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Control of Symmetrically Interconnected Plants*

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The symmetric circulant structure of many plants, including paper machines and distribution networks, is used to simplify the controller design for both H_2- and H_∞-control, and to obtain super-optimality in the H_∞-case.

Key Words—Control system design; linear systems; system theory; large scale systems; H-infinity control; optimal control; robust control; decentralized control; paper industry; process control.

Abstract—This paper is concerned with control of plants composed of n similar interacting subsystems. Such plants are common in practice and include paper machines, distribution networks, coating processes, and plants consisting of units operating in parallel. The transfer function matrices for these systems are block symmetric circulant. For H_∞- and H_2-optimal control, controller synthesis is simplified by considering $n/2+1$ independent problems of dimension n times smaller than the original problem. For the case of H_∞-optimal control this also yields 'super-optimality', where the H_∞ criterion is optimized in n directions, and not only in the worst direction. If the offdiagonal blocks ('interactions') are identical the matrix is termed block parallel, and controller synthesis involves only two independent subproblems of the same dimension as the subsystems. This leads to a dramatic reduction in dimension for systems composed of many subsystems.

1. INTRODUCTION

WHENEVER POSSIBLE, one should make use of any special properties of the system in order to simplify control system analysis or design. In this paper we study systems consisting of symmetrically interconnected subsystems, which we define as systems whose transfer function matrix has the following symmetric circulant structure:

$$G(s)=\begin{bmatrix} \alpha(s) & \beta_1(s) & \beta_2(s) & \beta_3(s) & \cdots & \beta_2(s) & \beta_1(s)\\ \beta_1(s) & \alpha(s) & \beta_1(s) & \beta_2(s) & \beta_3(s) & \cdots & \beta_2(s)\\ \beta_2(s) & \beta_1(s) & \alpha(s) & \beta_1(s) & \beta_2(s) & \ddots & \ddots\\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots\\ \ddots & \ddots & \beta_2(s) & \beta_1(s) & \alpha(s) & \beta_1(s) & \beta_2(s)\\ \beta_2(s) & \cdots & \beta_3(s) & \beta_2(s) & \beta_1(s) & \alpha(s) & \beta_1(s)\\ \beta_1(s) & \beta_2(s) & \cdots & \beta_3(s) & \beta_2(s) & \beta_1(s) & \alpha(s) \end{bmatrix}. \quad (1)$$

The n diagonal elements $\alpha(s)$ denote the transfer function of the individual subsystems, and the offdiagonal elements $\beta_i(s)$, $i=1, k-1$, denote the interactions. The number of different transfer function elements in $G(s)$ is $k=\frac{n}{2}+1$ (n even) or $k=\frac{n-1}{2}+1$ (n odd). For MIMO subsystems, both $\alpha(s)$ and $\beta_i(s)$ are matrices, and we term the corresponding $G(s)$ in equation (1) a block symmetric circulant matrix consisting of $n \times n$ blocks.

Systems consisting of symmetrically interconnected subsystems constitute an important class of large-scale systems where there may be a clear advantage of using multivariable control and where it is actually used in practice. The class includes a large number of systems that have some kind of symmetric planar arrangement. One important industrial example is the cross-directional control of paper machines if edge effects are neglected (Wilhelm and Fjeld,

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1983, Laughlin *et al.*, 1992). Other examples include multizone crystal growth furnaces (Abraham and Lunze, 1991) and dyes for plastic films (Martino, 1991). Brockett and Willems (1974) point out that this type of systems arise in lumped approximations to partial differential equations. The burner furnace of Rosenbrock (1974, p. 197) is also on a form very similar to (1). Additional examples are given in the discussion of parallel systems below.

1.1. *Parallel systems*

To simplify the analysis, we will concentrate the discussion in this paper on an important subclass of these systems where the interactions between the subsystems are identical, i.e.

$$\beta_i(s) = \beta(s)\ \forall i. \tag{2}$$

With n nominally identical subsystems the transfer matrix of the plant consisting of $n \times n$ blocks may be written

$$G(s) = \begin{bmatrix} \alpha(s) & \beta(s) & \beta(s) & \cdots & \beta(s) \\ \beta(s) & \alpha(s) & \beta(s) & \cdots & \beta(s) \\ \beta(s) & \beta(s) & \alpha(s) & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \beta(s) \\ \beta(s) & \beta(s) & \cdots & \beta(s) & \alpha(s) \end{bmatrix}. \tag{3}$$

We have not found any name for the matrix (3) in the literature, but we shall refer to it as a block parallel matrix in the following, as we believe that transfer function matrices of the form of $G(s)$ in equation (3) occur predominantly for nominally identical, interacting processes in parallel. The type of system in equation (3) is termed a 'symmetrically interconnected system' by Sundareshan and Elbanna (1991), and 'symmetrically composite systems' by Lunze (1986). However, as already noted above, we will use these terms in a more general sense to include also the block symmetric circulant systems in equation (1), and we will refer to the special cases described by equation (3) as 'parallel systems'.

Parallel systems occur whenever there are similar, interacting subsystems operating in parallel. Examples are found in distribution networks, when there are parallel units (e.g. reactors, compressors, pumps, heat exchangers) in a chemical plant (Shinskey, 1979, 1984), for electric power systems operating in parallel (Lunze, 1986, 1991), for adhesive coating processes (Braatz *et al.*, 1992), or for communication between ships (Hazewinkel and Martin, 1983). Additional examples are given by Sundareshan and Elbanna (1991) and in Section 2.

1.2. *Previous work*

A number of the important properties of circulant systems, which we make use of in this paper, are given in the paper by Brockett and Willems (1974), who study in particular state controllability and observability of such systems. They also point out that the special structure of the plant may be used to simplify the controller design for the case of least squares (H_2) optimal control. Circulant systems are also discussed by Hazewinkel and Martin (1983). Lunze (1986) studies parallel systems and points out they may be transformed into one subsystem describing the average state and $n-1$ identical subsystems. He makes use of this property to study state controllability and observability, as well as stability of the closed-loop system under decentralized control. Lunze (1989) extends these results to include a robust stability analysis. Lunze's model formulation can take account of a variety of different uncertainties and model errors. However, he does not account for the structure of the uncertainty, which can lead to very conservative results, as shown in Hovd (1992). Sundareshan and Elbanna (1991) also study conditions for state controllability and observability of parallel systems and they present in addition results for controller synthesis. They present solutions to the matrix Riccati and Lyapunov equations, and show that the solution is considerably simplified since one can solve the problems of considerably lower dimensions. In an application paper Trächtler (1991) studies some plants with structure similar to (1), including a flexible structure, and makes the assumption that the optimal controller has the same structure as the plant. Decentralized control of symmetrically interconnected systems is considered by Lundström *et al.* (1991).

1.3. *Insights for parallel systems.*

We now want to give the reader some insight into how we, as described in the references above, may make use of the special structure in equation (3) to simplify analysis and design. Consider the case when α and β are scalars and rewrite the system as

$$\begin{aligned} y_1 &= \alpha u_1 + \beta u_2 + \beta u_3 + \cdots \\ y_2 &= \beta u_1 + \alpha u_2 + \beta u_3 + \cdots \\ y_3 &= \beta u_1 + \beta u_2 + \alpha u_3 + \cdots \\ &\vdots \end{aligned} \tag{4}$$

Adding together all the n outputs yields

$$\begin{aligned} \sum_i y_i &= (\alpha + (n-1)\beta)u_1 + (\alpha + (n-1)\beta)u_2 + \cdots \\ &= (\alpha + (n-1)\beta)\sum_i u_i. \end{aligned} \tag{5}$$

We thus note that we have a completely decoupled subsystem between the input $u_1' = \sum_i u_i$ and the output $y_1' = \sum_i y_i$, and it is also clear that $\alpha + (n-1)\beta$ is an eigenvalue for the overall system $G(s)$. Next consider differences between two outputs. We get

$$\begin{aligned} y_2 - y_1 &= (\beta - \alpha)u_1 + (\alpha - \beta)u_2 \\ &= (\alpha - \beta)(u_2 - u_1). \end{aligned} \tag{6}$$

Thus we have another decoupled subsystem in terms of the input $u_2' = u_2 - u_1$ and the output $y_2' = y_2 - y_1$ with eigenvalue $(\alpha - \beta)$. Similar arguments for the other differences gives that there are a total of $(n-1)$ subsystems with eigenvalues $(\alpha - \beta)$. Thus we have found that the plant in equation (3) has at most two distinct eigenvalues given by $(\alpha + (n-1)\beta)$ and $(\alpha - \beta)$ (Lunze, 1986), and the physical reasoning concerning the sum and $(n-1)$ differences directly provides a physically motivated set of eigenvectors which may be used for diagonalizing $G(s)$ in equation (3).

