p}$ and $C' = C_{N_p}$. The (all pass) filter $B_\mathrm{I}(s)$ has all singular values $\sigma_i(s)$ and absolute *value of all eigenvalues $\lambda_i(s)$ equal to one for $s = j\omega$. The all pass filter B_I is given by*

$$B_\mathrm{I} = B_{N_p}(s)B_{N_p-1}(s)\cdots B_1(s) = \prod_{i=0}^{N_p-1} B_{N_p-i}(s) \quad (40)$$

where

$$B_i(s) = I + \frac{2\mathrm{Re}(p_i)}{s-p_i}\hat{u}_{pi}\hat{u}_{pi}^H \quad (41)$$

4. REFERENCES

Havre, K. and S. Skogestad (1996). Effect of RHP zeros and poles on performance in multivariable systems. In: *Proc. from Control'96*.

MacFarlane, A. G. J. and N. Karcanias (1976). Poles and zeros of linear multivariable systems: A survey of algebraic, geometric and complex variable theory. *International Journal of Control* **24**, 33–74.

Wall, J. E., J. C. Doyle and C. A. Harvey (1980). Tradeoffs in the design of multivariable feedback systems. In: *Proc. 18'th Allerton Conference*.

Zhou, K., J. C. Doyle and K. Glover (1996). *Robust and Optimal Control*. Prentice-Hall. Upper Saddle River, New Jersey.

Appendix A. PROOFS OF THE RESULTS

Proof of Lemma 1. We have that $G(s) = C(sI-A)^{-1}B + D$ and for $s = p$, $G(p) = C(pI-A)^{-1}B + D$. Since p is an eigenvalue of A and x_R is the eigenvector corresponding to the pole p, $Ax_R = px_R$ implies $(pI-A)x_R = 0$. This means that the eigenvector x_R is the zero input direction of $(pI-A)$. Form the singular value decomposition we have that the output direction with infinite gain for $(pI-A)^{-1}$ is x_R. The output direction becomes $y_p = Cx_R$ as long as $||D||$ is finite. Assume that G is square and D is non-singular. Then G^{-1} is given by (10). If p is a pole of G then p is a zero of G^{-1} and the input pole directions x_p and u_p are the output zero directions of G^{-1} then we have

$$\begin{bmatrix} x_p^H & u_p^H \end{bmatrix} \begin{bmatrix} A - BD^{-1}C - pI & -BD^{-1} \\ D^{-1}C & D^{-1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix} \quad (A.1)$$

which gives

$$x_p^H(A-pI) - (x_p^H B - u_p^H)D^{-1}C = 0 \quad (A.2)$$

$$(-x_p^H B + u_p^H)D^{-1} = 0 \quad (A.3)$$

Clearly, $x_p = x_L$ and $u_p = B^H x_L$ are the output zero directions corresponding to the zero $s = p$ for G^{-1}. Then it follows that $u_p^H G^{-1}(p) = 0$, which is desirable. Note that these relations are not restricted to square G with non-singular D matrix since the relations are not dependent on the D matrix. For a singular square D one can modify D to become $\tilde{D}$ in (A.1) without affecting the relationship between the directions. For non-square G with less inputs than outputs one need to add fictitious inputs with zero effect from u to x. This corresponds to to add columns of zeros in the B matrix. For non-square G with less outputs than inputs one need to add fictitious outputs with zero effect from x to y. This corresponds to add rows of zeros in the B matrix. □

REMARK. Note that for a system with real realization ($A \in \mathbb{R}^{n_x \times n_x}$, $B \in \mathbb{R}^{n_x \times m}$, $C \in \mathbb{R}^{l \times n_x}$ and $D \in \mathbb{R}^{l \times m}$) $u_p = B^H x_L = B^T x_L$ since $B^H = B^T$. An alternative proof of the input direction u_p is then to calculate it as the conjugate of the output direction for the transposed system.

