SELECTION OF VARIABLES FOR REGULATORY CONTROL USING POLE DIRECTIONS.

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Abstract. This paper considers the design of a stabilizing control structure, using the information given in the pole directions. It is shown how the input/output pole directions are related to the minimum input energy needed to stabilize a given unstable mode.

1. INTRODUCTION

In this paper we consider selecting inputs and outputs to obtain a stabilizing control structure. That is, we want to answer questions like:

- How many loops must be closed to stabilize a given unstable plant?
- Which outputs should be controlled?
- Which input should be used for control?

To answer these questions we need to look into the directionality of the poles. Pole directions for the case with distinct poles are defined in (Havre and Skogestad, 1996), where it is also shown how these directions can be computed from the left and the right eigenvalue problems. In some cases poles with multiplicity larger than one, i.e. a repeated pole, may actually occur in chemical process plants. In this paper we extend the definition of pole directions to the case where the poles have multiplicity larger than two and we show how to compute the pole directions by using the Normal Jordan Form. Furthermore, we derive a relationship between pole directions and the minimum input energy needed to stabilize a given unstable mode.

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Notation. We consider linear time invariant systems on state-space form

$$\dot{x} = Ax + Bu \tag{1}$$
$$y = Cx + Du \tag{2}$$

where $A \in \mathbb{R}^{n_x \times n_x}$, $B \in \mathbb{R}^{n_x \times m}$, $C \in \mathbb{R}^{l \times n_x}$ and $D \in \mathbb{R}^{l \times m}$ where n_x is the number of states, l is the number of outputs and m is the number of inputs. These equations may be rewritten as

$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}$$

This gives rise to the short-hand notation

$$G \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \tag{3}$$

which is frequently used to describe a state-space model of a system G. The transfer function of G defined by (3) can be evaluated as a function of the complex variable $s \in \mathbb{C}$, $G(s) = C(sI - A)^{-1}B + D$. We often omit to show the dependence on the complex variable s for transfer functions.

Outline. The outline of the paper is a follows; the second section defines input/output pole directions and show how they can be computed using the Normal Jordan form. This section also gives a couple of examples on pole directions. In the third section we consider the problem of finding one input and one output (and a SISO controller) which stabilizes a given unstable mode with minimum input usage. The fourth section concerns repeated poles. The result given in this section, treats the case where the system has a repeated pole with v linearly independent eigenvectors, and it says that one at least need v inputs and v outputs to control this mode. In Section 5 we discuss some limitations on the usage of pole directions to input/output selection, and we demonstrate through an example that parallel unstable system are difficult to control using less control loops than the number of instabilities. Section 6 contains some relevant control engineering problems from chemical process plants. Appendix A contains some relevant information about left/right eigenvalue problems, left/right Normal Jordan Forms and shows how these can be combined. Appendix B contains the proofs of the main results. Some minor proofs are also given in the main text.

Related work. Some related work are given in (Wang and Davison, 1973; Benninger, 1986; Tarokh, 1985; Tarokh, 1992; Hovd, 1992; Lunze, 1992; Li *et al.*, 1994*a*; Li *et al.*, 1994*b*).

2. POLE DIRECTIONS

The poles are defined as the eigenvalues of the A matrix in the state-space description, and the pole or characteristic polynomial $\phi(s)$ is defined as

$$\phi(s) = \det(sI - A) = \prod_{i=1}^{n_x} (s - p_i) \tag{4}$$

The gain of the system $G(s)$ evaluated at $s = p$, $G(p)$, is infinite in some directions at the input and the output. This is the basis for the following definition of input and output pole directions.

DEFINITION 1. (INPUT AND OUTPUT POLE DIRECTIONS). *If $s = p \in \mathbb{C}$ is a pole of $G(s)$ with multiplicity q then there exists q input and output directions, $u_{p,i} \in \mathbb{C}^m$ $y_{p,i} \in \mathbb{C}^l$, with infinite gain for $s = p$.*

REMARK 1. When pole p has multiplicity one, we know that there exists one input and one output direction. If the mode is unobservable then $y_p = 0$, and if the mode is uncontrollable then $u_p = 0$.
REMARK 2. We have stated that there exists q input and output directions. However, each of the input directions may not be linearly independent and each of the output direction may not be linearly independent.
REMARK 3. As mentioned above the pole directions at the input and the output may not be linearly independent, but as we shall see there exists q linearly independent input state directions and q linearly independent output state directions. These state directions are associated with a particular state-space realization of G, so any similarity transformation applied to give a different state-space realization also gives a new set of input and output state directions.

In Havre and Skogestad (1996) it is shown that for a system with state-space realization (3) the pole input (u_p) and output (y_p) directions associated with a distinct pole p can be computed using

$$u_p = B^H x_L; \quad y_p = C x_R \tag{5}$$

where $x_R \in \mathbb{C}^{n_x}$ and $x_L \in \mathbb{C}^{n_x}$ are the eigenvectors corresponding to the two eigenvalue problems $Ax_R = px_R$ and $x_L^H A = x_L^H p$.

For poles with multiplicity $q > 1$ it may happen that the number of linearly independent eigenvectors v are less than q. In such cases the state-space A matrix can not be diagonalized. Instead, we use the *Normal Jordan Form*. Section A.2 defines and shows how the left and right Jordan forms can be combined into

$$M_L^H A M_R S^{-H} = S^{-H} M_L^H A M_R = J \tag{6}$$

where M_R and M_L are the non-singular similarity transformations which gives $M_R^{-1} A M_R = J$, $M_L^H A M_L^{-H} = J$ and the columns in M_R and M_L which are eigenvectors are scaled such that their norms are equal to one. Furthermore, S has the structure given in (A.15) and $M_L = M_R^{-H} S$.

LEMMA 1. (POLE DIRECTIONS). *If p is a pole with multiplicity q of the system G with state-space realization (3) then the q output directions for the pole p can be computed from*

$$y_{p,i} = C m_{R,i}, \quad \forall i \in [1, \ldots, q] \tag{7}$$

where $m_{R,i}$ $\forall i \in [1, \ldots, q]$ corresponds to the columns in M_R associated with the pole p. The q input directions for the pole p can be computed from

$$u_{p,i} = B^H m_{L,i}, \quad \forall i \in [1, \ldots, q] \tag{8}$$

where $m_{L,i}$ $\forall i \in [1, \ldots, q]$ corresponds to the columns in M_L associated with the pole p.

REMARK 1. If A has n_x linearly independent eigenvectors, each Jordan block is of size 1×1, $M_L = X_L$, $M_R = X_R$ and $J = \Lambda$ and the matrix A is diagonalizable.
REMARK 2. If A has distinct eigenvalues, then A has n_x linearly independent eigenvectors and A can be diagonalized.
REMARK 3. The pole directions are independent of the state-space realization, to see this define a new state vector with the similarity transformation T

$$z = Tx$$

which leads to the state-space realization

$$\dot{z} = \underbrace{TAT^{-1}}_{A'} z + \underbrace{TB}_{B'} u; \quad y = \underbrace{CT^{-1}}_{C'} z + Du$$

From the the construction of the Jordan form we have $M_R^{-1} A M_R = J$ inserting $A = T^{-1} A' T$ gives

$$M_R^{-1} T^{-1} A' \underbrace{T M_R}_{M'_R} = J$$

and we have $M'_L = M'^{-H}_R = T^{-H} M^{-H}_R = T^{-H} M_L$ or

$$m'_{R,i} = T m_{R,i}; \quad m'_{L,i} = T^{-H} m_{L,i}$$

The new output direction becomes

$$y'_p = C' m'_{R,i} = C T^{-1} T m_{R,i} = C m_{R,i} = y_p$$

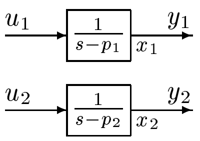
and the new input direction becomes

$$u'_p = B'^H m'_{L,i} = B^H T^H T^{-H} m_{L,i} = B^H m_{L,i} = u_p$$

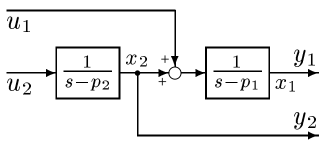
REMARK 4. The input and output directions can of course be normalized.

EXAMPLE 1. POLE DIRECTIONS FOR SYSTEMS IN SERIES AND PARALLEL.

Systems in parallel.



Systems in series.



$$p_1 = 1, \;\; p_2 = 2 \qquad\qquad p_1 = 1, \;\; p_2 = 2$$

$$G(s) = \begin{bmatrix} \frac{1}{s-1} & 0 \\ 0 & \frac{1}{s-2} \end{bmatrix} \qquad G(s) = \begin{bmatrix} \frac{1}{s-1} & \frac{1}{(s-1)(s-2)} \\ 0 & \frac{1}{s-2} \end{bmatrix}$$

$$U_p = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad U_p = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & 1 \end{bmatrix}$$

$$Y_p = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad Y_p = \begin{bmatrix} 1 & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} \end{bmatrix}$$

For the systems in parallel the two modes p_1 and p_2 appear in different channels both at the input and at the output. This can easily be seen from both $G(s)$ and the pole directions. The first column in U_p corresponds to $p_1 = 1$ and the second column corresponds to $p_2 = 2$. We see from U_p that only u_1 can affect the mode p_1 and only u_2 can affect p_2. In a similar way we see from Y_p that mode p_1 can only be observed in y_1 and the mode p_2 can only be observed in y_2. For the system in parallel it is obvious that we need to control both outputs by using both inputs in order to stabilize the system.

