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Representation of uncertain time delays in the H_∞ framework

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How to represent an uncertain time delay in a form suitable for robust control in terms of the H_∞-norm and the structured singular value (μ) is discussed. To use μ-synthesis the uncertain delay has to be approximated to yield a proper rational interconnection matrix $M(s)$ and a purely complex perturbation matrix $\Delta(s)$. The parametric average of the delay is usually included in the nominal model, while the uncertain perturbation covers variation around the average delay. It is proposed to model the nominal time delay as uncertainty, i.e. using a larger uncertainty set. This yields a delay-free nominal model, which simplifies the controller synthesis. For the cases studied the delay free nominal model does not yield a more conservative design than the average delay model, despite its larger uncertainty set.

Nomenclature

$c(s)$	controller
$g(s)$	nominal plant
$g_p(s)$	perturbed plant
$J(\omega)$	skewed μ
k	gain
k_c	controller gain
$l(\omega)$	irrational uncertainty weight
RP	Robust Performance
RS	Robust Stability
$S(s)$	$=(I+g(s)c(s))^{-1}$ nominal sensitivity function
$S_p(s)$	$=(I+g_p(s)c(s))^{-1}$ perturbed sensitivity function
$w(s)$	rational uncertainty weight
$w_P(s)$	performance weight
Δ	perturbation matrix
δ_θ	time delay uncertainty (min)
θ	nominal time delay (min)
μ	structured singular value
$\bar{\sigma}$	maximum singular value
τ	time constant (min)
ω	frequency (rad min^{-1})

1. Introduction

H_∞ control theory has gained a lot of attention and progress in the past decade. With the introduction of the structured singular value, μ (Doyle 1982)

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structured uncertainty can be handled in the H_∞ framework, and hence the criticized conservatism is substantially reduced. μ can deal not only with robust stability (RS) but also robust performance (RP). The standard '$M - \Delta$' structure is shown in Fig. 1, where M is a stable transfer matrix, comprising plant, controller as well as uncertainty and performance weights, and $\Delta = \text{diag}\{\Delta_i\}$ represents the uncertainty structure. In the case of robust performance, Δ also includes a performance block. μ is defined as

$$\mu_\Delta^{-1}(M) = \min_\Delta \{\bar{\sigma}(\Delta) | \det(I + M\Delta) = 0\} \tag{1}$$

unless no Δ makes $(I + M\Delta)$ singular, in which case $\mu_\Delta(M) = 0$. Robust stability or robust performance is equivalent to

$$\sup_\omega \mu_\Delta(M) < 1 \tag{2}$$

H_∞ and μ methods are now well developed, and cover both analysis and synthesis, although the μ synthesis problem is still not fully solved. (The present μ synthesis algorithm, called DK iteration, is a combination of H_∞ synthesis and optimal D-scaling. It only allows for complex uncertainty and does not guarantee global convergence.) However, more work on practical applications is needed. In particular, this applies to the selection of performance weights and uncertainty weights (Laughlin *et al.* 1986, 1987, Postlethwaite *et al.* 1990, Lundström *et al.* 1991 *a*, *b*).

In the H_∞ framework, we generally need to transform (approximate) the original uncertainty description to obtain the required linear fractional form, i.e. a form that fits the $M - \Delta$ structure of Fig. 1. The set of possible plants is modelled by two blocks: an interconnection matrix M which includes the nominal model and uncertainty weights, and a norm-bounded normalized uncertainty block Δ. If μ is used the uncertainty can be 'structured' in the sense that the perturbation Δ is block-diagonal. Parametric uncertainty can, in general, be directly rearranged into a linear fractional $(M - \Delta)$ structure, using real valued perturbations; however, this is not the case for time-delay uncertainty. Moreover, real perturbations cannot be used with the present μ synthesis algorithm (DK-iteration), which can be applied to complex perturbations only. Promising algorithms for μ analysis with mixed real/complex perturbations are being developed (Fan *et al.* 1991, Young *et al.* 1991). An important advantage of the H_∞/μ-framework is that the model uncertainty in non-parametric form, including unknown model order, may be handled. Thus, in this paper we also use the non-parametric framework when the original description for model