Proof of Lemma 2. $B_i(s)$ can be written

$$B_i(s) = V\tilde{\Sigma}V^H = \begin{bmatrix} u_1 & \cdots & u_{k-1} & v_i \end{bmatrix} \begin{bmatrix} 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & \frac{s-z_i}{s+\bar{z}_i} \end{bmatrix} \begin{bmatrix} u_1^H \\ \vdots \\ u_{k-1}^H \\ v_i^H \end{bmatrix} \quad (A.4)$$

where we select the set of vectors $\{u_1,\ u_2, \cdots u_{k-1}\}$ such that they form an orthonormal basis for $\mathbb{C}^k$ together with v_i. It follows that $\sigma_i(B_i(s)) = 1\ i \in [1 \ldots k-1]\ \forall s \in \mathbb{C}$ and $\sigma_k(B_i(s)) = \frac{|s-z_i|}{|s+\bar{z}_i|}\ \forall s \in \mathbb{C}$. Equation (A.4) gives $B_i(s)V = V\tilde{\Sigma}$, consequently $B_i(s)$ has $k-1$ eigenvalues equal to 1 and one eigenvalue is $\lambda_k = \frac{s-z_i}{s+\bar{z}_i}$. To prove $|\lambda_l(B_i(s))| = \underline{\sigma}(B_i(s)) = \frac{|s-z_i|}{|s+\bar{z}_i|} \leq 1$ when s and z are in the same half plane assume that $|s+\bar{z}_i| < |s-z_i|$, set $s = x_s + jy_s$ and $z_i = x_z + jy_z$. Then $|s+\bar{z}_i| < |s-z_i|$ implies $(x_s+x_z)^2 + (y_s-y_z)^2 < (x_s-x_z)^2 + (y_s-y_z)^2$ which gives $4x_z x_s < 0$. It follows that when s and z are in the same half plane then $\frac{|s-z_i|}{|s+\bar{z}_i|} \leq 1$ otherwise $\frac{|s-z_i|}{|s+\bar{z}_i|} \geq 1$. When either s or z is on the imaginary axis then $\frac{|s-z_i|}{|s+\bar{z}_i|} = 1$.

The inverse of $B_i(s)$ is given by

$$\begin{aligned} B_i^{-1}(s) &= V\tilde{\Sigma}^{-1}V^H \\ &= \begin{bmatrix} u_1 & \ldots & u_{k-1} & v_i \end{bmatrix} \begin{bmatrix} 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & \frac{s+\bar{z}_i}{s-z_i} \end{bmatrix} \begin{bmatrix} u_1^H \\ \vdots \\ u_{k-1}^H \\ v_i^H \end{bmatrix} \\ &= VV^H + \frac{2\mathrm{Re}(z_i)}{s-z_i}v_i v_i^H = I + \frac{2\mathrm{Re}(z_i)}{s-z_i}v_i v_i^H \end{aligned} \quad (A.5)$$

It follows that $\sigma_i(B_i^{-1}(s)) = \lambda_i(B_i^{-1}(s)) = 1\ i \in [1 \ldots k-1]\ \forall s \in \mathbb{C}$, $\lambda_k(B_i^{-1}(s)) = \frac{s+\bar{z}_i}{s-z_i}$, and $\sigma_l(B_i^{-1}(s)) = |\lambda_1(B_i^{-1}(s))| = \frac{|s+\bar{z}_i|}{|s-z_i|}$. It follows that when $s, z_i \in \mathbb{C}^+$ or $s, z_i \in \mathbb{C}^-$ then $\bar{\sigma}(B_i^{-1}(s)) = \frac{|s+\bar{z}_i|}{|s-z_i|} \geq 1$ and otherwise $\underline{\sigma}(B_i^{-1}(s)) = \frac{|s+\bar{z}_i|}{|s-z_i|} \leq 1$. □