For the systems in series we can see from the figure, from $G(s)$ and from the pole directions U_p and Y_p that both modes can be stabilized by controlling y_1 using u_2. From U_p we see that the mode p_1 (the first column of U_p) can be affected by using both u_1 and u_2, whereas the mode p_2 can only be affected by using u_2. In a similar way p_1 can only be observed in y_1, whereas $p_2 = 2$ can be observed in both y_1 and y_2.

EXAMPLE 2. Consider the following system:

$$G(s) \stackrel{s}{=} \left[\begin{array}{c|c} A & I \\ \hline I & \mathbf{0} \end{array}\right] \text{ where } A = \begin{bmatrix} -10 & 0 & 0 & -9 & -9 \\ 0 & -1 & 0 & 9 & -9 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -10 \end{bmatrix}$$

$$G(s) = \begin{bmatrix} \frac{1}{s+10} & 0 & 0 & \frac{-9}{(s+1)(s+10)} & \frac{9}{(s+10)^2} \\ 0 & \frac{1}{s+1} & 0 & \frac{9}{(s+1)^2} & \frac{-9}{(s+1)(s+10)} \\ 0 & 0 & \frac{1}{s+1} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{s+1} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{s+10} \end{bmatrix}$$

The system is stable with two poles at $p_1 = -10$ and three poles at $p_2 = -1$. For $p_1 = -10$ we have the following two left and right eigenvectors

$$X_{R,p_1} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad X_{L,p_1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & -1 \end{bmatrix}$$

which are linearly dependent. Similar for $p_2 = -1$ we have two linearly independent left and right and eigenvectors

$$X_{R,p_2} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad X_{L,p_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

To compute the pole directions we must compute the Jordan form. For the given A matrix, the matrices M_R, M_L and S

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 1/9 & 0 \\ 0 & 1/9 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1/9 & 0 \\ 0 & 1/9 & 0 & 0 & 0 \end{bmatrix}, \quad M_L = \begin{bmatrix} 1/9 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1/9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1/9 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1/9 & 0 & 0 \end{bmatrix},$$

$$S = \begin{bmatrix} 1/9 & 0 & 0 & 0 & 0 \\ 0 & 1/9 & 0 & 0 & 0 \\ 0 & 0 & 1/9 & 0 & 0 \\ 0 & 0 & 0 & 1/9 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

transform A into Normal Jordan Form

$$J = S^{-H} M_L^H A M_R = \begin{bmatrix} -10 & 1 & 0 & 0 & 0 \\ 0 & -10 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

The input and output pole directions corresponding to the poles $P = [-10 \quad -10 \quad -1 \quad -1 \quad -1]$ are

$$Y_p = \begin{bmatrix} 1 & 0 & 0 & 1/9 & 0 \\ 0 & 1/9 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1/9 & 0 \\ 0 & 1/9 & 0 & 0 & 0 \end{bmatrix}; \quad U_p = \begin{bmatrix} 1/9 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1/9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1/9 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1/9 & 0 & 0 \end{bmatrix}$$

In Figure 1 the step response to $u = [1 \quad 1 \quad -0.5 \quad -0.1 \quad 0.5]^T$ is given. The analytical solution to a step change in u with zero initial state, $x_0 = 0$ is given by

$$y(t) = \begin{bmatrix} 0.1 - 0.1e^{-10t} & 0 & 0 & e^{-t} - 0.1e^{-10t} - 0.9 & 0.09(1 - e^{-10t}) - 0.9te^{-10t} \\ 0 & 1 - e^{-t} & 0 & 9 - (9 + 9t)e^{-t} & e^{-t} - 0.1e^{-10t} - 0.9 \\ 0 & 0 & 1 - e^{-t} & 0 & 0 \\ 0 & 0 & 0 & 1 - e^{-t} & 0 \\ 0 & 0 & 0 & 0 & 0.1 - 0.1e^{-10t} \end{bmatrix} u$$

By studying the analytical solution of the step response, the elements of $G(s)$ and the pole directions it can be seen that

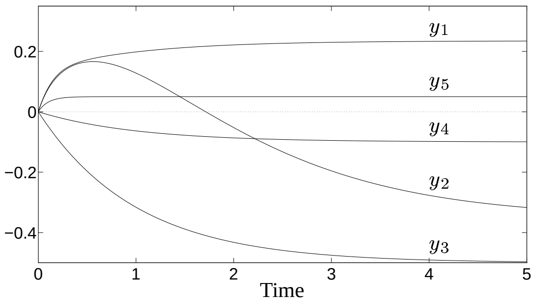


Figure 1. Response to step in $u = [\,1 \quad 1 \quad -0.5 \quad -0.1 \quad 0.5\,]^T$.

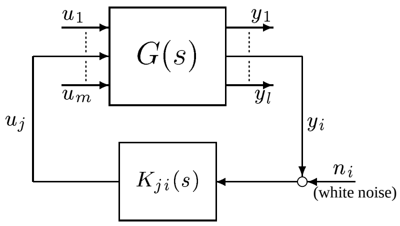


Figure 2. Plant G and stabilizing control loop with pairing $u_j \leftrightarrow y_i$.

there is a correspondence between the non-zero elements of the pole directions and in which inputs/outputs the two modes appear.

3. STABILIZING CONTROL WITH MINIMUM INPUT

In this section we consider the following problem, see also Figure 2:

PROBLEM 1. *Given a plant with one unstable mode p ($\mathrm{Re}\, p > 0$), with white measurement noise n_i of unit intensity at each output y_i; Find the pairing $u_j \leftrightarrow y_i$ such that the plant is stabilized with minimum input usage:*

$$J = E\left\{ \lim_{T\to\infty} \frac{1}{T} \int_0^T u_j^2(t)dt \right\} \tag{9}$$

At first sight it is not clear that the output selection problem is included at all, the reason is that the outputs do not enter into the objective (9) explicitly. However, the output selection problem is included implicitly through the measurement noise and the expectation operator E. We assume that the noise are uncorrelated zero-mean Gaussian stochastic processes with power spectral density matrix equal to the identity I. That is, each n_i are white noise processes with covariance

$$E\left\{ n(t)n^T(\tau) \right\} = I \cdot \delta(t-\tau) \tag{10}$$

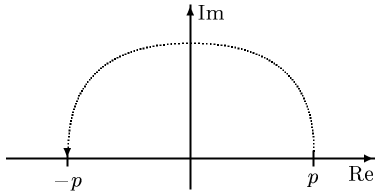


Figure 3. Mapping of pole from RHP to LHP with state feedback and minimum input usage.

where $n = [\,n_1 \ \cdots \ n_l\,]^T$. This problem can be cast into several LQG problems, one for each possible pairing, and solved numerically using a solver for Algebraic Riccati Equations or some specialized functions for LQG, LQR or LQE problems (see for example the corresponding names in the Control System Toobox in MATLAB). This problem is however so simple that an analytical solution to the ARE's can be found. As for LQG design we will use the Separation Theorem (Certainty Equivalence Principle) and find the best input using state feedback control (LQR) under the assumption of perfect measurement of all states. As the next step we will construct a state-observer (LQE) and find the best output so that

$$E\left\{ (x(t) - \hat{x}(t))^T (x(t) - \hat{x}(t)) \right\}$$

is minimized using one output y_i only.

Let us first state the solution:

SOLUTION TO PROBLEM 1.*The minimum of the objective J, for a specified input u_j and a specified output y_i is*

$$\boxed{J = \frac{8p^3}{u_{p,j}^2 y_{p,i}^2}}$$

where p is the pole, $u_{p,j}$ is the j'th element in the input pole direction and $y_{p,i}$ is the i'th element in the output pole direction. To minimize the control effort to stabilize the pole p, one should

- *use input u_j, where j corresponds to $\max_j u_{p,j}$,*
- *control output y_i, where i corresponds to $\max_i y_{p,i}$.*

It is well-known (Kwakernaak and Sivan, 1972) that minimum input to stabilize an unstable plant with state feedback $u = -Kx(t)$ mirrors the unstable poles across the imaginary axis, see Figure 3.

Optimal state feedback. For details regarding the LQR problem see Skogestad and Postlethwaite (1996). In this case, the problem is to minimize the deterministic cost

$$J_{LQR} = \int_0^\infty u_j^2(t)dt$$

That is, LQR problem with zero weight on the states and unity weight on the control u_j. The optimal solution (for any initial state) is $u_j(t) = -K_j x(t)$, where

$$K_j = \underbrace{e_j^T B^T}_{b_j^T} X = b_j^T X$$

where e_j is a vector of length m with zeros in all elements except for element j which equals one, b_j is the j'th column in the B matrix and $X = X^T \geq 0$ is the unique positive-semidefinite solution of the algebraic Riccati equation

$$A^T X + XA - XBe_j e_j^T B^T X = 0 \tag{11}$$

The solution to (11) is

$$X = \frac{2p}{u_{p,j}^2} x_L x_L^T \geq 0 \tag{12}$$

where x_L is the left eigenvector corresponding to p.