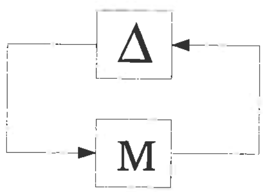

Figure 1. Standard '$M - \Delta$' structure.

uncertainty is in parametric form. In particular, we consider a set of linear time invariant plants with different time delays defined by

$$\text{plant set:} \qquad g_p(s) = \frac{k}{\tau s + 1} e^{-(\theta + \delta_\theta \Delta)s} = g(s)\, e^{-\delta_\theta \Delta s}, \quad -1 \leqslant \Delta \leqslant 1 \tag{3}$$

$$\text{nominal model:} \quad g(s) = \frac{k}{\tau s + 1} e^{-\theta s} \tag{4}$$

and study how to obtain a model suitable for μ synthesis.

We study (1) how to approximate the time delay uncertainty using simple analytical approximations resulting in a linear fractional form (§ 2); (2) how to choose a multiplicative uncertainty weight which covers the time delay uncertainty using a complex perturbation (§ 3); and (3) how to choose the nominal model (§ 4). In § 5, we make some final remarks.

2. Analytical approximations of a delay uncertainty

In this section we study how well the delay uncertainty is described by commonly used delay approximations. The different approximations are compared based on how well they predict the smallest delay required to destabilize a certain closed-loop system.

The delay uncertainty

$$e^{-\delta_\theta \Delta s}, \quad -1 \leqslant \Delta \leqslant 1 \tag{5}$$

has to be approximated into linear fractional uncertainty before μ methods can be applied to the uncertain system in (3). With linear fractional uncertainty we mean an uncertainty description which allows the perturbation matrix Δ to be 'pulled' out from the rest of the system such that the $M - \Delta$ form of Fig. 1 is obtained.

We consider the following five approximations of (5).

Zero—Power series expansion of numerator:

$$e^{-\delta_\theta \Delta s} \approx 1 - \delta_{\theta 1} s \Delta; \quad -1 \leqslant \Delta \leqslant 1 \tag{6}$$

Pole—Power series expansion of denominator:

$$e^{-\delta_\theta \Delta s} \approx \frac{1}{1 + \delta_{\theta 2} s \Delta}; \quad -1 \leqslant \Delta \leqslant 1 \tag{7}$$

Padé—Combination of the two approximations above:

$$e^{-\delta_\theta \Delta s} \approx \frac{1 - \dfrac{\delta_{\theta 3}}{2} s \Delta}{1 + \dfrac{\delta_{\theta 3}}{2} s \Delta} = 1 - \frac{\delta_{\theta 3} s \Delta}{1 + \dfrac{\delta_{\theta 3}}{2} s \Delta}, \quad -1 \leqslant \Delta \leqslant 1 \tag{8}$$

Complex0—Same as **Zero** but with Δ complex:

$$e^{-\delta_\theta \Delta s} \approx 1 - \delta_{\theta 4} s \Delta = 1 + w_0(s) \Delta, \quad |\Delta| \leqslant 1 \tag{9}$$

Complex1—Similar to **Padé** but with Δ complex:

$$e^{-\delta_\theta \Delta s} \approx 1 - \frac{\delta_{\theta 5} s}{1 + \frac{\delta_{\theta 5}}{2} s} \Delta = 1 + w_1(s)\Delta, \quad |\Delta| \leq 1 \tag{10}$$