Proof of Theorem 1. The proof is only given for $N_z = 1$ (note that in this case $u_z = \hat{u}_z$ and $x_z = \hat{x}_z$). The proof for $N_z > 1$ is to apply the proof of $N_z = 1$ repeatedly. To see that the minimum phase representation can be written as (21) with the matrix B' given by (22) one has to use the generalized eigenvalue problem (5). For $G(s)$, $s = z$ is a zero so (5) becomes

$$\begin{bmatrix} A - zI & B \\ C & D \end{bmatrix} \begin{bmatrix} x_z \\ u_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (A.6)$$

For the minimum phase system, $G_\mathrm{I}(s)$, the zero $s = -\bar{z} = -\mathrm{Re}(z) + \mathrm{Im}(z)j$ has the same input direction and the same state direction as $G(s)$ for $s = z$. The generalized eigenvalue problem becomes

$$\begin{bmatrix} A + \bar{z}I & B' \\ C & D \end{bmatrix} \begin{bmatrix} x_z \\ u_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (A.7)$$

By subtracting (A.7) from (A.6) one obtains

$$\begin{bmatrix} A - A - zI - \bar{z}I & B - B' \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_z \\ u_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (A.8)$$

from the first equation one gets

$$-zIx_z - \bar{z}Ix_z + Bu_z - B'u_z = 0$$

By extracting u_z on the right and solving for B' one obtains

$$B' = B - 2\mathrm{Re}(z)x_z u_z^H$$

which proves (21) and (22). The all pass filter with RHP-zero for $s = z$ with input and output zero directions u_z, and LHP-pole for $s = -\bar{z}$ with input and output pole directions u_z is given by Lemma 2, (16) and (17) with $N_z = 1$, $v_i = u_z$ and $z_i = z$. From the construction of $G_\mathrm{I}(s)$ we know there is a zero for $s = -\bar{z}$ with input direction u_z. We may therefore cancel the pole for $s = -\bar{z}$ in $B_\mathrm{I}(s)$ with the zero in same location in $G_\mathrm{I}(s)$ and it follows that $G_\mathrm{I}B_\mathrm{I}(s) = G(s)$. □

Proof of Theorem 2. The proof is given for $N_z = 1$ the proof for $N_z > 1$ is to apply the proof of $N_z = 1$ repeatedly. Since z is a RHP-zero for $G(s)$ it follows that z is a RHP-zero for $G^T = \left[\begin{array}{c|c} A^T & C^T \\ \hline B^T & D^T \end{array}\right]$. From Theorem 1 we have $G^T = G_\mathrm{I}B_\mathrm{I}$, $G_\mathrm{I} = \left[\begin{array}{c|c} A^T & C'^T \\ \hline B^T & D^T \end{array}\right]$ and $B_\mathrm{I} = I - \frac{2\mathrm{Re}(z)}{s+\bar{z}}yy^H$ where $C'^T = C^T - 2\mathrm{Re}(z)xy^H$, x and y are defined from $\begin{bmatrix} A^T - zI & C^T \\ B^T & D^T \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$. However, $x = \bar{x}_{z,\mathrm{O}}$ and $y = \bar{y}_z$. $G_\mathrm{O} = G_\mathrm{I}^T = \left[\begin{array}{c|c} A & B \\ \hline C' & D \end{array}\right]$, $C' = C - 2\mathrm{Re}(z)\bar{y}x^T = C - 2\mathrm{Re}(z)y_z x_{z,\mathrm{O}}^H$. $B_\mathrm{O} = B_\mathrm{I}^T = I - \frac{2\mathrm{Re}(z)}{s+\bar{z}}\bar{y}y^T = I - \frac{2\mathrm{Re}(z)}{s+\bar{z}}y_z y_z^H$. □