In terms of controller design it seems likely that the optimal controller will have the same structure as that of the plant. This assumption limits the degrees of freedom, but does not necessarily simplify the controller design. However, equations (5) and (6) directs us towards a procedure where we may independently design controllers for two subplants: one describing the 'average' behavior (or sum) and one describing the 'distance from the average'. This idea also follows from the work of Fagnani and Willems (1991) who approach the problem from a mathematical point of view by considering more general symmetries. The idea of designing one controller for the 'average' and $n-1$ (or n) identical controllers for the 'distance from average' has also been used in practice [e.g. for paper machine control (Wilhelm and Fjeld, 1983) and for control of fired heaters with several parallel passes (Shinskey, 1984, p. 104)].

Clearly, if control performance is defined in terms of the 'new' outputs (i.e. the 'average' and 'difference from the average') then an independent controller design for each subsystem is optimal. However, if performance is defined in terms of the 'original' outputs y_i then this may not be the case. The transformation from the 'original' outputs to the 'new' outputs is given by the eigenvector matrix for the diagonalization. It turns out that a key step is to use an orthogonal eigenvector matrix, for example the Fourier matrix, rather than the physically motivated eigenvector matrix introduced above [since we have $(n-1)$ eigenvalues that are not distinct, we have freedom in selecting the eigenvector matrix]. With an orthogonal eigenvector matrix we can make use of the fact that the H_2 and H_∞-norms are invariant under unitary scalings.

These results on parallel systems can be extended directly to include the more general block symmetric circulant transfer matrices in equation (1). For a $n \times n$ plant, we show that one may simplify the controller design for the H_2- and H_∞-norm by designing $n/2+1$ independent controllers (for n even) rather than a large $n \times n$ controller.

For the H_2-case the resulting simplifications in terms of controller synthesis are known from previous work (Brockett and Willems, 1974; Sundareshan and Elbanna, 1991), but our approach is quite different and somewhat more general. The main emphasis in this paper is on H_∞-case for which the results are new, and where we in addition to achieving a much simplified controller design, also directly get a 'super-optimal' controller that optimizes all directions. For the H_2-norm the controller is always unique and we do not have the additional advantage of super-optimality.

1.4. *Extensions*

Two other generalizations are considered in Hovd and Skogestad (1992) and Hovd *et al.* (1993): (1) The results may be generalized to plants that may be diagonalized using constant unitary matrices. We then generally find that the controller has the form of a Singular Value Decomposition (SVD) controller. This is indeed the case for the plants considered in this paper, but this is of somewhat minor significance in our case since the SVD is not unique in our case. (2) It is also possible to include robustness considerations and still achieve significant simplifications in the controller design. We show that this is the case when the objective is to optimize robust performance within the H_∞ framework, i.e. using the Structured Singular Value of Doyle (1982).

2. NOTATION

In this paper, $G(s)$ denotes the plant, which is assumed to consist of n symmetrically interconnected subsystems, each subsystem of dimension $n_o \times n_i$. The plant $G(s)$ therefore has the dimension $n \cdot n_o \times n \cdot n_i$. The controller is denoted $K(s)$, and $S(s) = (I + G(s)K(s))^{-1}$ denotes the sensitivity function. The Laplace variable s will often be suppressed to simplify the notation. The reference signal is denoted r, the manipulated inputs u, and the controlled outputs y. A block diagram of a feedback system is shown in Fig. 1.

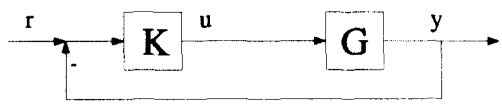


FIG. 1. Block diagram of a feedback system.

Circulant and block circulant matrices are denoted C, and parallel and block parallel matrices are denoted P. Eigenvalues are denoted λ, with two subscripts: a letter referring to the matrix of which λ is an eigenvalue, and a number to distinguish the different eigenvalues of a matrix. Thus λ_{X1} is the first eigenvalue of the matrix X. The blocks on the diagonal of a matrix that is transformed to be block diagonal [see equation (33)] are denoted γ; subscripts are used for γ in the same way as for λ. The matrix M is in general a matrix consisting of blocks which are block circulant, but M is also used for the matrix in the design objective, i.e. we want to optimize the H_∞-norm or H_2-norm of M. The matrix N is the matrix in the design objective (M) expressed as a linear fractional transformation (LFT) of the controller K. The matrix $\tilde{A}$ denotes a matrix A that has been transformed such that it consists of blocks that are block diagonal.

3. SIMPLE EXAMPLES OF PARALLEL PROCESSES

In this section we present two simple examples of a parallel process of the form (3). To simplify notation we define for a system consisting of SISO subsystems the degree of interaction at steady state as

$$a = \beta(0)/\alpha(0). \tag{7}$$

Example 1. *Flow splitting*. Consider controlling the flows of n parallel streams from a single source as shown in Fig. 2 (Shinskey, 1979, p. 201). The manipulated inputs u_i are the n valve positions z_i and the controlled outputs y_i are the n flows q_i [$\mathrm{m}^3\,\mathrm{s}^{-1}$]. Opening value one causes q_1 (flow 1) to increase and q_2 (flow 2) to decrease because of the reduction in header pressure. If there are two parallel streams the steady state value of a is expected to lie between -1 and 0. The value of zero would be obtained if the source was a large tank such that the header pressure was unaffected by increasing flow 1, and the value of -1 would be obtained if the source was a pump with constant total flow q. For n parallel streams from a single source similar arguments yield at steady state

$$-1/(n-1) \le a \le 0. \tag{8}$$

A value less than the lower bound $-1/(n-1)$

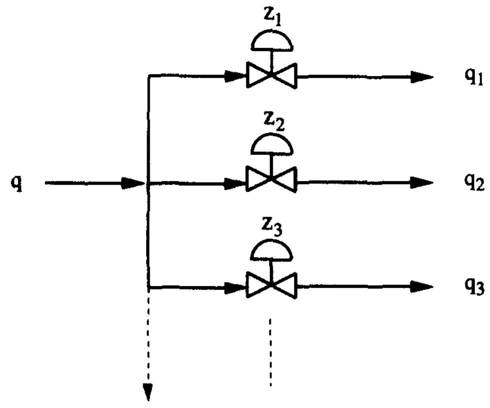


FIG. 2. Splitting into parallel streams (Example 1).

would imply that the total flow q is reduced by opening a valve and is unlikely in a practical situation.