Proof of (12). Since, p is real (only one unstable pole) $x_L = \bar{x}_L, u_p = \bar{u}_p$. x_L is a left eigenvector, $x_L^H A = p x_L^H$, by taking the transposed we get $A^T x_L = p x_L$. Inserting X into (11) we obtain

$$\underbrace{A^T x_L}_{p x_L} x_L^T \frac{2p}{u_{p,j}^2} + \frac{2p}{u_{p,j}^2} x_L \underbrace{x_L^T A}_{p x_L^T} - x_L \underbrace{x_L^T B e_j}_{u_{p,j}} \underbrace{e_j^T B^T x_L}_{u_{p,j}} x_L^T \frac{4p^2}{u_{p,j}^4} = 0$$

□

The controller gain matrix K_j becomes

$$K_j = \overbrace{e_j^T \underbrace{B^T x_L}_{u_p}}^{u_{p,j}} x_L^T \frac{2p}{u_{p,j}^2} = \frac{2p}{u_{p,j}} x_L^T \tag{13}$$

Kalman filter. In this case the Kalman filter is updated by only using the information in output y_i, and we have no process noise. The structure is similar to the structure in an ordinary state-estimator, see Skogestad and Postlethwaite (1996, page 355), where

$$\dot{\hat{x}} = A\hat{x} + Bu + K_{f,i}(y_i - e_i^T C\hat{x}) \tag{14}$$

The optimal choice of $K_{f,i}$, which minimizes

$$E\left\{[x - \hat{x}]^T [x - \hat{x}]\right\}$$

is given by

$$K_{f,i} = YC^T e_i$$

where $Y = Y^T \geq 0$ is the unique positive-semidefinite solution of the algebraic Riccati equation

$$YA^T + AY - YC^T e_i e_i^T CY = 0 \tag{15}$$

The solution to (15) is

$$Y = \frac{2p}{y_{p,i}^2} x_R x_R^T \geq 0 \tag{16}$$

where x_R is the right eigenvector corresponding to p.

Proof of (16). Since, p is real (only one unstable pole) $x_R = \bar{x}_R, y_p = \bar{y}_p$. x_R is a right eigenvector, $Ax_R = p x_R$, by taking the transposed we get $x_R^T A^T = p x_R^T$. Inserting Y into (15) we obtain

$$\frac{2p}{y_{p,i}^2} x_R \underbrace{x_R^T A}_{p x_R^T} + \underbrace{A x_R}_{p x_R} x_R^T \frac{2p}{y_{p,i}^2} - x_R \underbrace{x_R^T C^T e_i}_{y_{p,i}} \underbrace{e_i^T C x_R}_{y_{p,i}} x_R^T \frac{4p^2}{y_{p,i}^4} = 0$$

□

The estimator gain matrix $K_{f,i}$ becomes

$$K_{f,i} = \frac{2p}{y_{p,i}^2} x_R \overbrace{\underbrace{x_R^T C^T}_{y_p^T} e_i}^{y_{p,i}} = \frac{2p}{y_{p,i}} x_R \tag{17}$$

Minimum value of objective. To prove the minimum value of the objective J (9), we use Theorem 5.4 part (d) in Kwakernaak and Sivan (1972, page 394–395). In this case, and with the notation used here, we get

$$\begin{aligned} J &= \operatorname{tr}\{XK_{f,i}K_{f,i}^T\} = \operatorname{tr}\{YK_j^T K_j\} \\ &= \operatorname{tr}\{\frac{2p}{u_{p,j}^2} x_L x_L^T \frac{2p}{y_{p,i}} x_R \frac{2p}{y_{p,i}} x_R^T\} = \frac{8p^3}{u_{p,j}^2 y_{p,i}^2} \end{aligned}$$

Implications on input/output selection. The pole input/output directions depends on scaling, so it is crucial to scale the inputs and outputs properly. One procedure for selecting inputs and outputs to stabilize a given unstable mode is:

(1) Scale the inputs so that a change in each input are of equal importance on the overall objective.
(2) Scale outputs relative to measurement noise.
(3) Select input u_j, where u_j corresponds to a large element in input pole direction vector u_p
(4) Select to control output y_i, where y_i corresponds to a large element in output pole direction vector y_p.

If the plant has several unstable modes to be stabilized after stabilizing one mode using one loop, the poles and the pole directions of the partially controlled system (closed loop system with the SISO controller included) can be recomputed. It may be that the SISO controller has stabilized several unstable poles. If there are remaining unstable poles then new control links can be identified from the recomputed pole directions and new controllers can be included, see the Tennessee Eastman example in Section 6.

4. REPEATED POLES

In this section we look into the special case when a system $G(s)$ has one or more poles with multiplicity greater than one. An important property connected to the mode, is the number of linearly independent eigenvectors. This gives information about the minimum number of outputs to be con-

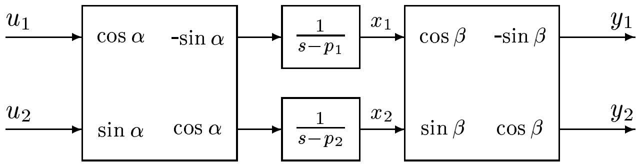


Figure 4. Unstable modes in parallel with input and output rotations.

trolled and the minimum number of inputs to be used for control in order to affect the mode p.

THEOREM 1. *Consider a system G with a repeated mode p (multiplicity q) with v linearly independent eigenvectors; in order to affect the mode p one need to control at least v outputs using v inputs.*

This results comes from Lemma 2 given in Appendix B.2 which states that all square subsystems containing the mode p with dimension less than $v \times v$ has at least one zero for $s = p$, so the mode p is not completely controllable or observable or both in square subsystems of dimension less than $v \times v$.

EXAMPLE 3. Consider the system shown in Figure 4, with $p_1 = p_2 = p = 1$ we have

$$G(s) \stackrel{s}{=} \left[\begin{array}{cc|cc} p & 0 & \cos\alpha & -\sin\alpha \\ 0 & p & \sin\alpha & \cos\alpha \\ \hline \cos\beta & -\sin\beta & 0 & 0 \\ \sin\beta & \cos\beta & 0 & 0 \end{array}\right]$$

The state-space matrix A has two linearly independent left and right eigenvectors for the mode $p = 1$

$$X_R = X_L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Input and output pole directions become

$$U_p = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}; \quad Y_p = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$$

The two pole directions are orthogonal both on the input and on the output. The transfer function $G(s)$ is given by

$$G(s) = \begin{bmatrix} \frac{\cos(\alpha+\beta)(s-p)}{(s-p)^2} & -\frac{\sin(\alpha+\beta)(s-p)}{(s-p)^2} \\ \frac{\sin(\alpha+\beta)(s-p)}{(s-p)^2} & \frac{\cos(\alpha+\beta)(s-p)}{(s-p)^2} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{\cos(\alpha+\beta)}{s-p} & -\frac{\sin(\alpha+\beta)}{s-p} \\ \frac{\sin(\alpha+\beta)}{s-p} & \frac{\cos(\alpha+\beta)}{s-p} \end{bmatrix}$$

Since we have two linearly independent eigenvectors corresponding to $p = 1$, a pole/zero cancellation occurs in all elements (predicted by Lemma 2), so that no elements contain the term $(s-1)^2$ in the denominator. So in order to control the mode p we need to use both inputs and both outputs, which also was stated in Theorem 1.

5. LIMITATIONS IN THE USE OF POLE DIRECTIONS

We have already seen one limitation in the use of pole directions to select inputs and outputs. This limitation is demonstrated in Example 3 where the system $G(s)$ has a repeated mode p with two linearly independent eigenvectors. As stated in Example 3 this system can not be stabilized by controlling one output and using one input. This is the fact despite that both input pole directions has a component in one of the inputs for all $\alpha \neq k \cdot 90^\circ,\ k \in \mathbb{N}$ and both output pole directions has a component in one of the outputs for all $\beta \neq k \cdot 90^\circ,\ k \in \mathbb{N}$. This problem is caused by pole/zero cancellation in each element. However, the situation can be identified by the fact that the system has a repeated mode p with multiplicity two and two linearly independent eigenvectors.

In the next example we consider the same system but in this case we have $p_1 \neq p_2$ so pole/zero cancellation does not occur for values of α and β between 0° and 90°. Thus, in theory the plant can be stabilized using one input and one output. However, in practice this may be impossible due to the presence of a RHP-zero in $g_{ij}(s)$ which is close to the two RHP-poles, see (Havre and Skogestad, 1996).