The three first approximations are derived for real valued perturbations and correspond to the z-transform approximations presented by Åström and Wittenmark (1984, p. 176), who denote the approximations 'Euler', 'Backward' and 'Trapezoidal', respectively. **Complex0** is obtained by using a complex perturbation in **Zero**. **Pole** and **Padé** cannot be used with complex perturbations, since that would allow for infinite uncertainty for frequencies above $1/\delta_{\theta 2}$ and $2/\delta_{\theta 3}$, respectively. However, by setting the Δ in the denominator of **Padé** equal to 1, we may let the remaining Δ become complex and obtain approximation **Complex1**. This approximation is commonly used in robust design (Laughlin *et al*. 1987, Skogestad and Lundström 1990). Both **Complex0** and **Complex1** yield multiplicative uncertainties; we can think that weight $w_0(s)$ is obtained by setting Δ in the denominator of **Padé** equal to 0, so it is always larger that weight $w_1(s)$ in **Complex1**.

To study the accuracy of these approximations we shall consider the smallest delay δ_θ which destabilizes a closed loop system. We use the following PI controller

$$c(s) = \frac{k_c}{k} \frac{\tau s + 1}{s} \tag{11}$$

where k, τ are the same as in the nominal plant model (3). The exact value of the destabilizing delay is easy to find from the open loop transfer function

$$g_p(s)c(s) = \frac{k_c}{s} e^{-(\theta + \delta_\theta \Delta)s}, \quad -1 \leq \Delta \leq 1 \tag{12}$$

The magnitude of this transfer function equals 1 at frequency $\omega = k_c$ and the delay free part of the transfer function yields a phase equal to $-\pi/2$, so the smallest δ_θ required to destabilize the closed loop system is

$$\delta_\theta = \frac{\pi}{2k_c} - \theta \tag{13}$$

Using the approximations above we can arrange the overall system into the desired $M - \Delta$ structure. The condition for robust stability (RS) for a real-valued Δ is

$$|M(j\omega_p)| < 1, \quad \forall\{\omega_p | \mathrm{Im}\,(M(j\omega_p)) = 0\} \tag{14}$$

and for a complex valued Δ

$$\sup_\omega |M(j\omega)| < 1 \tag{15}$$

The exact δ_θ and its various estimated values from the five different approximations are shown in Table 1 for a constant nominal delay ($\theta = 1$) and different k_c values.

k_c	Exact, δ_θ	**Zero**, $\delta_{\theta 1}$	**Pole**, $\delta_{\theta 2}$	**Padé**, $\delta_{\theta 3}$	**Complex0**, $\delta_{\theta 4}$	**Complex1**, $\delta_{\theta 5}$
0·1	14·71	9·496	∞	18·09	9·143	17·89
0·2	6·854	4·491	∞	8·176	4·187	7·774
0·5	2·142	1·476	∞	2·373	1·257	1·646
1·0	0·571	0·441	∞	0·587	0·319	0·328
1·5	0·047	0·045	0·050	0·047	0·025	0·025

Table 1. Destabilizing delays estimated using different approximations for time delay uncertainty (nominal $\theta = 1$).

We see from Table 1:

(1) **Zero** is always conservative.

(2) **Pole** is always optimistic (overestimating the stability margin).

(3) **Padé** is the best approximation, in the sense that $\delta_{\theta 3}$ obtained from **Padé** is close to the exact δ_θ. However, **Padé** is always a bit optimistic, which shows that it does not include all plants in the set (3).

(4) **Complex0** is even more conservative than **Zero**, since Δ is relaxed to be complex.

(5) **Complex1** is optimistic when k_c is small (detuned), and conservative when k_c is large (overtuned). However, it is good in the range for k_c where robust stability is a more reasonable concern. Thus, **Complex1** may be a practically good approximation. Also note that $\delta_{\theta 5}$ is always larger than $\delta_{\theta 4}$ as expected.