Example 2. *Parallel reactors with combined precooling*. In processing plants it is common to have units in parallel, either because one single unit would be too large or to add flexibility. Figure 3 shows n identical mixing tank reactors in parallel, with a common precooler. The cooling medium comes from a single source which is split into n streams and then completely evaporated by heat exchange with the reactors. The streams are then combined and this stream is superheated by precooling the reactor feed. At steady state all temperatures and flows in the parallel streams are assumed equal. Consider the transfer matrix $G(s)$ between the flows q_i (inputs) and the reactor temperatures T_i (outputs). By neglecting the dynamics of the evaporator and the superheater, the model $G(s)$ can be shown (Skogestad *et al.*, 1989) to be of the form

$$G(s) = \frac{k}{\tau s + 1}\begin{bmatrix} 1 & a & a & \cdots & a \\ a & 1 & a & \cdots & a \\ a & a & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & a \\ a & a & \cdots & a & 1 \end{bmatrix}, \tag{9}$$

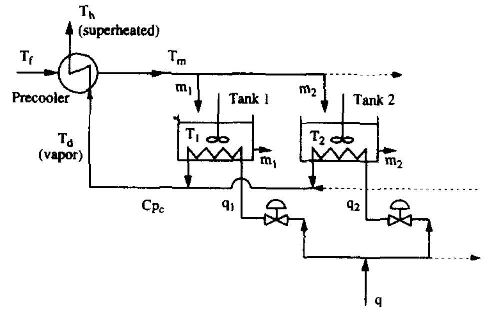


FIG. 3. Cooling system for parallel reactors (Example 2).

where k and a are real constants and τ is the time constant for holdup in each of the reactors. Based on physical arguments we have

$$-1/(n-1) \leq a \leq 1. \quad (10)$$

The lower limit is obtained by considering the case with no precooler and assuming constant total flow (recall the parallel flow example above). The upper limit is obtained by considering the case with no heat exchange taking place in the tanks. In this case the streams q_i are split and then recombined without changing their temperatures, and an increase in any cooling stream will affect all reactor temperatures equally and we have $a = 1$.

Note that $G(s)$ is singular both for $a = -1/(n-1)$ (rank $(G) = n - 1$) and for $a = 1$ (rank $(G) = 1$). Obviously, when G is singular, independent control of the controlled outputs is not possible. An example of this is found in Braatz *et al.* (1992), who consider the cross directional control of a coating process for which $a = -1/(n-1)$. In this case we need only design one controller since we cannot do anything about the average.

In many cases we have multi-input multi-output (MIMO) subsystems. For example, in Fig. 3, one may have 2×2 subsystems where we want to control temperature (T_i) and concentration (c_i) in each tank, using cooling (q_i) and flow (m_i) as manipulated variables.

4. RESULTS FROM MATRIX THEORY

4.1. *Circulant matrices*

The results in this section on circulant matrices are from Bellman (1970) and Davis (1979), and some of their important properties for control are given by Brockett and Willems (1974). The general form of a circulant matrix C is:

$$C = \begin{bmatrix} c_1 & c_2 & c_3 & \cdots & c_n \\ c_n & c_1 & c_2 & \cdots & c_{n-1} \\ c_{n-1} & c_n & c_1 & \cdots & c_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_2 & c_3 & c_4 & \cdots & c_1 \end{bmatrix}. \quad (11)$$

Circulant matrices belong to the class known as Toeplitz matrices, as all elements along any one diagonal are identical. Introduce $v_l = \exp(2\pi(l-1)i/n)$ where $i = \sqrt{-1}$ and $l = 1, \ldots, n$. That is, v_l is a root of the equation $v^n = 1$. The eigenvalues of the circulant matrix C are given by Davis (1979):

$$\lambda_{Cl} = c_1 + c_2 v_l + c_3 v_l^2 + \cdots + c_n v_l^{n-1}. \quad (12)$$

The eigenvector corresponding to λ_{Cl} is:

$$m_l = [1 \; v_l \; v_k^2 \cdots v_l^{n-1}]^T. \quad (13)$$

Since v_l can take n distinct values, C will always have a complete set of eigenvectors, and will thus always be diagonalizable. In fact, all circulant matrices of the same order have the same eigenvectors, and are therefore diagonalized by the same eigenvector matrix, the Fourier matrix. The Fourier matrix of order n is given by Davis (1979).

$$F^H = \frac{1}{\sqrt{n}}[m_1 \; m_2 \cdots m_n]. \quad (14)$$

F is unitary ($FF^H = F^H F = I$), and we have for any circulant matrix C

$$C = F^H \Lambda_C F; \quad \Lambda_C = \text{diag}\{\lambda_{C1}, \ldots, \lambda_{Cn}\}. \quad (15)$$

Furthermore, we have for the singular value decomposition $C = U\Sigma_C V^H$ that the singular values $\sigma_l = |\lambda_l|$, and $V = F^H$, the eigenvector matrix, and $U = F^H D$, where $D = \text{diag}\{d_l\}$, $d_l = \lambda_l/|\lambda_l| = \lambda_l/\sigma_l$($d_l$ contains the phase of the lth eigenvalue).

4.2. *Symmetric circulant matrices*

Consider the case when $c_2 = c_n$, $c_3 = c_{n-1}$, etc in equation (11), such that C is symmetric as well as circulant. Note that equation (1) is on this form with $\alpha = c_1$, $\beta_1 = c_2 = c_n$, $\beta_3 = c_3 = c_{n-1}$, etc. Calculating the eigenvalues of a symmetric circulant matrix from equation (12), we find that it is only eigenvalues number 1 and $n/2 + 1$ (if n is even) that are distinct. All the other eigenvalues appear in pairs such that $\lambda_{C_p} = \lambda_{C(n+2-p)}$, for $p = \{2, 3, \ldots, v\}$, where $v = (n+1)/2$ if n is odd and $v = n/2$ if n is even.

The eigenvectors corresponding to the distinct eigenvalues are real. Eigenvectors corresponding to repeated eigenvalues are not unique, and we may use this to construct a real eigenvector matrix R for symmetric circulant matrices. In the theory that follows it will be required that the eigenvector matrix is unitary. One possible choice for the real orthogonal eigenvector matrix is $R = 1/\sqrt{n}[r_1 \cdots r_n]$ with

$$r_1 = m_1 = [1 \; 1 \cdots 1]^T \quad (16)$$

$$r_{(n/2)+1} = m_{(n/2)+1} \text{ if } n \text{ is an even number} \quad (17)$$

$$r_p = \frac{1}{\sqrt{2}}(m_p + m_{n+2-p}) \quad (18)$$

$$r_{n+2-p} = \frac{i}{\sqrt{2}}(m_p - m_{n+2-p}) \quad \text{for } p = \{2, 3, \ldots, v\}. \quad (19)$$

Note that m_p is the complex conjugate of

TABLE 1. Fourier matrix, F_n, and real, orthogonal eigenvector matrix, R_n, for symmetric circulant matrices of dimension $n \times n$ [see equation (16)–(19) for general case].

n	F_n	R_n
$n=2$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$
$n=3$	$\frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1 & 1\\1 & -0.5(1+i\sqrt{3}) & -0.5(1-i\sqrt{3})\\1 & -0.5(1-i\sqrt{3}) & -0.5(1+i\sqrt{3})\end{bmatrix}$	$\frac{1}{\sqrt{3}}\begin{bmatrix}1 & \sqrt{2} & 0\\1 & -0.5\sqrt{2} & 0.5\sqrt{6}\\1 & -0.5\sqrt{2} & -0.5\sqrt{6}\end{bmatrix}$
$n=4$	$\frac{1}{2}\begin{bmatrix}1 & 1 & 1 & 1\\1 & -i & -1 & i\\1 & -1 & 1 & -1\\1 & i & -1 & -i\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}1 & -0.5\sqrt{2} & 1 & 0\\1 & 0 & -1 & \sqrt{2}\\1 & -\sqrt{2} & 1 & 0\\1 & 0 & -1 & -\sqrt{2}\end{bmatrix}$

m_{n+2-p}. In Table 1 we show the Fourier matrix F and the corresponding real orthogonal eigenvector matrix R resulting from equations (16)–(19) for $n=2$, $n=3$ and $n=4$. For diagonalization one has $\Lambda_P = XPX^{-1}$ where $X=F$ (Fourier matrix) or $X=R$ (real, orthogonal matrix).