EXAMPLE 4. Again we consider two subsystems in parallel, however, in this case the two modes of the subsystems are different, see Figure 4. We have

$$G(s) \stackrel{s}{=} \left[\begin{array}{cc|cc} p_1 & 0 & \cos\alpha & -\sin\alpha \\ 0 & p_2 & \sin\alpha & \cos\alpha \\ \hline \cos\beta & -\sin\beta & 0 & 0 \\ \sin\beta & \cos\beta & 0 & 0 \end{array}\right]$$

The left and right eigenvectors corresponding to the modes p_1 and p_2 are

$$X_R = X_L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Input and output pole directions are

$$U_p = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}; \quad Y_p = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$$

The transfer function $G(s)$ is given by

$$G(s) = \begin{bmatrix} \frac{n_{11}(s)}{(s-p_1)(s-p_2)} & \frac{n_{12}(s)}{(s-p_1)(s-p_2)} \\ \frac{n_{21}(s)}{(s-p_1)(s-p_2)} & \frac{n_{22}(s)}{(s-p_1)(s-p_2)} \end{bmatrix}$$

where

$$n_{11}(s) = (s-p_2)\cos(\alpha)\cos(\beta) - (s-p_1)\sin(\alpha)\sin(\beta)$$
$$= \cos(\alpha+\beta)s - p_2\cos(\alpha)\cos(\beta) + p_1\sin(\alpha)\sin(\beta)$$
$$n_{12}(s) = -(s-p_2)\sin(\alpha)\cos(\beta) - (s-p_1)\cos(\alpha)\sin(\beta)$$
$$= -\sin(\alpha+\beta)s + p_2\sin(\alpha)\cos(\beta) + p_1\cos(\alpha)\sin(\beta)$$
$$n_{21}(s) = (s-p_2)\cos(\alpha)\sin(\beta) + (s-p_1)\sin(\alpha)\cos(\beta)$$
$$= \sin(\alpha+\beta)s - p_2\cos(\alpha)\sin(\beta) - p_1\sin(\alpha)\cos(\beta)$$
$$n_{22}(s) = -(s-p_2)\sin(\alpha)\sin(\beta) + (s-p_1)\cos(\alpha)\cos(\beta)$$

$$= \cos(\alpha+\beta)s + p_2 \sin(\alpha)\sin(\beta) - p_1\cos(\alpha)\cos(\beta)$$

Zeros in the individual transfer function elements are

$$z_{11} = p_1 \frac{p_2/p_1 \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)}{\cos(\alpha+\beta)}$$
$$z_{12} = p_1 \frac{p_2/p_1 \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)}{\sin(\alpha+\beta)}$$
$$z_{21} = p_1 \frac{p_2/p_1 \cos(\alpha)\sin(\beta) + \sin(\alpha)\cos(\beta)}{\sin(\alpha+\beta)}$$
$$z_{22} = p_1 \frac{\cos(\alpha)\cos(\beta) - p_2/p_1 \sin(\alpha)\sin(\beta)}{\cos(\alpha+\beta)}$$

When $p_1 = p_2 = p$ all elements of $G(s)$ has a RHP-zero for $s = p$ and we have pole/zero cancellation. Set $p_1 = 1$ and consider

$$p_2 \in [1.1, 1.01, 1, 0.99, 0.9]$$

The zeros of the transfer function elements are given in Table 1. We observe that all elements has RHP-zeros.

Table 1. Zeros of the transfer function elements.

α	β	p_2	z_{11}	z_{12}	z_{21}	z_{22}
		1.1	1.15	1.05	1.05	0.95
		1.01	1.015	1.005	1.005	0.995
30°	30°	1	1	1	1	1
		0.99	0.985	0.995	0.995	1.005
		0.9	0.85	0.95	0.95	1.05
		1.1	1.321	1.032	1.067	0.779
		1.01	1.032	1.003	1.007	0.978
30°	50°	1	1	1	1	1
		0.99	0.968	0.997	0.993	1.022
		0.9	0.679	0.967	0.933	1.221

Table 2 summarizes the pole directions, and the controllability/observability results for the case with parallel systems. Except for $\alpha, \beta = 0°$ and $\alpha, \beta = 90°$ there is no warning given in this table that stabilization using one input and one output may be difficult due to the presence of nearby RHP-zeros.

This example demonstrates one limitation on the use of pole directions for input/output selection. The reason to this is that the information about the zeros is not taken into account. The example can also be viewed as a counter example on the usefulness of the controllability and observability measures defined in Tarokh (1992), which also fails to signal the problems with SISO "controllability" for α, β different from $0°$ and $90°$.

6. CASE STUDIES

In the first example we consider a CSTR with two unstable modes.

EXAMPLE 5. Consider a CSTR with the chemical reaction; $A(l) \rightarrow B(l)$. We will assume that $\rho_A = \rho_B = \rho =$ const.,

Table 2. Controllability, observability and pole directions.

Observability			
β	Y_p	y_1 [a]	y_2 [b]
0°	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	No	No
30°	$\begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$	Yes	Yes
45°	$\begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$	Yes	Yes
60°	$\begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$	Yes	Yes
90°	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	No	No

[a] Observable with y_1 only.
[b] Observable with y_2 only.

Controllability			
α	U_p	u_1 [c]	u_2 [d]
0°	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	No	No
30°	$\begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$	Yes	Yes
45°	$\begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$	Yes	Yes
60°	$\begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}$	Yes	Yes
90°	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	No	No

[c] Controllable with u_1 only.
[d] Controllable with u_2 only.

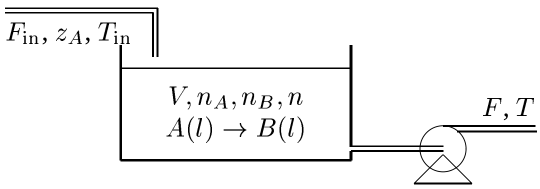


Figure 5. CSTR - liquid phase.

$C_{p,A} = C_{p,B} = C_p =$ const., the flow out of the tank is independent of liquid height in the tank (the material is pumped out), and we will assume a zero order reaction rate; $r = k(T) = k_0\, e^{-\frac{E}{R}(\frac{1}{T} - \frac{1}{T_{\text{ref}}})}$. The setup is shown in Figure 5. The material balances are

$$\dot{n} = F_{\text{in}} - F \tag{18}$$
$$\dot{n}_A = F_{\text{in}} z_A - F\frac{n_A}{n} - k(T)n \tag{19}$$

where

n	[kmol]	total mass/mole in the CSTR,
n_A	[kmol]	mass/mole of component A,
F_{in}	[kmol/min]	flow into the CSTR,
F	[kmol/min]	flow out of the CSTR,
z_A	[-]	mole fraction of component A in F_{in},
$k(T)$	[-]	mole of component A reacted divided by total moles in reactor.

The energy balance becomes

$$\overbrace{\dot{U} + p\dot{V}}^{\dot{H}} = \dot{n}C_p\,(T - T_{\text{ref}}) + nC_p\dot{T}$$
$$= F_{\text{in}}C_p\,(T_{\text{in}} - T_{\text{ref}}) - FC_p\,(T - T_{\text{ref}}) +$$
$$k(T)n\,(\text{-}\Delta H_{\text{rx}})$$

Rearranging yields

$$\dot{T} = \frac{F_{\text{in}}}{n}\,(T_{\text{in}} - T) + \frac{k(T)}{C_p}\,(\text{-}\Delta H_{\text{rx}}) \qquad (20)$$

where the additional symbols are

T	[°K]	temperature in the CSTR,
T_{in}	[°K]	temperature of F_{in},
C_p	[kJ/kmol °K]	heat capacity of the mixture in the CSTR and
-ΔH_{rx}	[kJ/kmol]	heat of reaction.

At steady state we have from $\dot{n} = 0$, $F = F_{\text{in}}$, from $\dot{T} = 0$, $F_{\text{in}}C_p\,(T_{\text{in}} - T^*) + k_0 n^*(\text{-}\Delta H_{\text{rx}}) = 0$ and from $\dot{n}_A = 0$ $n_A^* = z_A n^* - \frac{k(T^*)}{F_{\text{in}}} n^{*2}$.

Linearizing the model around the operating point yields

$$\dot{x} = Ax + Bu + B_d d$$
$$y = Cx$$

where $x = \begin{bmatrix} \Delta n \\ \Delta n_A \\ \Delta T \end{bmatrix}$, $u = \begin{bmatrix} \Delta F \\ \Delta T_{\text{in}} \end{bmatrix}$, $d = \begin{bmatrix} \Delta F_{\text{in}} \\ \Delta z_A \end{bmatrix}$, $y = \begin{bmatrix} \Delta n \\ \Delta T \end{bmatrix}$,

$$A = \begin{bmatrix} 0 & 0 & 0 \\ F\frac{n_A^*}{n^{*2}} - k_0 & -\frac{F}{n^*} & -\frac{E k_0}{RT^{*2}} \\ \frac{F_{\text{in}}}{n^{*2}}(T_{\text{in}} - T^*) & 0 & -\frac{F_{\text{in}}}{n^*} + \frac{E k_0}{RT^{*2}}\frac{\text{-}\Delta H_{\text{rx}}}{C_p} \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & 0 \\ -\frac{n_A^*}{n^*} & 0 \\ 0 & \frac{F_{\text{in}}}{n^*} \end{bmatrix}, \quad B_d = \begin{bmatrix} 1 & 0 \\ z_A & F_{\text{in}} \\ \frac{T_{\text{in}} - T^*}{n^*} & 0 \end{bmatrix}$$

and

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The eigenvalues of A are

$$\lambda_1 = -\frac{F}{n^*}, \quad \lambda_2 = 0 \quad \text{and}$$
$$\lambda_3 = -\frac{F_{\text{in}}}{n^*} + \frac{Ek_0}{RT^{*2}}\frac{\text{-}\Delta H_{\text{rx}}}{C_p}$$

The operating point is specified in Table 3, and the physical process constants are given in Table 4. Note that the

Table 3. CSTR operating point.