Theoretically, we can get arbitrary high accuracy by using multiple real perturbations. For example, we may divide the delay uncertainty $e^{-\delta_\theta \Delta s}$ into n parts $(e^{-(\delta_\theta/n)\Delta s})^n$ and then use any approximation for each of the smaller uncertainties $e^{-(\delta_\theta/n)\Delta s}$. Another alternative (Lundström *et al.* 1993, Postlethwaite *et al.* 1991) is to include the nominal delay in the approximation

$$e^{-(\theta+\delta_\theta\Delta)s} \approx \left(\frac{1 - \dfrac{\theta}{2n}s - \dfrac{\delta_\theta \Delta}{2n}s}{1 + \dfrac{\theta}{2n}s + \dfrac{\delta_\theta \Delta}{2n}s} \right)^n \tag{16}$$

In both cases we get n repeated real perturbations, which makes the μ computation much harder than for a complex perturbation. The latter alternative (16) has the advantage of avoiding additional approximations for the nominal delay and yielding a minimal state-space representation.

The conclusions of this subsection are that **Padé** and **Complex1** approximations seem reasonably good. However, they cannot be used to guarantee robustness, since they may be 'optimistic', i.e. do not cover the original set. Another drawback with **Padé** is that it requires a real-valued Δ, which makes it unsuitable for DK-iteration. This motivates us to study refined versions of **Complex1**, which cover the original set and are suitable for μ synthesis. That is the topic of the next section.

3. Complex perturbations that cover the time delay

In robust design it is preferable to use a norm-bounded 'upper-bound' uncertainty set which covers all possible plants of the original set. One reason for this is that μ-optimal controllers seem to be sensitive to unconsidered uncertainty (as we will show later in this paper).

We approximate the set of delay uncertain plants (3) using the following model

$$g_p(s) = g(s)(1 + l(\omega)\Delta(s)), \quad |\Delta| \leqslant 1 \tag{17}$$

Here, the delay uncertainty is approximated by complex multiplicative uncertainty, i.e. Δ is a complex perturbation and $l(\omega)$ is a multiplicative uncertainty weight. The tightest bound $l(\omega)$ that covers a delay is (Owens and Raya 1982)

$$l(\omega) = \begin{cases} |e^{-j\delta_\theta \omega} - 1|, & \forall\ \omega < \pi/\delta_\theta \\ 2, & \forall\ \omega \geqslant \pi/\delta_\theta \end{cases} \tag{18}$$

This bound is irrational, it may be used for analysis but not for synthesis. For synthesis the first-order weight $w_1(s)$, derived for approximation **Complex 1** in the previous section is commonly used

$$w_1(s) = \frac{\delta_\theta s}{1 + \delta_\theta s/2} \tag{19}$$

It is possible to derive an even tighter description (smaller uncertainty set) of the set in (3) if the nominal model with $\Delta = 0$ is *not* restricted to be equal to $g(s)$ (Lundström *et al.* 1991 b, 1993). However, this tightest description yields an irrational nominal model and will not be considered in this paper.

From Fig. 2 we can see that $w_1(s)$ approximates the tightest bound $l(\omega)$ very well at both low and high frequencies, but at intermediate frequencies it is a little smaller than the tightest bound and thus does *not* cover all possible plants.

In the following we consider five multiplicative uncertainty weights for the time-delay uncertainty which are all upper bounds on $l(\omega)$, i.e. they do contain all possible plants.

$$w_0(s) = \delta_\theta s \tag{20}$$

$$w_{1h}(s) = \frac{\delta_\theta s}{1 + \delta_\theta s/3{\cdot}465} \tag{21}$$

$$w_{1a}(s) = \frac{1{\cdot}216\delta_\theta s}{1 + \delta_\theta s/2} \tag{22}$$

$$w_2(s) = \frac{\delta_\theta s(2 \times 0{\cdot}2152^2\delta_\theta s + 1)}{(0{\cdot}2152\delta_\theta s + 1)^2} \tag{23}$$

$$w_3(s) = \frac{\delta_\theta s}{1 + \delta_\theta s/2}\,\frac{(s/\omega_0)^2 + 1{\cdot}676(s/\omega_0) + 1}{(s/\omega_0)^2 + 1{\cdot}370(s/\omega_0) + 1}, \quad \omega_0 = 2{\cdot}363/\delta_\theta \tag{24}$$