4.3. *Parallel matrices*

Consider the case with $c_1=\alpha$ and $c_2=c_3=\cdots=c_n=\beta$. This yields a parallel $n\times n$ matrix, P, with the general form in (3). All the properties derived above for circulant matrices and symmetric circulant matrices hold for the parallel matrix P. Specifically, the parallel matrix P has eigenvalues λ_{Pl} given by the formula:

$$\lambda_{Pl} = \alpha + \beta(v_l + v_l^2 + \cdots + v_l^{n-1}). \qquad (20)$$

Note that

$$1 + v_l + v_l^2 + \cdots + v_l^{n-1} = 0 \quad \text{for} \quad v_l \neq 1. \qquad (21)$$

From equations (20) and (21) we see that that the matrix P will have at most two distinct eigenvalues given by

$$\lambda_{P1} = \alpha + (n-1)\beta \qquad (22)$$

$$\lambda_{P2} = \lambda_{P3} = \cdots = \lambda_{Pn} = \alpha - \beta. \qquad (23)$$

The eigenvector matrix for P can be any non-singular matrix with (some multiple of) m_1 in the first column and columns 2-n orthogonal to column 1. One possible eigenvector matrix may be found from the physical considerations in the introduction. However, for controller design we shall need a unitary eigenvector matrix, and we will use the Fourier matrix F [we may instead use the real orthogonal eigenvector matrix R from equations (16)–(19) if we for computational reasons need to work with real matrices]. For realization of controllers we require the use of a real eigenvector matrix, but for this purpose the eigenvector matrix need not be orthogonal, as will become clear below.

5. RESULTS FOR SYMMETRICALLY INTERCONNECTED SYSTEMS

5.1. *Systems with SISO subsystems*

5.1.1. *Diagonalization of parallel systems.* The matrix results above can be applied in a straightforward manner to the diagonalization of parallel transfer function matrices. For example, consider the parallel transfer matrix $G(s)$ in equation (3). From the results above we have

$$G(s) = X^{-1}\Lambda(s)X \qquad (24)$$

$$\Lambda(s) = \begin{bmatrix}\lambda_{G1}(s) & & & \\ & \lambda_{G2}(s) & & \\ & & \ddots & \\ & & & \lambda_{G2}(s)\end{bmatrix}, \qquad (25)$$

where $\lambda_{G1}(s) = \alpha(s) + (n-1)\beta(s)$ and $\lambda_{G2}(s) = \alpha(s) - \beta(s)$. We see that the $n\times n$ process described by $G(s)$ has been decomposed into n non-interacting 1×1 subprocesses. One of these subprocesses, $\lambda_{G1}(s) = \alpha(s) + (n-1)\beta(s)$ is distinct, and the $n-1$ other subprocesses $\lambda_{G2}(s) = \cdots = \lambda_n(s) = \alpha(s) - \beta(s)$ are equal. For a fixed value of s, $\lambda_{Gi}(s)$ is the ith eigenvalue of $G(s)$. Three possible choices for X are:

(1) $X=F$ from equation (14);

(2) $X=R$ from equations (16)–(19); and

(3) From the physical reasoning in the Introduction:

$$X = \begin{bmatrix}1 & 1 & 1 & \cdots & 1\\ 1 & -1 & 0 & \cdots & 0\\ 0 & \ddots & \ddots & \ddots & \vdots\\ \vdots & \ddots & 1 & -1 & 0\\ 0 & \cdots & 0 & 1 & -1\end{bmatrix}. \qquad (26)$$

5.1.2. *Combinations of parallel systems.* If A and B are parallel matrices of the same dimension and k_i a scalar, then A^T, A^H, $k_1A + k_2B$, AB, $\sum_i k_iA^i$ are parallel matrices and A and B commute, that is, $AB = BA$. Note that A^{-1} is also a parallel matrix.

For example, if both the process G and the controller C are parallel transfer function matrices, the sensitivity function $S = (I + GC)^{-1}$ and the complementary sensitivity function $H = I - S$ are both parallel matrices.

5.1.3. *Systems consisting of circulant blocks.* Consider a matrix M consisting of $m_1 \times m_2$ blocks which in general are different, but each block is a circulant matrix of order n. We shall call the class of such matrices $CB_{m_1,m_2,n}$. The many results from the theory of circulant matrices do not hold for matrices consisting of circulant blocks. However, we can find some results which will prove helpful.

- If M_1 and M_2 both belong to the class $CB_{m_1,m_2,n}$ and α_1 and α_2 are scalars, then $\alpha_1 M_1 + \alpha_2 M_2$ also belongs to the class $CB_{m_1,m_2,n}$ and M_1^H belongs to the class $CB_{m_2,m_1,n}$. If M_1 belongs to the class $CB_{m_1,m_2,n}$ and M_2 belongs to the class $CB_{m_2,m_1,n}$ then $M_1 M_2$ belongs to the class $CB_{m_1,m_1,n}$.
- For 'diagonalization' we have

$$\tilde{M}_1 = (I_{m_1} \otimes F_n) M_1 (I_{m_2} \otimes F_n)^H, \qquad (27)$$

where $\otimes$ denotes the Kronecker product, and $\tilde{M}_1$ is a matrix with the same block structure as M_1, each block in $\tilde{M}_1$ being the (diagonal) eigenvalue matrix of the corresponding block in M_1. This is illustrated by an example. Consider

$$M_1 = \begin{bmatrix} C_1 & C_2 & C_3 \\ C_4 & C_5 & C_6 \end{bmatrix}, \qquad (28)$$

where $C_1, C_2, \ldots, C_6$ all are circulant matrices of order n. M_1 then belongs to the class $CB_{2,3,n}$. We then have

$$M_1 = (I_2 \otimes F_n)^H \tilde{M}_1 (I_3 \otimes F_n) \qquad (29)$$

$$(I_2 \otimes F_n)^H = \begin{bmatrix} F_n^H & 0 \\ 0 & F_n^H \end{bmatrix} \qquad (30)$$

$$(I_3 \otimes F_n) = \begin{bmatrix} F_n & 0 & 0 \\ 0 & F_n & 0 \\ 0 & 0 & F_n \end{bmatrix} \qquad (31)$$

$$\tilde{M}_1 = \begin{bmatrix} \Lambda_{C1} & \Lambda_{C2} & \Lambda_{C3} \\ \Lambda_{C4} & \Lambda_{C5} & \Lambda_{C6} \end{bmatrix}, \qquad (32)$$

where Λ_{Ci} is the (diagonal) eigenvalue matrix of block C_i.

5.2. *Systems with MIMO subsystems*

5.2.1. *Block circulant and block parallel systems.* The matrix C in equation (11) is block circulant if $c_1, c_2, \ldots, c_n$ all are blocks of dimension $n_o \times n_i$. Note that the individual blocks c_i need not be circulant, thus in general block circulant matrices are not matrices consisting of circulant blocks. For such a block circulant matrix C we can generate a block diagonal matrix

$$\tilde{C} = (F_n \otimes I_{n_o}) C (F_n \otimes I_{n_i})^H, \qquad (33)$$

where $\tilde{C} = \operatorname{diag}\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$, and $\gamma_1, \gamma_2, \ldots, \gamma_n$ all have dimension $n_o \times n_i$ and can be calculated from the blocks of C using

$$\begin{bmatrix} \gamma_{C1} \\ \gamma_{C2} \\ \vdots \\ \gamma_{Cn} \end{bmatrix} = (\sqrt{n} F_n \otimes I_{n_o})^H \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}. \qquad (34)$$

Proof. Follows from the proof of Theorem 5.6.4 in Davis (1979), by setting $B_k = I_{n_o} A_{k+1} I_{n_i}$. If $c_2 = c_3 = \cdots = c_n$ then we term the matrix C block parallel, and we have

$$\gamma_{C1} = c_1 + (n+1) c_2 \qquad (35)$$

$$\gamma_{C2} = \gamma_3 = \cdots = \gamma_n = c_1 - c_2. \qquad (36)$$

In this way, a symmetrically interconnected system C consisting of n units in parallel can be decomposed into one distinct subprocess γ_{C1} and $n-1$ equal subprocesses γ_{C2}.