Variable	Value	Unit	Variable	Value	Unit
n^*	1	[kmol]	T_{in}	300	[°K]
n_A^*	0.2	[kmol]	z_A	1	[-]
T^*	370	[°K]	F_{in}	1	[kmol/min]

state $x_2 = \Delta n_A$ is not observable. The full linear state-space model becomes

$$G_F(s) \overset{s}{=} \left[\begin{array}{c|cc} A & B & B_d \\ \hline C & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc|cccc} 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ -0.6 & -1 & -0.0514 & -0.2 & 0 & 1 & 1 \\ 70 & 0 & 3.5 & 0 & 1 & -70 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right]$$

Table 4. Physical constants for CSTR.

Variable	Value	Unit	Variable	Value	Unit
k_0	0.8	[min^{-1}]	E/R	8807	[°K]
C_p	40	[kJ/kmol °K]	-ΔH_{rx}	3500	[kJ/kmol]
$T_{\text{ref}} = T^*$	370	[°K]			

A minimal realization is given below, we have also scaled the inputs using; $\Delta F = 1$ (100% variation), $\Delta T_{\text{in}} = 20$ °K, the outputs using; $\Delta n = 0.05$ (5% variation), $\Delta T = 1$ °K, and the disturbances using; $\Delta F_{\text{in}} = 0.5$ and $\Delta z_A = 1$.

$$G(s) \overset{s}{=} \left[\begin{array}{cc|cc} 0 & 0 & -1 & 0 \\ 70 & 3.5 & 0 & 20 \\ \hline 20 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \quad \text{and} \quad G_d(s) \overset{s}{=} \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 70 & 3.5 & -35 & 0 \\ \hline 20 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

In the minimal realization, state $x_2 = \Delta n_A$ is removed and the new x_2 is the temperature $x_2 = \Delta T$. The pole directions corresponding to the poles $P = [\,0 \quad 3.5\,]$ are

$$U_p = \begin{bmatrix} 1 & -0.9988 \\ 0 & 0.9988 \end{bmatrix} \quad \text{and} \quad Y_p = \begin{bmatrix} -0.9988 & 0 \\ 0.9988 & 1 \end{bmatrix}$$

From the pole directions we consider to control the temperature (y_2) using the flow F (u_1). By using a LQG controller based on $g_{21}(s)$, it is verified that both modes can be stabilized with this control link. However, we did not manage to stabilize both modes using one PI-controller.

In the next example we consider the Tennessee Eastman problem, where we use the pole directions to find a stabilizing control structure.

EXAMPLE 6. TENNESSEE EASTMAN PROBLEM. The Tennessee Eastman problem is shown in Figure 6. For details about the Tennessee Eastman problem refer to (Downs and Vogel, 1993). The figure contains both measurements y_i and manipulated variables u_j. Also given in the figure are candidate outputs (y_i) for stabilizing control. A separate numbering scheme is given for those outputs. Table 5 summarizes the selected candidate outputs for stabilizing control and the corresponding variable number in the full model (referred to as PID No.). Also given in the table is the scaling of the outputs used in this analysis. The manipulated variables are summarized in Table 6, also given in the table is the suggested scaling of the inputs used in this analysis. The linearized model in the in the base case (mode 1, 50/50 G/H mass ratio) is used in this example.

The model has six unstable poles in the operating point considered

$$P_u = [\,0 \quad 0.001 \quad 0.023 \pm 0.156i \quad 3.066 \pm 5.079i\,]$$

The pole output directions are

Figure 6. Tennessee Eastman test problem

Table 5. Candidate outputs for stabilizing control of the Tennessee Eastman problem.

Variable name	No. [a]	PID No. [b]	Scaling
Reactor pressure	y_1	y_7	54.1 [kPa]
Reactor level	y_2	y_8	1.5 %
Reactor temperature	y_3	y_9	1.2 [°C]
Separator temperature	y_4	y_{11}	1.0 [°C]
Separator level	y_5	y_{12}	1.0 %
Separator pressure	y_6	y_{13}	52.6 [kPa]
Stripper level	y_7	y_{15}	1.0 %
Stripper pressure	y_8	y_{16}	62.0 [kPa]
Stripper temperature	y_9	y_{18}	1.0 [°C]
Reactor cooling water outlet temperature	y_{10}	y_{21}	0.2 [°C]
Separator coolingwater outlet temperature	y_{11}	y_{22}	0.2 [°C]

[a] Variable number in the smaller model used in the analysis.
[b] Variable number in the full model provided by Downs and Vogel.

$$|Y_p| = \begin{bmatrix} 0.000 & 0.001 & 0.041 & 0.112 \\ 0.000 & 0.004 & 0.169 & 0.065 \\ 0.000 & 0.000 & 0.013 & 0.366 \\ 0.000 & 0.001 & 0.051 & 0.410 \\ 0.009 & 0.580 & 0.488 & 0.315 \\ 0.000 & 0.001 & 0.041 & 0.115 \\ 1.605 & 1.192 & 0.754 & 0.131 \\ 0.000 & 0.001 & 0.039 & 0.107 \\ 0.000 & 0.001 & 0.038 & 0.217 \\ 0.000 & 0.001 & 0.055 & 1.485 \\ 0.000 & 0.002 & 0.132 & 0.272 \end{bmatrix} \begin{matrix} \\ \\ \\ \\ \\ \\ \leftarrow y_{15} \\ \\ \\ \leftarrow y_{21} \\ \\ \end{matrix}$$

We have taken the absolute value to avoid complex conjugate directions. The first column corresponds to the pole $p_1 = 0$, the second column corresponds to the pole $p_2 = 0.001$, the third column corresponds to the complex conjugate pair $p_{3,4} = 0.023 \pm 0.156i$ and the fourth column corresponds to the complex conjugate pair $p_{5,6} = 3.066 \pm 5.079i$. We see

Table 6. Manipulated variables in the Tennessee Eastman problem.

Variable name	No. [a]	Stream no.	Scaling
D feed flow	u_1	2	10%
E feed flow	u_2	3	10%
A feed flow	u_3	3	10%
A and C feed flow	u_4	4	10%
Compressor recycle valve	u_5		10%
Purge valve	u_6	9	10%
Separator pot liquid flow	u_7	10	10%
Stripper liquid product flow	u_8	11	10%
Stripper steam valve	u_9	Stm	10%
Reactor cooling water flow	u_{10}	CWS	10%
Condenser cooling water flow	u_{11}	CWS	10%
Agitator speed	u_{12}		10%

[a] Variable number in both the full model and the model used in the analysis.

that output y_{15} in the full model (row 7) has the largest component in the output direction of pole $p_1 = 0$, and non of the other outputs has significant components in this direction. In a similar way output y_{21} (row 10) has a large component in the pole output direction corresponding to the complex conjugate pair $p_{5,6} = 3.066 \pm 5.079i$. The input pole directions are

$$|U_p| = \begin{bmatrix} 6.815 & 6.909 & 2.573 & 0.964 \\ 6.906 & 7.197 & 2.636 & 0.246 \\ 0.148 & 1.485 & 0.768 & 0.044 \\ 3.973 & 11.550 & 5.096 & 0.470 \\ 0.012 & 0.369 & 0.519 & 0.356 \\ 0.597 & 0.077 & 0.066 & 0.033 \\ 0.132 & 1.850 & 1.682 & 0.110 \\ 22.006 & 0.049 & 0.000 & 0.000 \\ 0.007 & 0.054 & 0.009 & 0.013 \\ 0.247 & 0.708 & 1.501 & 2.020 \\ 0.109 & 0.976 & 1.446 & 0.753 \\ 0.033 & 0.094 & 0.201 & 0.302 \end{bmatrix} \begin{matrix} \\ \\ \\ \\ \\ \\ \\ \leftarrow u_8 \\ \\ \leftarrow u_{10} \\ \\ \\ \end{matrix}$$

Table 7. Tunings of PI-controllers

Loop	k_p	T_i
$y_{15} \leftrightarrow u_8$	-0.1 [1/°C]	1 [min]
$y_{21} \leftrightarrow u_{10}$	-0.05 [m^3/h]	300 [min]
$y_{12} \leftrightarrow u_7$	-0.0025 [m^3/h]	200 [min]

By considering both input and output pole directions at the same time we arrive at the suggested pairings; $y_{15} \leftrightarrow u_8$ and $y_{21} \leftrightarrow u_{10}$ which corresponds to controlling the stripper level using the stripper liquid product flow and controlling reactor cooling water outlet temperature using the reactor cooling water flow. It can also be seen from the pole directions that these two loop will interact very little since the common elements in the two directions are almost zero. It is worth noting that both of these loops was also included by McAvoy and Ye (1994) in their study.