All these weights (except $w_0(s)$) are proper rational transfer functions and may be used for *DK*-iteration. The weights are plotted in Fig. 2. $w_0(s)$ is the weight obtained from approximation **Complex0** in the previous section. It approximates the tightest bound $l(\omega)$ very well at low frequencies but is much larger at high frequencies. The next two weights are derived based on $w_1(s)$ (from **Complex1**)

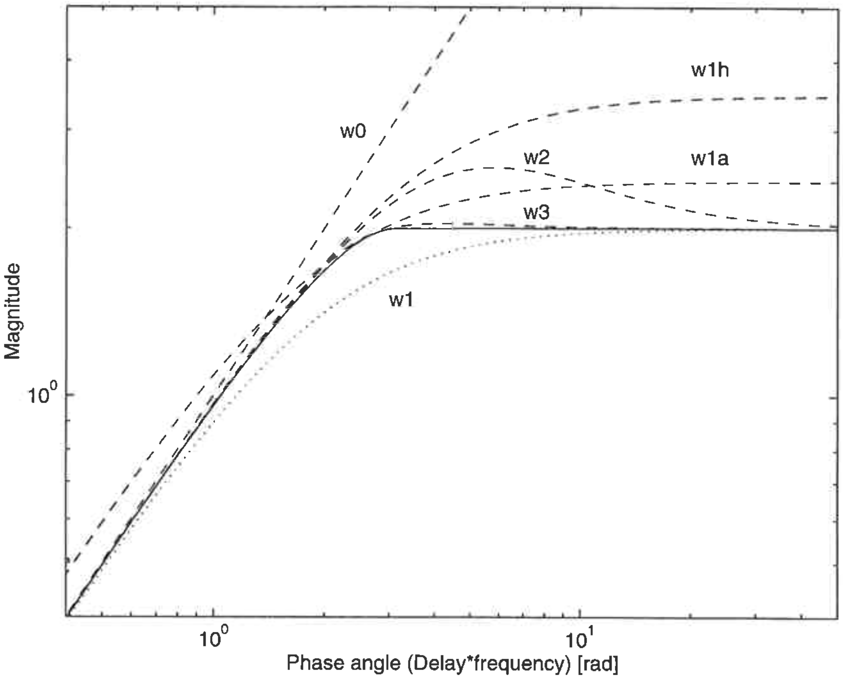

Figure 2. Different multiplicative weights for time delay uncertainty. Solid curve: $l(\omega)$ from Equation (18).

which is not an upper bound on $l(\omega)$. To modify it and make it an upper bound, but still restrict it to be of first-order, we have three choices. (1) Increase $w_1(s)$ at high frequencies without changing the value at low frequencies. This yields $w_{1h}(s)$. (2) Conversely, increase $w_1(s)$ at low frequencies without changing the value at high frequencies. However, this is not possible since $l(\omega) = 2$ at finite frequencies $\omega \geqslant \pi/\delta_\theta$. (3) Increase $w_1(s)$ at all frequencies to cover $l(\omega)$. This gives weight $w_{1a}(s)$. The second-order weight $w_2(s)$ is a refinement of $w_0(s)$. The third-order weight $w_3(s)$ is obtained by covering the mismatch between $l(\omega)$ and $w_1(j\omega)$ with a second-order transfer function (Lundström *et al.* 1993).

We now want to compare these alternative weights by computing the value of μ for robust stability, μ_{RS}, using the same PI controller and values of k_c as in the last section. In each case, the time-delay uncertainty δ_θ is set to equal to the largest delay the system can tolerate without getting unstable, so the 'exact' μ value for robust stability '$\mu_{RSexact}$' is equal to 1 for all cases.