5.2.2. *Combinations of block circulant systems.* If A is a block circulant matrix with $n \times n$ blocks, each of size $n_o \times n_i$ we have:

- A^H and A^T are block circulant matrices with $n \times n$ blocks of size $n_i \times n_o$.
- If A^{-1} exists, it is a block circulant matrix.
- If B is block circulant, consisting of $n \times n$ blocks of size $n_i \times n_b$, then AB is a block circulant matrix with blocks of size $n_o \times n_b$.
- In general, block circulant matrices do not commute, $AB \neq BA$.

5.2.3. *Systems consisting of blocks which are block circulant.* Consider a matrix N consisting of $m_r \times m_c$ blocks, each block being a block circulant matrix with $n \times n$ sub-blocks. Sub-blocks belonging to the same column of main blocks must have the same number of columns. Let n_i^c denote the number of columns of the sub-blocks in column c of main blocks. Likewise, sub-blocks belonging to the same row of main blocks must have the same number of rows, and we use n_o^r to denote the number of rows in the sub-blocks of blocks in row r of main blocks. To illustrate, consider

$$N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}. \qquad (37)$$

Let N_{11}, N_{12} N_{21} and N_{22} all be block circulant matrices consisting of $n \times n$ sub-blocks, and let the sub-blocks of N_{ij} have dimension $n_o^{ij} \times n_i^{ij}$. Then $n_o^{11} = n_o^{12} = n_0^1$, $n_o^{21} = n_o^{22} = n_o^2$, $n_i^{11} = n_i^{21} =$

n_i^1 and $n_i^{12} = n_i^{22} = n_i^2$, but n_o^1 may be different from n_o^2, and n_i^1 may be different from n_i^2. Introduce the matrices

$$\mathcal{F}_{\mathcal{L}} = \text{diag}\,\{F_n \otimes I_{n_o^r}\} \tag{38}$$

$$\mathcal{F}_{\mathcal{R}} = \text{diag}\,\{F_n \otimes I_{n_i^r}\}. \tag{39}$$

We then have that N can be 'diagonalized' by the following transformation

$$\tilde{N} = \mathcal{F}_{\mathcal{L}} N \mathcal{F}_{\mathcal{R}}^H, \tag{40}$$

where $\tilde{N}$ is a matrix consisting of blocks which are block diagonal, where each block of $\tilde{N}$ can be calculated from the corresponding block of N using equation (34).

6. CONTROLLER DESIGN FOR BLOCK PARALLEL SYSTEMS

In this section we consider control of plants described by block parallel transfer function matrices, that is, matrices with the block structure shown in equation (3). Note that the blocks $\alpha(s)$ and $\beta(s)$ need not be parallel. We also assume that all weighting matrices used to represent uncertainty or performance are block parallel matrices. Physically this means that the performance requirements and uncertainties are the same for all subsystems. We will find that the result of the controller design is that the optimal controller is block parallel, such that the distinct subprocess $\gamma_{G1}(s)$ of the plant $G(s)$ is controlled by the controller $\gamma_{K1}(s)$ and the $n-1$ identical subprocesses $\gamma_{G2}(s)$ are controlled by $n-1$ controllers all equal to $\gamma_{K2}(s)$, and that $\gamma_{K1}(s)$ and $\gamma_{K2}(s)$ both have dimension $n_i \times n_o$. The block parallel feedback controller $K(s)$ will then have diagonal blocks $k_{ii} = [\gamma_{K1} + (n-1)\gamma_{K2}]/n$ and offdiagonal blocks $k_{ij} = [\gamma_{K1} - \gamma_{K2}]/n$.

All H_∞-computations in this section are performed with the μ-Analysis and Synthesis Toolbox for MATLAB™.

6.1. *Optimal* H_∞ *control*

H_∞ control theory can be used for designing controllers which ensures that the closed-loop system satisfies singular value loop shaping specifications. For example, the standard 'mixed sensitivity' H_∞ problem is to minimize

$$\|M\|_\infty = \left\| \begin{matrix} W_O H \\ W_P S \end{matrix} \right\|_\infty, \tag{41}$$

where $S = (I + GK)^{-1}$ is the sensitivity function and $H = GK(I + GK)^{-1}$ the complementary sensitivity function. This objective may correspond to simultaneous optimization of robust stability with respect to output multiplicative uncertainty and nominal performance in terms of weighted sensitivity. For controller synthesis, the

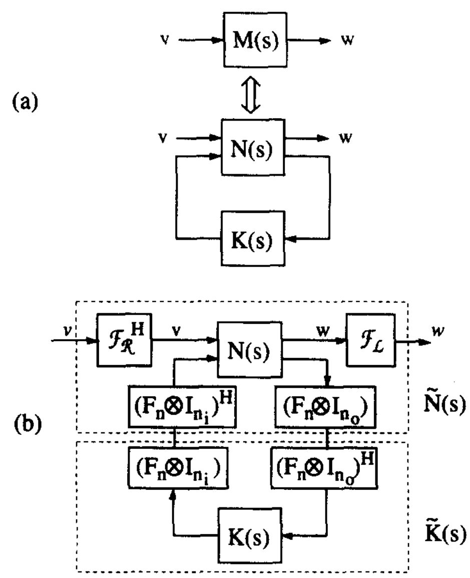


FIG. 4. (a) Expressing $M(s)$ as a linear fractional transformation of the controller $K(s)$. (b) Pre and postmultiplication of $M(s)$ by unitary matrices.

controller is 'pulled out' of M, by writing M as a Linear Fractional Transformation (LFT) of the controller (see Fig. 4a):

$$M(s) = N_{11}(s) + N_{12}(s)K(s) \times [I - N_{22}(s)K(s)]^{-1}N_{21}(s). \tag{42}$$

For example, for the mixed sensitivity problem in (41)

$$N_{11} = \begin{pmatrix} 0 \\ W_P \end{pmatrix}, \quad N_{12} = \begin{pmatrix} W_O G \\ -W_O G \end{pmatrix}, \quad N_{21} = I, \quad N_{22} = -G. \tag{43}$$

Theorem 1. Consider the design of a controller in order to minimize $\|M\|_\infty$ where the interconnection matrix M is a function of the plant $G(s)$, the controller $K(s)$ and some weights $W_i(s)$, and may be written as an LFT of the controller as given in (42). Assume that

(1) $G(s)$ is described by a block parallel transfer function matrix [equation (3)], consisting of $n \times n$ blocks, each block [$\alpha(s)$ or $\beta(s)$] of dimension $n_o \times n_i$.
(2) All weights $W_i(s)$ are block parallel matrices with blocks with dimensions compatible with the dimension of the blocks of $G(s)$.
(3) $M(s)$ has overall dimension $n \cdot r \times n \cdot c$.

Then the $n \cdot n_i \times n \cdot n_o$ optimal controller K is block parallel and is obtained by solving two independent H_∞-optimal controller problems, each involving minimization of the H_∞-norm of a $r \times c$ interconnection matrix to obtain a $n_i \times n_o$ controller.

Proof

(1) After expressing the matrix $M(s)$, whose H_∞ norm is to be minimized as a LFT of the controller $K(s)$, we find that N_{11}, N_{12}, N_{21} and N_{22} consist of blocks which are block parallel. This is so because all blocks of N_{11}, N_{12}, N_{21} and N_{22} can depend only on the plant G, the weights, identity matrices of appropriate dimension, and/or zero matrices of appropriate dimension, all of which are block parallel (see Section 5.2.2). In general $N_{22} = -G$.