Using two PI-controllers with tunings given in Table 7, we manage to stabilize all the unstable modes except the mode $p_2 = 0.001$. By recomputing the pole directions with the controllers included we get

$$Y_p = \begin{bmatrix} -0.001 \\ -0.005 \\ 0.000 \\ 0.001 \\ -0.867 \\ -0.001 \\ 0.000 \\ -0.001 \\ 0.001 \\ 0.000 \\ -0.002 \end{bmatrix} \begin{matrix} \\ \\ \\ \\ \leftarrow y_{12} \\ \\ \\ \\ \\ \\ \\ \end{matrix} \quad U_p = \begin{bmatrix} -7.363 \\ -7.536 \\ 1.410 \\ 11.515 \\ -0.346 \\ -0.065 \\ 2.465 \\ 0.000 \\ -0.062 \\ 0.008 \\ 0.901 \\ -0.078 \end{bmatrix} \begin{matrix} \\ \\ \\ \\ \\ \\ \leftarrow u_7 \\ \\ \\ \\ \\ \\ \end{matrix}$$

We see that the output pole direction has a large element in y_{12} and only small elements in the other outputs. From the input direction we see that input u_4 is the best choice, however, this is a feed stream. We would like to avoid (if possible) to use the feed streams to stabilizing control and rather use these to set the production rate. The feed streams are all gas, so it makes sens from a physical point of view that these manipulated variables have large effect. Also u_1 and u_2 are feed streams so this leaves us with input u_7 which is the separator liquid flow. The pairing; $y_{12} \leftrightarrow u_7$ then corresponds to controlling the separator level using the separator liquid flow. The controller parameters are given in Table 7. After closing this loop the plant is stabilized. Figure 7 shows the Tennessee Eastman plant with the controllers included. The mode closest to the imaginary axis is about 0.07 which corresponds to a time constant about 14 hours, which from the operators point of view may seems like a unstable mode drifting away. The procedure can also be repeated on the stable modes which one wants to affect. The intention with this example was to demonstrate a systematic approach to the problem of control structure design and not to design a complete control structure for the Tennessee Eastman problem. Also note that no effort has been put into tuning of the controllers by us, we have used the tunings given in (McAvoy and Ye, 1994).

7. SUMMARY

- Input/output pole directions is defined, and it is shown how to compute these directions using Jordan forms.
- The input/output pole directions is related to the minimum input energy needed to stabilize one unstable pole using a single loop controller.
- When selecting the manipulated variable and the output to be controlled pole directions provide useful information. However, one also need to consider if the selected transfer function has zeros in the region nearby the pole.

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Appendix A. EIGENVALUE PROBLEMS AND NORMAL JORDAN FORM

A.1 Left and right eigenvalue problems

Assume in the following that the dimensions of A is $n \times n$. The standard eigenvalue problem (referred to in this context as the right eigenvalue prob-

Figure 7. Tennessee Eastman plant with control loops included

lem) is to find the eigenvalue λ and the eigenvector (referred to as the right eigenvector) x_R which satisfy

$$Ax_R = \lambda x_R \tag{A.1}$$

In a similar way, the left eigenvector problem is to find eigenvalue λ and the left eigenvector x_L which satisfy

$$x_L^H A = x_L^H \lambda \tag{A.2}$$

It is well known that any scalar $s \in \mathbb{C}$ times an eigenvector also is an eigenvector and that eigenvectors $x_1, \ldots, x_k$ corresponding to *different eigenvalues* $\lambda_1, \ldots, \lambda_k$, then those eigenvectors are linearly independent. These properties are of course valid for both types of eigenvalue problems.

The following property relates the the right eigenvector $x_{R,1}$, corresponding to an eigenvalue λ_1, to the left eigenvector $x_{L,2}$, corresponding to an eigenvalue λ_2, when $\lambda_1 \neq \lambda_2$.

PROPERTY 1. *Left and right eigenvectors which corresponds to different eigenvalues, are orthogonal to one another.*

Proof. Let $x_{R,1}$ be a right eigenvector corresponding to the eigenvalue λ_1 and $x_{L,2}$ be a left eigenvector corresponding to the eigenvalue λ_2 we then have

$$Ax_{R,1} = \lambda_1 x_{R,1} \quad \text{and} \quad x_{L,2}^H A = \lambda_2 x_{L,2}^H$$

Multiplying the latter on the right by $x_{R,1}$ gives

$$\lambda_2 x_{L,2}^H x_{R,1} = x_{L,2}^H A x_{R,1} = x_{L,2}^H \lambda_1 x_{R,1}$$

which implies $x_{L,2}^H x_{R,1} = 0$ since $\lambda_2 \neq \lambda_1$. □

The eigenvalue problems (A.1) and (A.2) can be written on matrix form by arranging $x_{R,i}$ and $x_{L,i}$ so that they both correspond to eigenvalue λ_i. We then form the matrices

$$X_R = [\,x_{R,1} \quad x_{R,2} \quad \cdots \quad x_{R,n}\,]$$

$$X_L = [\,x_{L,1} \quad x_{L,2} \quad \cdots \quad x_{L,n}\,]$$

and the diagonal matrix $\Lambda = \mathrm{diag}\{\lambda_i\}$. Then we have the following relationships

$$AX_R = X_R \Lambda \quad \text{and} \quad X_L^H A = \Lambda X_L^H \tag{A.3}$$

The right eigenvalue problem of the transpose of A (A^T) becomes

$$A^T X = X\Lambda \tag{A.4}$$

Taking the transpose of (A.4) gives

$$X^T A = \Lambda X^T \tag{A.5}$$

Comparing (A.5) and the last equation in (A.3) gives that $X_L = \bar{X}$, that is, the left eigenvectors are equal to the conjugate of the right eigenvectors to A^T. In MATLAB the left eigenvectors can therefore be computed as the conjugate of the right eigenvectors of A^T.

Scaling. Since any scalar times an eigenvector is an eigenvector, the eigenvectors can be scaled independently. It is usual to scale the eigenvectors so that the norm of the vectors are equal to one. Note that the eigenvectors are still not unique they can be multiplied with a complex number with magnitude one and arbitrary phase. In this work we assume that both left and right eigenvalues are scaled so that their norm are one. Consider next a pair of left and right eigenvectors $(x_{L,i}, x_{R,i})$ corresponding to the eigenvalue λ_i, define the scalar $s_i = x_{L,i}^H x_{R,i}$ and the diagonal matrix $S = \mathrm{diag}\{s_i\}$. In the case of n linearly independent eigenvectors we can write the diagonalization of A in terms of X_R, X_L and S, see (A.9).

n linearly independent eigenvectors. It is well known that a matrix A with n linearly independent eigenvectors $x_{R,i}$ can be diagonalized by the matrix X_R

$$X_R^{-1} A X_R = \Lambda \quad \text{or} \quad A = X_R \Lambda X_R^{-1} \tag{A.6}$$

In a similar way A can be diagonalized by X_L if A has n linearly independent left eigenvectors.

$$X_L^H A X_L^{-H} = \Lambda \quad \text{or} \quad A = X_L^{-H} \Lambda X_L^H \tag{A.7}$$

From (A.6) and (A.7) we have

$$X'_L = X_R^{-H} \tag{A.8}$$

when A has n linearly independent right eigenvectors. When computing the left eigenvectors according to (A.8) it follows that the left eigenvectors are scaled so that $x_{R,i}^H x_{L,i} = 1$. To see this multiply X'_L on the left with X_R^H to obtain $X_R^H X'_L = X_R^H X_R^{-H} = I$. It is therefore necessary to normalize the left eigenvectors, $X_L = X'_L S$, where S contains the inverse of the norm of the columns in X'_L on the diagonal. Multiplying X_L on the left by X_R^H reveals

$$X_R^H X_L = X_R^H X'_L S = X_R^H X_R^{-H} S = S$$

which is desired. Multiplying (A.3) on the left by X_L^H, using (A.8) and $X_L = X'_L S$ we obtain

$$\boxed{X_L^H A X_R = S^H \Lambda = \Lambda S^H} \tag{A.9}$$

The last identity follows since two diagonal matrices commute. In a similar way we can multiply (A.3) on the right with $X_R^{-1} = S^{-H} X_L^H$ to obtain

$$\boxed{A = X_R \Lambda S^{-H} X_L^H = X_R S^{-H} \Lambda X_L^H} \tag{A.10}$$

n distinct eigenvalues. In the case of n distinct eigenvalues there exists n linearly independent eigenvectors and we have

$$X_L^H X_R = X_R^H X_L = S = \mathrm{diag}\{x_{L,i}^H x_{R,i}\}$$

A.2 Normal Jordan Form

It is not our intention to show how to compute the Normal Jordan Form or to derive it. However, the intention is to show how we can use the Normal Jordan Form to obtain both left and right generalized vectors which can be used to describe the directionality of those poles or eigenvalues which do not have sufficient number of linearly independent eigenvectors.

A defective matrix A is a matrix which does not possess n linearly independent eigenvectors and can not be diagonalized. Those matrices which cannot be diagonalized can be brought into Normal Jordan Form.

If a matrix has s linearly independent eigenvectors, then it is similar to a matrix which is in Normal Jordan Form with s square blocks on the diagonal:

$$J = M^{-1} A M = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_s \end{bmatrix}$$

Each block has one eigenvector, one eigenvalue, and 1's just above the diagonal:

$$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \lambda_i & 1 \\ & & & \lambda_i \end{bmatrix}$$

It follows that $AM = MJ$, the single eigenvector for each block satisfy $Aw_i = \lambda_i w_i$ and for each block J_i of size greater than one there exists $\mathrm{size}\{J_i\} - 1$ additional vectors w_i which satisfy $Aw_i = \lambda_i w_i + w_{i-1}$. These additional vectors are called *generalized vectors*. The eigenvectors together with the generalized vectors form the matrix M which has rank equal to n. Since M has rank equal to n, M^{-1} exists and $J = M^{-1} A M$.