The μ_{RS} computed for the complex uncertainty description (17) with the irrational weight $l(\omega)$ (18) and the five rational uncertainty weights (20)–(24) are shown in Table 2. We see from Table 2

(1) μ_{RSl} computed with $l(\omega)$ is the best obtainable result using complex multiplicative uncertainty to cover the time-delay uncertainty. The difference between μ_{RSl} and $\mu_{RSexact}$ is the conservatism introduced by the complex multiplicative uncertainty assumption;

(2) the differences between μ_{RS1h}, μ_{RS2} and μ_{RS3} are minor, so increasing the weight order does not have a large effect on μ_{RS}. This is because μ_{RS1h} is

k_c	δ_θ	Weight						
		Exact, μ_{RSexact}	$l(\omega)$, μ_{RSl}	$w_0(s)$, μ_{RS0}	$w_{1h}(s)$, μ_{RS1h}	$w_{1a}(s)$, μ_{RS1a}	$w_2(s)$, μ_{RS2}	$w_3(s)$, μ_{RS3}
0·1	14·708	1	1·0694	1·6086	1·1148	1·0972	1·0983	1·0882
0·2	6·8540	1	1·1146	1·6370	1·1660	1·1392	1·1474	1·1333
0·5	2·1416	1	1·3808	1·7031	1·4254	1·4175	1·4102	1·4064
1·0	0·5708	1	1·7403	1·7862	1·7417	2·0214	1·7409	1·7460
1·5	0·0472	1	1·8530	1·8534	1·8530	2·2522	1·8530	1·8530

Table 2. Robust stability μ_{RS} for different complex multiplicative uncertainty approximations of the delay uncertainty.

already quite close to μ_{RSl}. The remaining conservativeness comes from the complex multiplicative uncertainty assumption. The differences between the different approximations are most easily understood by considering at what phase-angle ($\delta_\theta \omega$) the peak value of μ (i.e. μ_{RS}) is obtained. For small k_c values the μ peak will occur at a relatively high phase-angle, while for large k_c values it will be at a low phase-angle. Hence, Table 2 demonstrates that w_0 is a poor approximation at high phase-angles and w_{1a} is poor at low phase-angles. This is also seen from Fig. 2.

(3) The conclusion is that $w_{1h}(s)$ is a simple and reasonably good approximation of the time-delay uncertainty.

4. Choice of nominal model

In this section we study how to choose the nominal delay θ in (3) such that the resulting model is well suited for μ-synthesis and $\theta \pm \delta_\theta$ covers a prespecified range of possible delays. We consider the following set of SISO plants

$$\text{Plant set: } g_p(s) = \frac{k}{\tau s + 1} e^{-\theta_p s}, \quad \theta_{\min} \leqslant \theta_p \leqslant \theta_{\max} \tag{25}$$

It can be modelled in two fundamentally different ways (for notational simplicity we assume $\theta_{\min} \geqslant 0$, i.e. no prediction).

Approach 1: *Average delay in the nominal model (same as (4)).*

$$\text{Nominal model:} \quad g(s) = \frac{k}{\tau s + 1} e^{-((\theta_{\max}+\theta_{\min})/2)s} \tag{26}$$

$$\text{Time-delay uncertainty:} \quad e^{-((\theta_{\max}-\theta_{\min})/2)\Delta s}, \quad -1 \leqslant \Delta \leqslant 1 \tag{27}$$

Approach 2: *Delay free nominal model.*

$$\text{Nominal model:} \quad g(s) = \frac{k}{\tau s + 1} \tag{28}$$

$$\text{Time-delay uncertainty:} \quad e^{-\theta_{\max}\Delta s}, \quad -1 \leqslant \Delta \leqslant 1 \tag{29}$$

Approach 1 covers all delays in (25) exactly, while Approach 2 allows delays between $-\theta_{\max}$ and $+\theta_{\max}$, i.e. Approach 2 allows more uncertainty. The main advantage with Approach 2 is that it leads to a simple nominal model with no delay. For example, this simplifies controller synthesis using DK-iteration. Since

Approach 1 uses a more complicated nominal model, one would expect it to yield better results than Approach 2. But as we shall see, this need not be the case.