(2) Premultiplication or postmultiplication of $M(s)$ by unitary matrices will not change the singular values, and will thus leave the H_∞ norm unchanged (Fig. 4b). We use the matrices $\mathscr{F}_{\mathscr{L}}$ and $\mathscr{F}_{\mathscr{R}}$ defined in equations (38) and (39).

$$\tilde{M} = \mathscr{F}_{\mathscr{L}} M \mathscr{F}_{\mathscr{R}}^H = \tilde{N}_{11} + \tilde{N}_{12}\tilde{K}[I - \tilde{N}_{22}\tilde{K}]^{-1}\tilde{N}_{21} \quad (44)$$

$$\tilde{N}_{11} = \mathscr{F}_{\mathscr{L}} N_{11} \mathscr{F}_{\mathscr{R}}^H \quad (45)$$

$$\tilde{N}_{12} = \mathscr{F}_{\mathscr{L}} N_{12} (F_n \otimes I_{n_i})^H \quad (46)$$

$$\tilde{N}_{21} = (F_n \otimes I_{n_o}) N_{21} \mathscr{F}_{\mathscr{L}}^H \quad (47)$$

$$\tilde{N}_{22} = (F_n \otimes I_{n_o}) N_{22} (F_n \otimes I_{n_i})^H. \quad (48)$$

Thus, since N_{11}, N_{12}, N_{21} and N_{22} all consist of blocks which are block parallel, $\tilde{N}_{11}$, $\tilde{N}_{12}$, $\tilde{N}_{21}$ and $\tilde{N}_{22}$ all consist of blocks which are block diagonal, the first sub-block in each block being distinct and the other $n-1$ sub-blocks equal (see upper part of Fig. 5).

(3) The structure of $\tilde{N}$ given in equations (45)–(48) implies that the controller synthesis problem can be decomposed into n non-interacting synthesis subproblems. To see this consider Fig. 5 where the matrix at the top may represent $\tilde{N}$ for the mixed-sensitivity problem in (41) where $\tilde{N}_{22} = \operatorname{diag}\{h_1, h_2, \ldots, h_2\}$ is the diagonalized plant. After permutations (reordering the inputs and outputs) we get the matrix at the bottom of Fig. 5. It is then apparent that the controller design problem consists of n independent subproblems, one distinct and $n-1$ identical.

(4) The controller $\tilde{K} = \operatorname{diag}\{\gamma_{K1}, \gamma_{K2}, \ldots, \gamma_{K2}\}$ in Fig. 4 will be block diagonal, the first block on the diagonal being distinct and the $n-1$ other blocks equal. Consequently,

$$\begin{aligned} K(s) &= (F_n \otimes I_{n_i})^H \tilde{K}(s)(F_n \otimes I_{n_o}) \\ &= (R_n \otimes I_{n_i})^T \tilde{K}(s)(R_n \otimes I_{n_o}) \quad (49) \end{aligned}$$

will be a block parallel matrix with the same structure as G. R_n is the real eigenvector matrix for a parallel matrix of dimension $n \times n$ (recall Section 4 and Table 1).

$$\begin{bmatrix} a_1 & & & & b_1 & & & \\ & a_2 & & & & b_2 & & \\ & & \ddots & & & & \ddots & \\ & & & a_2 & & & & b_2 \\ d_1 & & & & e_1 & & & \\ & d_2 & & & & e_2 & & \\ & & \ddots & & & & \ddots & \\ & & & d_2 & & & & e_2 \\ g_1 & & & & h_1 & & & \\ & g_2 & & & & h_2 & & \\ & & \ddots & & & & \ddots & \\ & & & g_2 & & & & h_2 \end{bmatrix}$$

$\updownarrow$ Permutations

$$\begin{bmatrix} a_1 & b_1 & & & & & & \\ d_1 & e_1 & & & & & & \\ g_1 & h_1 & & & & & & \\ & & a_2 & b_2 & & & & \\ & & d_2 & e_2 & & & & \\ & & g_2 & h_2 & & & & \\ & & & & \ddots & & & \\ & & & & & & a_2 & b_2 \\ & & & & & & d_2 & e_2 \\ & & & & & & g_2 & h_2 \end{bmatrix}$$

FIG. 5. Top: matrix with a special block structure. Bottom: same matrix after permuting the order of the inputs and outputs.

Remark 1. The same theorem holds for the H_2-optimal problem, since the Frobenius norm is also unitary invariant. The result for the H_2-case has been proved before (Sundareshan and Elbanna, 1991) using properties of the Riccati equations. However, the results of Sundareshan and Elbanna (1991) are stated for the somewhat less general case where both the matrices B and C of the state-space realization are block-diagonal. Note, for example, that the parallel system in (9) cannot be realized in this form.

Remark 2. We have shown that the optimal controller K has the same structure as $N_{22} = -G$, that is, $\tilde{K}$ has the same structure as $\tilde{N}_{22}$. If we use this as a starting point, then $\tilde{M}$ must have the same structure as the matrix at the top of Fig. 5 which may be permuted to give the block-diagonal structure at the bottom of Fig. 5. From this it is trivial to confirm that the H_∞-optimal controller for the overall problem may be obtained by solving two independent subproblems of much smaller dimension.

Remark 3. In general, $N_{22} = -G$ and $\tilde{N}_{22} = -\mathrm{diag}\{\gamma_{G1}, \gamma_{G2}, \ldots, \gamma_{G2}\}$ [See equations (33)–(36)]. Thus the controller design corresponds to designing controllers for each of the two subsystems corresponding to the two 'plants' $\gamma_{G1}(s)$ and $\gamma_{G2}(s)$.

Remark 4. Note that this corresponds to optimizing the H_∞ objective for both the systems corresponding to γ_{G1} and γ_{G2}. If $m_c = n_i = 1$ or $m_r = n_o = 1$, all directions in the H_∞ criterion are optimized. In contrast, the controller Doyle *et al.* (1989) terms the 'central solution' to the H_∞ synthesis problem only optimizes the worst direction in the overall H_∞ criterion. In general, the solution to the H_∞ controller synthesis problem is non-unique (Doyle *et al.*, 1989), since many controllers will achieve the optimum H_∞ norm in the worst direction, while doing equally well or better in the other directions. How to minimize the peak values of the singular values corresponding to directions other than the worst direction is a line of research, called super-optimal H_∞ control (Tsai *et al.*, 1988; Kwakernaak, 1986). Thus, we have here found a class of problems where the solution to the super-optimal H_∞ control problem is very simple.

Remark 5. The interconnection matrix $\tilde{N}(s)$ in Fig. 4b has the same number of states as $N(s)$ in Fig. 4a. The number of states in the controller $\tilde{K}$ will therefore equal the number of states in a controller based on regular H_∞ synthesis.

Remark 6. Recall from Section 4 that the choice of real eigenvector matrix R_n is not unique. However, if an eigenvector matrix R_n that is not orthogonal is chosen, the transpose in equation (49) and Fig. 6 must be replaced by inversion. A block diagram for $K(s)$ is shown in Fig. 6.

Example 3. Consider the process

$$G(s) = \frac{1}{(20s+1)(100s+1)} \times \begin{bmatrix} -0.25s+1 & 0.75s \\ 0.75s & -0.25s+1 \end{bmatrix} \quad (50)$$

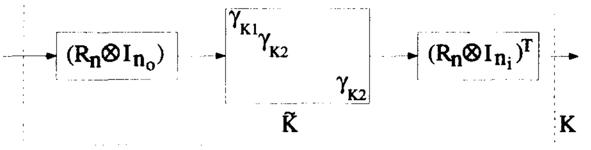


Fig. 6. Realization of a full block parallel controller K.

corresponding to the two subplants

$$\lambda_{G1}(s) = \frac{0.5s+1}{(20s+1)(100s+1)}; \quad \lambda_{G2}(s) = \frac{-s+1}{(20s+1)(100s+1)}. \quad (51)$$