We will denote M for M_R since it multiplies A on the right in $AM_R = M_R J$. Following the same arguments for constructions of M_R and J (see Strang, 1986) it follows that there also exists a M_L such that

$$M_L^H A = J M_L^H \tag{A.11}$$

Since both M_R and M_L are nonsingular we can write

$$J = M_R^{-1} A M_R \quad \text{or} \quad A = M_R J M_R^{-1} \tag{A.12}$$

$$J = M_L^H A M_L^{-H} \quad \text{or} \quad A = M_L^{-H} J M_L^H \tag{A.13}$$

From (A.12) and (A.13) it follows that

$$M'_L = M_R^{-H} \tag{A.14}$$

REMARK 1. We can split up both $AM_R = M_R J$ and $M_L^H A = J M_L^H$. To do this let $M_{R,i}$ be the columns in M_R corresponding to J_i and let similarly $M_{L,i}$ be the columns in M_L corresponding to J_i. From the right Jordan form we get

$$A \left[M_{R,1} \;\; \cdots \;\; M_{R,s} \right] = \left[M_{R,1} \;\; \cdots \;\; M_{R,s} \right] \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_s \end{bmatrix}$$

which gives $AM_{R,i} = M_{R,i} J_i$, and from (A.11) we get

$$\begin{bmatrix} M_{L,1}^H \\ \vdots \\ M_{L,s}^H \end{bmatrix} A = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_s \end{bmatrix} \begin{bmatrix} M_{L,1}^H \\ \vdots \\ M_{L,s}^H \end{bmatrix}$$

which gives $M_{L,i}^H = J_i M_{L,i}^H$.

REMARK 2. From $AM_R = M_R J$ and for Jordan block number i, $AM_{R,i} = M_{R,i} J_i$ it follows that the first column $m_{R,i}$ in $M_{R,i}$ is the eigenvector and the remaining columns in $M_{R,i}$ are the generalized vectors. Let $m_{R,i}$ denote column i in M_R and assume that the size of the Jordan block is three, we get

$$A \left[m_{R,i} \;\; m_{R,i+1} \;\; m_{R,i+2} \right] = \left[m_{R,i} \;\; m_{R,i+1} \;\; m_{R,i+2} \right] \begin{bmatrix} \lambda_i & 1 & 0 \\ 0 & \lambda_i & 1 \\ 0 & 0 & \lambda_i \end{bmatrix}$$

or $Am_{R,i} = \lambda_i m_{R,i}$, $Am_{R,i+1} = \lambda_i m_{R,i+1} + m_{R,i}$ and $Am_{R,i+2} = \lambda_i m_{R,i+2} + m_{R,i+1}$.

REMARK 3. For the left Jordan problem the ordering of the vectors is opposite to that of the right Jordan problem. That is, the last column in $M_{L,i}$ is the eigenvector and the remaining columns are the generalized vectors. We have $M_L^H A = J M_L^H$ and for Jordan block number i, $M_{L,i}^H A = J M_{L,i}^H$. Let $m_{L,i}$ denote column i in M_L and assume that the size of of the Jordan block is three, we get

$$\begin{bmatrix} m_{L,i}^H \\ m_{L,i+1}^H \\ m_{R,i+2}^H \end{bmatrix} A = \begin{bmatrix} \lambda_i & 1 & 0 \\ 0 & \lambda_i & 1 \\ 0 & 0 & \lambda_i \end{bmatrix} \begin{bmatrix} m_{L,i}^H \\ m_{L,i+1}^H \\ m_{L,i+2}^H \end{bmatrix}$$

or $m_{L,i}^H A = \lambda_i m_{L,i}^H + m_{L,i+1}^H$, $m_{L,i+1}^H A = \lambda_i m_{L,i+1}^H + m_{L,i+2}^H$ and $m_{L,i+2}^H A = \lambda_i m_{L,i+2}^H$

Scaling. We know that each eigenvector can be scaled independently, however, the generalized vectors described by $Aw_i = \lambda_i w_i + w_{i-1}$ must be scaled with the same scalar as the eigenvector which starts the string. As an example, suppose that we have found an eigenvector w_1 and two generalized vectors w_2 and w_3 so that $Aw_1 = \lambda w_1$, $Aw_2 = \lambda w_2 + w_1$ and

$Aw_3 = \lambda w_3 + w_2$ are all satisfied. Next assume that we scale the eigenvector w_1 to become $w_1' = sw_1$ where $s \in \mathbb{C}$. In order to satisfy $Aw_2 = \lambda w_2 + w_1'$ we must scale w_2 with the same scalar s to obtain $w_2' = sw_2$ and $Aw_2' = \lambda w_2' + w_1'$ which again imply that we must scale w_3 with s. So, for each Jordan block we have one degree of freedom for scaling. The structure of the scaling matrix is then for a matrix with s linearly independent eigenvectors

$$S = \begin{bmatrix} s_1 I_1 & & \\ & \ddots & \\ & & s_s I_s \end{bmatrix} \tag{A.15}$$

where $s_i \in \mathbb{C}$ and the sizes of the identity matrices $I_1, \dots, I_s$ are equal to the sizes of the corresponding Jordan blocks $J_1, \dots, J_s$. It follows from (A.14) that selecting $M_L = M_L' S$,

$$M_R^H M_L = M_R^H M_L' S = M_R^H M_R^{-H} = S \tag{A.16}$$

Usually we select the scalings $s_1, \dots, s_s$ to be real and equal to the inverse of the norms of the columns in M_L' corresponding to the left eigenvectors. This implies that all $s_1, \dots, s_s$ are real, and by taking the complex conjugate transpose of (A.16) we obtain

$$M_L^H M_R = S^H = S = M_R^H M_L \tag{A.17}$$

Multiplying (A.11) on the right with M_R gives

$$M_L^H A M_R = J M_L^H M_R = J S^H \tag{A.18}$$

which is valid for all diagonal scaling matrices S with the predescribed structure given in (A.15). In a similar way we can multiply $AM_R = M_R J$ on the left by M_L^H to get

$$\boxed{M_L^H A M_R = S^H J = J S^H} \tag{A.19}$$

So, S and J commute, which can also be seen from

$$\begin{aligned} JS &= \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_s \end{bmatrix} \begin{bmatrix} s_1 I_1 & & \\ & \ddots & \\ & & s_s I_s \end{bmatrix} \\ &= \begin{bmatrix} J_1 s_1 I_1 & & \\ & \ddots & \\ & & J_s s_s I_s \end{bmatrix} = \begin{bmatrix} s_1 I_1 J_1 & & \\ & \ddots & \\ & & s_s I_s J_s \end{bmatrix} \\ &= \begin{bmatrix} s_1 I_1 & & \\ & \ddots & \\ & & s_s I_s \end{bmatrix} \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_s \end{bmatrix} = SJ \end{aligned}$$

When the scalers $s_1, \dots, s_s$ are all real $S^H = S$.

Multiplying $AM_R = M_R J$ on the left with $M_R^{-1} = M_L'^H = S^{-H} M_L^H$ gives

$$\boxed{A = M_R J S^{-H} M_L^H = M_R S^{-H} J M_L^H} \tag{A.20}$$

A.3 Inverse of J

The inverse of J consisting of s square blocks along the main diagonal is the matrix

$$J^{-1} = \begin{bmatrix} J_1^{-1} & & \\ & \ddots & \\ & & J_s^{-1} \end{bmatrix}$$

and the inverse of a Jordan block of size v

$$J_i = \left.\begin{bmatrix} p & 1 & \cdots & 0 & 0 \\ 0 & p & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & p & 1 \\ 0 & 0 & \cdots & 0 & p \end{bmatrix}\right\} v \tag{A.21}$$

is

$$J_i^{-1} = \begin{bmatrix} 1/p & -1/p^2 & \cdots & (-1)^{v-2}/p^{v-1} & (-1)^{v-1}/p^v \\ 0 & 1/p & \cdots & (-1)^{v-3}/p^{v-2} & (-1)^{v-2}/p^{v-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1/p & -1/p^2 \\ 0 & 0 & \cdots & 0 & 1/p \end{bmatrix} \tag{A.22}$$

Appendix B. PROOFS OF THE RESULTS

B.1 Proof of Lemma 1

From (A.19), $A = M_L^{-H} S^H J M_R^{-1}$, then we have

$$\begin{aligned} (sI - A) &= (sI - M_L^{-H} S^H J M_R^{-1}) \\ &= (M_L^{-H} S^H M_R^{-1} s - M_L^{-H} S^H J M_R^{-1}) \\ &= M_L^{-H} S^H (sI - J) M_R^{-1} \end{aligned} \tag{B.1}$$

and $G(s)$ can be written

$$\begin{aligned} G(s) &= C(sI - A)^{-1} B + D \\ &= C M_R (sI - J)^{-1} S^{-H} M_L^H B + D \end{aligned} \tag{B.2}$$

as an alternative we could extract the scalings on the other side of $(sI - J)^{-1}$

$$G(s) = C M_R S^{-H} (sI - J)^{-1} M_L^H B + D \tag{B.3}$$

Consider $(sI - J)^{-1}$ for $s = p$

$$(sI - J)^{-1} = \begin{bmatrix} (sI_1 - J_1)^{-1} & & \\ & \ddots & \\ & & (sI_s - J_s)^{-1} \end{bmatrix}$$

for the Jordan blocks involving p

$$(sI_i - J_i)^{-1} = \begin{bmatrix} 1/(s-p) & -1/(s-p)^2 & \cdots & (-1)^{v-2}/(s-p)^{v-1} & (-1)^{v-1}/(s-p)^v \\ 0 & 1/(s-p) & \cdots & (-1)^{v-3}/(s-p)^{v-2} & (-1)^{v-2}/(s-p)^{v-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1/(s-p) & -1/(s-p)^2 \\ 0 & 0 & \cdots & 0 & 1/(s-p) \end{bmatrix} \tag{B.4}$$