If Approach 1 is used for synthesis, one needs to approximate the delay in the nominal model (26) by a rational transfer function, since H_∞ synthesis and μ synthesis in both existing toolboxes (μ-Analysis and Synthesis Toolbox (Balas *et al.* 1991) and Robust Control Toolbox (Chiang and Safonov 1988)) are not capable of dealing with time-delays in an exact manner. The approximation of the nominal model may cause significant deterioration in robust performance because it introduces additional uncertainty, as shown by Wang and Skogestad (1993). Approach 2, on the other hand, does not have this problem since it uses a delay free nominal model (28).

We now proceed to compare these two choices of nominal models by considering J_{RP}, the 'skewed μ' (Packard 1988) for robust performance

$$J_{RP} = \sup_{\omega} \sup_{\Delta} |w_p S_p| \tag{30}$$

where

$$S_p(s) = (1 + g_p(s)c(s))^{-1} \tag{31}$$

With performance weight

$$w_p(s) = \frac{s+1}{2s} \tag{32}$$

and uncertainty weight $w_{1h}(s)$ (21) we get

$$J_{RP} = \sup_{\omega} \sup_{\Delta} |w_p S(1 + \Delta w_{1h} g c S)^{-1}| = \sup_{\omega} \frac{|w_p S|}{1 - |w_{1h} g c S|} \tag{33}$$

J_{RP} is similar to the structured singular value for robust performance, μ_{RP}, in the sense that $J_{RP} < 1$ iff $\mu_{RP} < 1$. However, they are not equivalent since $J_{RP} \geqslant \mu_{RP}$ for $\mu_{RP} \geqslant 1$ and $J_{RP} \leqslant \mu_{RP}$ for $\mu_{RP} \leqslant 1$. When $\mu_{RP} < 1$, both the performance weight and uncertainty weight can be increased by a factor $1/\mu_{RP}$ while still retaining robust performance, but when $J_{RP} < 1$, only the performance weight is allowed to be increased by a factor $1/J_{RP}$. The reason we use J_{RP} here is that it measures robust performance with respect to a fixed uncertainty set and thereby allows us to compare different uncertainty models. Another reason is that we are able to calculate the 'exact' J_{RP} (J_{RP0}) for this simple case (25) and this J_{RP0} provides a reference for the comparison.

$$J_{RP0} = \sup_{\omega} \sup_{\theta_p} |w_p S_p| = \sup_{\omega} (|w_p| \sup_{\theta_p} |(1 + c(j\omega)\frac{1}{j\omega + 1} e^{-j\omega\theta_p})^{-1}|) \tag{34}$$

The peak values J_{RP} for different controllers and different modelling approaches are shown in Table 3 for the specific model parameters $k = 1$, $\tau = 1$, $\theta_{min} = 0$ and $\theta_{max} = 1$. The controllers used in Table 3 are three μ-optimal controllers and PI controllers of the form

$$c(s) = k_c \frac{s+1}{s} \tag{35}$$

with different k_c. Two of the μ optimal controllers are synthesized using Approach 1: $C_{\mu 1}$ where the nominal time-delay is approximated by a first-order

Controller	k_c	Exact, J_{RP0}	Approach 1, J_{RP1}	Approach 2, J_{RP2}
$C_{\mu 1}$		1·286	**1·334**	1·325
$C_{\mu 14}$		1·034	**1·036**	1·113
$C_{\mu 2}$		0·976	2·337	**0·986**
PI	0·2	2·500	2·5166	2·5538
	0·5	1·139	1·1471	1·2200
	1·0	1·938	3·5945	2·2325

Table 3. Comparison based on robust performance J_{RP} for two different modelling approaches of uncertain time-delay systems.

Padé approximation, and $C_{\mu 14}$ where a fourth-order Padé approximation is used. $C_{\mu 2}$ is obtained using Approach 2 with the delay free nominal model.