We want to minimize the H_∞ norm of M in (41), with weights $W_O(s) = w_O(s)I$; $w_O(s) = 0.25s + 1/0.5s + 1$ and $W_P(s) = w_P(s)I$; $w_P(s) = 0.52s + 1/2s$. Using the approach outlined above we may solve this problem by considering the two SISO plants $\lambda_{G1}(s)$ and $\lambda_{G2}(s)$, and solve the two design subproblems

$$\min_{\lambda_{Ki}} \left\| \begin{matrix} w_O\lambda_{Gi}\lambda_{Ki}/(1+\lambda_{Gi}\lambda_{Ki}) \\ w_P/(1+\lambda_{Gi}\lambda_{Ki}) \end{matrix} \right\|_\infty; \quad i = 1, 2. \quad (52)$$

H_∞ synthesis for these two subplants gave H_∞ norms of 0.56 and 0.89, respectively. State space descriptions of the resulting 'controllers' λ_{K1} and λ_{K2} are given in the Appendix. The 'controllers' λ_{K1} and λ_{K2} may be combined into a regular controller with eight states using equation (49). Conventional H_∞ synthesis for the overall system also gave a controller with eight states which achieved a H_∞ norm of 0.89. However, in this case the peak of the singular value corresponding to the 'easy' direction was 0.73 (for the central solution), whereas our controller gave 0.56. In Fig. 7 are shown the responses to setpoint changes for the two controllers. We also show the response for an inverse-based controller which gives peak values of 0.89 for both singular values (see section on inverse-based controllers below). When the setpoint enters in the 'difficult' direction, the three responses are indistinguishable, whereas when the setpoint enters in the 'easy' direction it is clear that the controller synthesized using our method is superior and the inverse-based controller is worst.

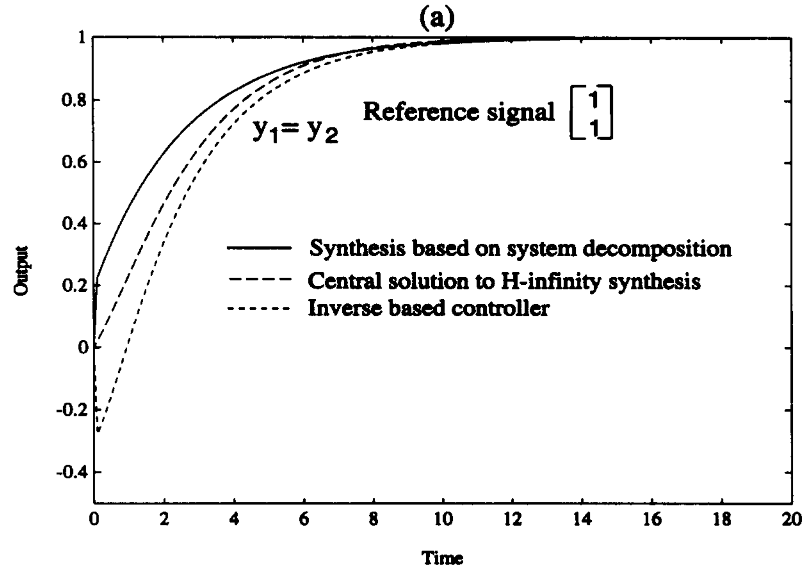
Example 4. Consider the 8×12 block parallel plant

$$G(s) = \begin{bmatrix} \alpha(s) & \beta(s) & \beta(s) & \beta(s) \\ \beta(s) & \alpha(s) & \beta(s) & \beta(s) \\ \beta(s) & \beta(s) & \alpha(s) & \beta(s) \\ \beta(s) & \beta(s) & \beta(s) & \alpha(s) \end{bmatrix}, \quad (53)$$

where $\alpha(s)$ and $\beta(s)$ are transfer function matrices with two outputs ($n_o = 2$) and three inputs ($n_i = 3$) with state space realizations

$$\alpha(s) = C_1(sI - A_1)^{-1}B_1 + D_1 \quad (54)$$

$$\beta(s) = C_2(sI - A_2)^{-1}B_2 + D_2 \quad (55)$$



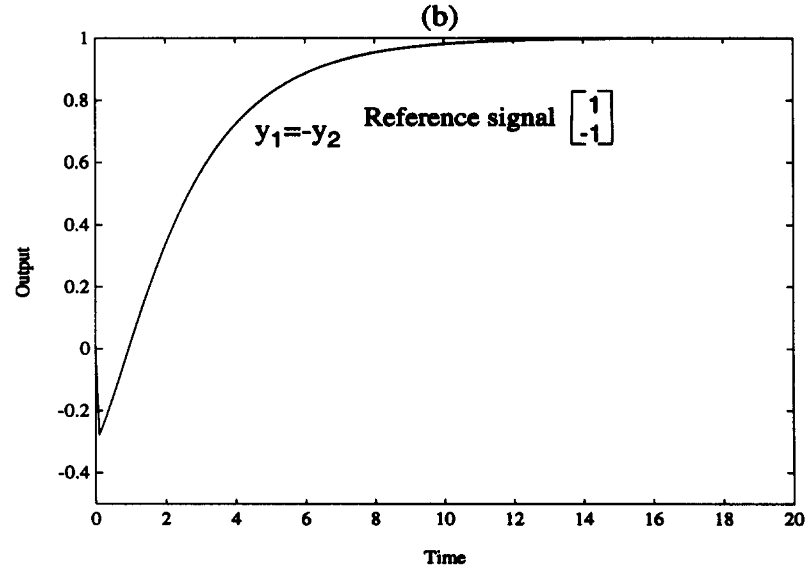


FIG. 7. Response of the closed-loop system to setpoint changes entering in the 'easy' (a) and 'difficult' (b) directions in Example 3. All three controllers yield an H_∞-norm of 0.89 for the overall problem.

and

$$A_1 = A_2 = \begin{bmatrix} -0.05 & 0 & 0 \\ 0 & -0.1 & 0 \\ 0 & 0 & -0.2 \end{bmatrix};$$

$$B_1 = \begin{bmatrix} 2 & 3 & 9 \\ 6 & 9 & 4 \\ 8 & 0 & 1 \end{bmatrix};$$

$$B_2 = \begin{bmatrix} 3 & -2 & 1 \\ -2 & 1 & 0 \\ -2 & 1 & 0 \end{bmatrix}; \tag{56}$$

$$C_1 = C_2 = \begin{bmatrix} 0.35 & 0.60 & 0.20 \\ 0.45 & -0.20 & 2.40 \end{bmatrix};$$

$$D_1 = D_2 = \begin{bmatrix} -1.25 & 0 & 0 \\ 0 & -1.25 & -1.25 \end{bmatrix}. \tag{57}$$

The design criterion is to minimize the H_∞-norm of

$$M = \begin{bmatrix} W_O GK(I+GK)^{-1} \\ W_P(I+GK)^{-1} \\ W_u K(I+GK)^{-1} \end{bmatrix} \tag{58}$$

with weights

$$W_O = 0.2\frac{4s+1}{0.4s+1} I_8$$

$$W_P = I_4 \otimes \begin{bmatrix} 0.5\frac{2.5s+1}{2.5s} & 0 \\ 0 & 0.5\frac{0.3s+1}{0.3s} \end{bmatrix}.$$

$$W_u = 0.1 I_{12}$$

We decompose the plant $G(s)$ into γ_{G1} and

$\gamma_{G2} = \gamma_{G3} = \gamma_{G4}$, each of dimension 2×3, using equation (33), and design one 3×2 controller γ_{K1} for the system corresponding to γ_{G1}, and one 3×2 controller γ_{K2} for the system corresponding to γ_{G2}. For both these design subproblems a H_∞-norm of 0.91 was achieved, and the same H_∞-norm of 0.91 was achieved for the overall system after calculating the controller K from γ_{K1} and γ_{K2} according to equation (49). The best value of the H_∞-norm achieved when using the same software to design a 12×8 controller for the overall plant was 0.99. The fact that the value was 0.99 instead of 0.91 demonstrates weaknesses in the synthesis software we have available.* However, more importantly it demonstrates that controller synthesis becomes simpler when the system is decomposed into problems with fewer states and of lower dimension. In our case, H_∞ synthesis for the overall plant gives a 36×8 interconnection matrix M with 28 states, whereas after decomposition we get two H_∞ synthesis problems, each with a 9×2 interconnection matrix with seven states [three states from the plant, two from $w_O(s)$ and two from $w_P(s)$]. The number of states in the final controller is 28 for both cases. State space descriptions of the 'controllers' γ_{K1} and γ_{K2} are given in the Appendix.