When inserting $s = p$ we see that the upper triangular part of $(sI_i - J_i)^{-1}$ becomes ∞. We partition M_R and M_L into blocks so that the columns in M_R and M_L corresponding to Jordan block number i, are collected in $M_{R,i}$ and $M_{L,i}$ then (B.2) can be rewritten

$$G(s) = C\,[\,M_{R,1} \quad \cdots \quad M_{R,s}\,] \cdot \begin{bmatrix} (sI_1 - J_1)^{-1} & & \\ & \ddots & \\ & & (sI_s - J_s)^{-1} \end{bmatrix} \cdot \begin{bmatrix} \bar{s}_1 I_1 & & \\ & \ddots & \\ & & \bar{s}_s I_s \end{bmatrix} \begin{bmatrix} M_{L,1}^H \\ \vdots \\ M_{L,s}^H \end{bmatrix} B + D$$

Assume that block number one is the block involving p, inserting $s = p$ in $G(s)$ gives

$$G(s = p) = C\,[\,M_{R,1} \quad \cdots \quad M_{R,s}\,] \cdot \begin{bmatrix} \infty \cdot T_1 \bar{s}_1 & & \\ & \ddots & \\ & & \mathrm{O} \cdot T_s \bar{s}_s \end{bmatrix} \cdot \begin{bmatrix} M_{L,1}^H \\ \vdots \\ M_{L,s}^H \end{bmatrix} B + D$$

where T_i is used to signal an upper triangular matrix compatible in size with J_i, ∞ is used to signal infinity gain and O is used to signal finite gain. The directions associated with infinite gain at the output are those contained in $M_{R,1}$, or if Jordan block i is involved, then those directions are contained in $M_{R,i}$. The output directions becomes

$$Y_p = C M_{R,i}, \quad \forall i \text{ whose Jordan block involves } p.$$

By considering $G^H(s = p)$ we find that the corresponding input directions become

$$U_p = B^H M_{L,i}, \quad \forall i \text{ whose Jordan block involves } p.$$

□

B.2 Proof of Theorem 1

In order to prove Theorem 1 we need some more notation. The plant $G(s)$ of size $l \times m$ has m inputs and l outputs. We consider next a subsystem of G, $G_{\gamma,\beta}(s)$ where the multiple γ describes the outputs and the multiple β describes the inputs. As an example, if we consider the 2×2 subsystem of G, consisting of the outputs 2 and 4 and the inputs 1 and 3 we get $\gamma = (2, 4)$ and $\beta = (1, 3)$. Let further n_γ and n_β denote the number of elements in the multiples γ and β. We define the matrix N_γ of size $l \times n_\gamma$, where column j in N_γ corresponds to element j in γ, is equal to the unit vector e_i of size l, with zeros in all positions except for position i which contains 1 and where i is element j in γ. We define the matrix N_β in a similar way as a $m \times n_\beta$ matrix consisting of unit vectors in each column with zeros in all elements except for element i which contains 1, where i is equal to element j in β for column j. Then we have that

$$G_{\gamma,\beta}(s) = N_\gamma^T G N_\beta \stackrel{s}{=} \left[\begin{array}{c|c} A & B_\beta \\ \hline C_\gamma & D_{\gamma,\beta} \end{array}\right]$$

where $B_\beta = BN_\beta$, $C_\gamma = N_\gamma^T C$ and $D_{\gamma,\beta} = N_\gamma^T D N_\beta$. If for example $\gamma = (2, 4)$, $\beta = (1, 3)$ and G is of size 4×3 then

$$N_\gamma = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad N_\gamma = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$G_{\gamma,\beta}(s) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \\ g_{41} & g_{42} & g_{43} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} g_{21} & g_{23} \\ g_{41} & g_{43} \end{bmatrix}$$

LEMMA 2. *Consider a system G with with a repeated mode p (multiplicity q) and with v linearly independent eigenvectors. For all square subsystems described by $G_{\gamma,\beta}^r \stackrel{s}{=} \left[\begin{array}{c|c} A & B_\beta^r \\ \hline C_\gamma^r & D_{\gamma,\beta}^r \end{array}\right]$ of G with dimension $r \times r$, where $r < v$, the mode p is not completely observable or not completely controllable or both at the same time. Furthermore, in $G_{\gamma,\beta}$ we have at least $v - r$ pole/zero cancellations.*

REMARK. This result says that if a system G has mode p with multiplicity q and $v \leq q$ linearly independent eigenvectors corresponding to the mode p, then the controller needs to take into account v outputs and v inputs to affect the mode p. So, the controller with minimum number of inputs and outputs which can affect all the v modes p, is a $v \times v$ controller, see Theorem 1.

Proof of Lemma 2. The selected square subsystem of size r, has state-space realization $G_{\gamma,\beta}^r(s) \stackrel{s}{=} \left[\begin{array}{c|c} A & B_\beta^r \\ \hline C_\gamma^r & D_{\gamma,\beta}^r \end{array}\right]$, where the r selected inputs (corresponding to the columns in the overall B matrix) are specified by the r-multiple β, and the r selected outputs (corresponding to rows in the overall C matrix) are specified by the r-multiple γ. $G_{\gamma,\beta}^r(s)$ is then given by

$$\begin{aligned} G_{\gamma,\beta}^r(s) &= C_\gamma^r (sI - A)^{-1} B_\beta^r + D_{\gamma,\beta}^r \\ &= \frac{C_\gamma^r \mathrm{adj}(sI - A) B_\beta^r + D_{\gamma,\beta}^r \phi(s)}{\phi(s)} \end{aligned}$$

Since, $\phi(p) = 0$ it is sufficient to show that the system described by $G'^r_{\gamma,\beta}(s) \stackrel{s}{=} \left[\begin{array}{c|c} A & B_\beta^r \\ \hline C_\gamma^r & 0 \end{array}\right]$ has $v - r$ zeros for $s = p$. The zeros of $G'^r_{\gamma,\beta}(s)$ are the values of s where the matrix

$$\begin{bmatrix} sI - A & B_\beta^r \\ C_\gamma^r & 0 \end{bmatrix}$$

is singular, $\det(\begin{bmatrix} sI - A & B_\beta^r \\ C_\gamma^r & 0 \end{bmatrix}) = 0$. The zeros and the input/output zero directions are independent of the state-space realization so we define a new state vector $z = M_R^{-1} x$ and consider

$$\begin{bmatrix} M_R^{-1} & 0 \\ 0 & I_r \end{bmatrix} \begin{bmatrix} (sI - A) & B_\beta^r \\ C_\gamma^r & 0 \end{bmatrix} \begin{bmatrix} M_R & 0 \\ 0 & I_r \end{bmatrix} = \begin{bmatrix} sI - J & M_R^{-1} B_\gamma^r \\ C_\gamma^r M_R & 0 \end{bmatrix} = \begin{bmatrix} sI - J & S^{-H} U_{p,\beta}^H \\ Y_{p,\gamma} & 0 \end{bmatrix}$$

where $Y_{p,\gamma}$ contains the output directions for the selected outputs described by γ, and $U_{p,\beta}$ contains the input directions for the selected inputs described by β. When inserting $s = p$, v columns and v rows in $pI - J$ become equal to zero since J has v Jordan blocks with p on the diagonal, one Jordan block for each independent eigenvector. So, the rank of $pI - J$ is $n_x - v$. We have r columns in $U_{p,\beta}$ and r rows in $Y_{p,\gamma}$. If $r < v$ then the matrix $\begin{bmatrix} pI - J & S^{-H} U_{p,\beta}^H \\ Y_{p,\gamma} & 0 \end{bmatrix}$ becomes singular, the rank can at maximum become $n_x - v + 2r$ and we need rank equal to $n_x + r$, if the matrix shall be non-singular, which leaves us with a zero of multiplicity $v - r$. Since, the poles span the whole v dimensional space described by the v independent eigenvectors in the state-space, it is sufficient to show that the state input zero direction is within this v dimensional space to achieve pole/zero cancellation. Set $s = p$ and consider the solution to the set of equations

$$\begin{bmatrix} pI - A & B_\beta^r \\ C_\gamma^r & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}$$

The obvious solution is $x = x_R$ and $u = 0$, so the space described by the eigenvectors contains the state input zero direction. Note that for a repeated mode p with v linearly independent eigenvectors, any linear combinations of the independent eigenvectors also are an eigenvector. As an example consider $v = 3$; $x_{R,1}$, $x_{R,2}$ and $x_{R,3}$ are all linearly independent eigenvectors corresponding to p

$$A \cdot (ax_{R,1} + bx_{R,2} + cx_{R,3}) = p \cdot (ax_{R,1} + bx_{R,2} + cx_{R,3})$$

In a similar way we can show that the space described by the left eigenvectors contain the state output zero direction. So, pole/zero cancellation of at least order $v - r$ occurs. Note that pole/zero cancellation of higher order may occur if the minimal realization of $G^r_{\gamma,\beta}(s)$ does not contain r poles p. As an example, if a minimal realization of $G^r_{\gamma,\beta}(s)$ contains no poles for $s = p$, we have v pole/zero cancellations (with the possibility of even more pole/zero cancellations if $q > v$). □