The results in Table 3 are somewhat unanticipated. Controller synthesis using model Approach 2 yields the lowest J_{RP0} value (0·976), although this modelling approach uses a larger uncertainty set than Approach 1. We also see that J_{RP2} is not much larger than J_{RP1} for any of the controllers, while J_{RP1} is much larger than J_{RP2} for controller $C_{\mu 2}$ and for the PI controller with gain $k_c = 1{\cdot}0$. This indicates that we can use a delay-free approach at almost no cost in terms of accuracy. The reason is that the H_∞-norm, μ and J_{RP} are worst case measures. It is not the size of the uncertainty set, but the worst uncertainty within the set that matters. Although Approach 1 covers a smaller uncertainty set than Approach 2, it may include a plant that is worse than any plant covered by Approach 2. This is graphically shown in Fig. 3 where a worst case plant in the shaded region yields J_{RP2} smaller than J_{RP1}.

The conclusions from Table 3 are as follows.

(1) J_{RP2} for $C_{\mu 2}$ is smaller than J_{RP1} for $C_{\mu 1}$ and $C_{\mu 14}$ and also J_{RP0} for $C_{\mu 2}$ is smaller than J_{RP0} for $C_{\mu 1}$ and $C_{\mu 14}$. This yields the surprising result that

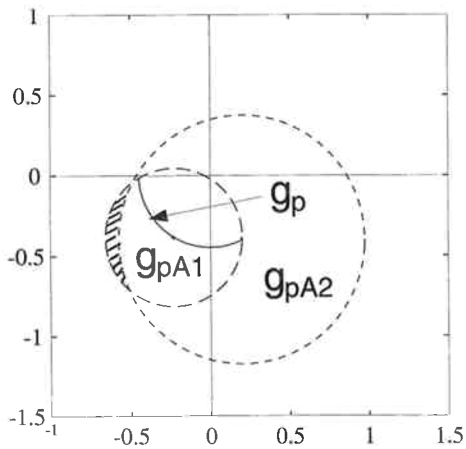

Figure 3. The exact set g_p (arc) and its disc approximations g_{pA1} (Approach 1) and g_{pA2} (Approach 2) plotted on the complex plane.

not only is the delay free approach (Approach 2) simpler for controller design, it also is better.

(2) The differences between J_{RP1} and J_{RP2} for two of the μ-optimal controllers are large. This shows that the μ-optimal controller can be very sensitive to the uncertainty not considered in the controller design.

Only time delay uncertainty has been considered in this section, however, results for simultaneous uncertainties in gain, time-constant and time-delay of (25) (Wang and Skogestad 1992) also support the conclusions above.

5. Conclusions

From the above results and discussions, we are ready to draw the following conclusions.

(1) μ-optimal controllers seem to be very sensitive to unconsidered (uncovered) uncertainty. Hence, uncertainty models which do not cover the original uncertainty (e.g. (19)) should be used with caution.

(2) For the multiplicative complex uncertainty approximation of time-delay uncertainty, the upper bound $w_{1h}(s)$ adjusted at high frequencies is a simple and good weight.

(3) For the modelling of uncertain time-delay systems, the approach of using a nominal model without time delay (Approach 2) is better since it gives comparable results but leads to a delay-free design problem for the time-delay systems.

(4) The transformation from original uncertainty description to the norm-bounded complex perturbation inevitably introduces conservatism. In order to reduce the conservatism, one usually chooses a nominal model which minimizes the uncertainty weight. However, μ is a worst case measure. Of most importance is not the size of uncertainty but the worst uncertainty. Consequently, the correct way to reduce conservatism is not to minimize the uncertainty size but to minimize the worst uncertainty introduced in transformation. Of course, minimizing the uncertainty size is easy and direct. However, which is the worst uncertainty is generally not obvious. Research work on identifying the worst uncertainty may be worthwhile.

(5) Note that all results are for SISO plants, and MIMO systems may behave entirely differently. For example, for a SISO plant a large delay is generally bad, while for a MIMO system a large delay in an off-diagonal element is good since it helps to reduce interaction.

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