Proof of Theorem 3. The proof is only given for $N_p = 1$, the proof for $N_p > 1$ is to apply the proof for $N_p = 1$ repeatedly (note that for $N_p = 1$, $\hat{y}_{pi} = y_p$ and $\hat{x}_{pi} = x_p$). Assume without loss of generality that G is square and that the state space matrix D is non-singular. Then D^{-1} exists and G^{-1} is given by (10). Furthermore, p is a RHP-zero of G^{-1} which can be factorized in an "input" factorization $G^{-1} = G_\mathrm{I}B_\mathrm{I}$ (Theorem 1). It then follows that $B_\mathrm{O} = B_\mathrm{I}^{-1}$ and $G_\mathrm{O} = G_\mathrm{I}^{-1}$ where

$$G_\mathrm{I}^{-1} = \left[\begin{array}{c|c} A - BD^{-1}C & -BD^{-1} + 2\mathrm{Re}(p_i)x_p y_p^H \\ \hline D^{-1}C & D^{-1} \end{array}\right]^{-1}$$

since $x_{z=p} = -x_p$ and $u_{z=p} = y_p$, which gives

$$G_\mathrm{O} = \left[\begin{array}{c|c} A - 2\mathrm{Re}(p_i)x_p y_p^H C & B - 2\mathrm{Re}(p_i)x_p y_p^H D \\ \hline C & D \end{array}\right]$$

The all pass filter B_O is given by (18) and (19) with $N_z = N_p = 1$, $z_i = p$ and $v_i = y_p$, $B_\mathrm{O} = I + \frac{2\mathrm{Re}(p)}{s-p_i}y_p y_p^H$. The output pole directions are independent of the matrix D in the state space description. Also the relation between output pole directions of G and the input zero directions of G^{-1} is independent of D, this means that we can add non-zero elements to D without affecting the the pole directions. So, if D is singular we add non-zero elements along the diagonal of D so that it becomes non-singular. Consider next the case where G has more outputs than inputs then fictitious inputs with zero effect on y can be included by adding columns with zeros in B and and D so that D becomes square. The next step is then to add non-zero elements to D so that D becomes non-singular. Similarly if G has more inputs than outputs one can add rows with zeros to the C and D matrices. Adding columns with zeros to the B matrix and rows with zeros to the C matrix does not change the direction of the pole, it only expands the dimension. □

Proof of Theorem 4. The proof is given for $N_p = 1$ the proof for $N_p > 1$ is to apply the proof of $N_p = 1$ repeatedly (note that for $N_p = 1$, $\hat{u}_{pi} = u_p$ and $\hat{x}_{pi} = x_p$). Since p is a RHP-pole for $G(s)$ it follows that p is a RHP-pole for $G^T = \left[\begin{array}{c|c} A^T & C^T \\ \hline B^T & D^T \end{array}\right]$. From Theorem 3 we have $G^T = B_\mathrm{O}G_\mathrm{O}$, $G_\mathrm{O} = \left[\begin{array}{c|c} A'^T & C'^T \\ \hline B^T & D^T \end{array}\right]$ and $B_\mathrm{O} = I + \frac{2\mathrm{Re}(p)}{s-p}yy^H$ where $A'^T = A^T - 2\mathrm{Re}(p)xy^H B^T$, $C'^T = C^T - 2\mathrm{Re}(p)xy^H D^T$, x and y are defined from $(A^T - pI)x = 0$ and $y = B^T x$. However, $x = \bar{x}_p$, $y = \bar{u}_p$, $G_\mathrm{I} = G_\mathrm{O}^T = \left[\begin{array}{c|c} A' & B \\ \hline C' & D \end{array}\right]$, $A' = A - 2\mathrm{Re}(p)B\bar{y}x^T = A - 2\mathrm{Re}(p)Bu_p x_p^H$, $C' = C - 2\mathrm{Re}(p)D\bar{y}x^T = C - 2\mathrm{Re}(p)Du_p x_p^H$ and $B_\mathrm{I} = B_\mathrm{O}^T = I + \frac{2\mathrm{Re}(p)}{s-p}\bar{y}y^T = I + \frac{2\mathrm{Re}(p)}{s-p}u_p u_p^H$. □