6.2. *Special case: inverse-based controllers*

Consider a parallel plant $G(s)$ consisting of SISO subsystems, and the special case when the following conditions hold:

Condition 1. The subplants $\lambda_{G1}(s)$ and $\lambda_{G2}(s)$, defined by equations (22) and (23), have the same RHP zeros.

Condition 2. All weights are scalar times identity matrices.

Condition 3. G and K only appear as products of each other in the problem statement [as in equation (41)].

Condition 4. The subplants $\lambda_{G1}(s)$ and $\lambda_{G2}(s)$ have the same pole excess.

In this case we need only perform one H_∞ design for the subplant λ_{G1} and obtain the controller λ_{K1}. Then the H_∞ norm that was obtained for the SISO 'system' corresponding to λ_{G1} can be obtained for the overall system by choosing

$$\lambda_{K2} = \cdots = \lambda_{Kn} = \lambda_{K1}\lambda_{G1}/\lambda_{G2}. \tag{59}$$

The result will be an inverse-based controller of the type

$$K(s) = k(s)G^{-1}(s), \tag{60}$$

thus effectively transforming the $n \times n$ H_∞ design problem to n identical SISO problems. Condition 1 ensures that any RHP zeros in the subplants cancel in equation (59), such that the controller is stable. If λ_{G2} has a larger pole excess than λ_{G1}, calculating λ_{K2} from equation (59) may result in an improper controller which is impossible to realize. Condition 4 ensures that this is not a problem. The Conditions 1–4 ensure that the same H_∞-norm is achievable for the subproblems corresponding to λ_{G1} and λ_{G2}. Thus, the H_∞ controller will be unique if the solution to the SISO subproblems corresponding to λ_{G1} and λ_{G2} are unique. Whereas Condition 2 will normally hold for SISO subsystems in parallel, Condition 3 may well be violated, e.g. if M contains a term like $W_u K(I + GK)^{-1}$ corresponding to a bound on the closed-loop transfer function from reference signal to manipulated variables.

Note that Conditions 1–4 imply that the same H_∞-norm is achievable for both design subproblems (corresponding to λ_{G1} and λ_{G2}), such that there is no need to consider superoptimality, as this is automatically achieved.

Example 5. To illustrate, consider Example 2 in Section 2, with four reactors in parallel ($n = 4$), and choose the values $k = 1$, $\tau = 100$ and $a = 0.7$. From equations (22) and (23) we then have that the subplants are

$$\lambda_{G1}(s) = \frac{3.1}{100s + 1}; \quad \lambda_{G2}(s) = \frac{0.3}{100s + 1}.$$

Consider the mixed H_∞-sensitivity problem in equation (41), with weights $W_P = w_P I$; $\{w_P = 0.5(10s + 1)/10s\}$ and $W_O = w_O I$; $\{w_O = 0.2(5s + 1)/(0.5s + 1)\}$. Thus Conditions 1–4 are all fulfilled. The magnitude of the weights w_P and w_O are shown together with the magnitude of the plant eigenvalues (= singular values) in Fig. 8.

A third order H_∞-optimal controller λ_{K1} was designed for the SISO 'plant' λ_{G1} according to equation (41) (with weights w_P and w_O substituted for W_P and W_O, respectively), achieving a H_∞-norm of 0.50. The magnitude of λ_{K1} is also shown in Fig. 8, and a state space description is given in the Appendix. After calculating λ_{K2} from λ_{K1} according to equation (59), and combining the two controllers λ_{K1} and λ_{K2} according to equation (49) to find the controller K for the full plant, we found that the same H_∞-norm was achieved for the full system as for the SISO case (end example 5).

There may exist an inverse-based controller achieving the optimal H_∞-norm also when

* The μ-tools and Robust Control toolboxes for MATLAB.

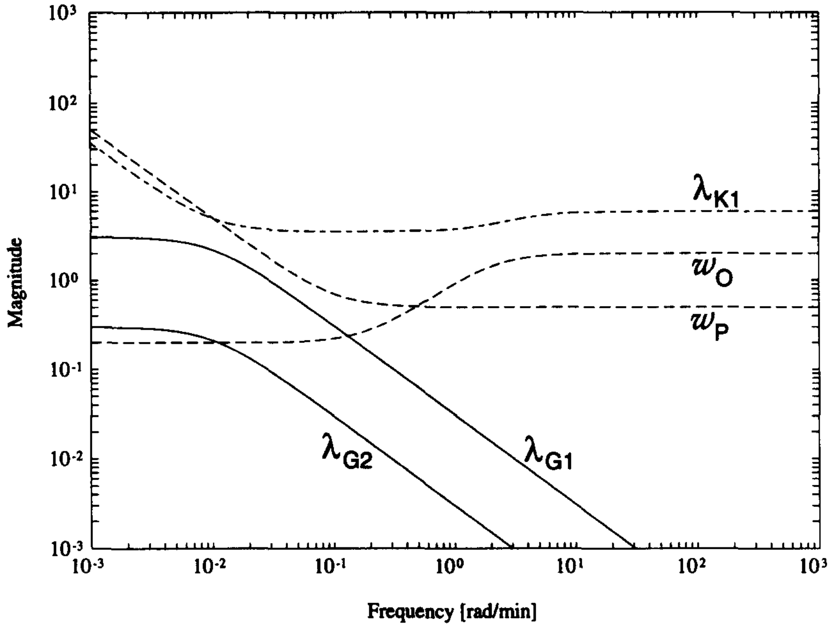


FIG. 8. Magnitudes of plant eigenvalues λ_{G1} and λ_{G2}, performance weight w_P, output weight w_O and first controller eigenvalue λ_{K1} for Example 5.

Conditions 1–4 are not fulfilled (or when the plant consists of MIMO subsystems), and in many cases this inverse-based controller can be found by first synthesizing the controller corresponding to the most difficult system direction and then use equation (59) to find the controller corresponding to the other direction. For such cases super-optimality will not be achieved by an inverse-based controller.

Example 3 continued. We see that Condition 1 above is violated, as λ_{G2} has a zero at $s = 1$, where λ_{G1} has no RHP zero. Designing an H_∞-controller for the most difficult subplant λ_{G2}, and calculating λ_{K1} from λ_{K2} using equation (59) (the reverse would yield an internally unstable closed-loop system) yields an equalizing solution with a H_∞-norm of 0.89 in both directions. The resulting poor response in the easy direction corresponding to λ_{G1} is shown by the 'Inverse based controller' in Fig. 7a.

6.3. *Controller implementation*

One possible controller implementation is given in equation (49). However, an infinite number of eigenvector matrices X may be used instead of R_n [provided the transpose in equation (49) is replaced with inversion whenever X is not orthogonal]. Possible choices for X for $n = \{2, 3, 4\}$ are given in Table 1, another possible choice is given by equation (26).

A distributed implementation of the optimal controller may be desirable in some cases. In such cases, one may use the physical insight from the Introduction, and control the 'distance from average' locally, and only correct for deviations in the 'average' output centrally. This is illustrated schematically in Fig. 9. Mathematically, this can be expressed as

$$K = \operatorname{diag}\{\gamma_{K2}\}\left(I_{n\cdot n_o} - \frac{1}{n}\begin{bmatrix} I_{n_o} & \cdots & I_{n_o} \\ \vdots & \vdots & \vdots \\ I_{n_o} & \cdots & I_{n_o}\end{bmatrix}\right) + \begin{bmatrix} I_{n_i} \\ \vdots \\ I_{n_i}\end{bmatrix}\gamma_{K1}\frac{1}{n}[I_{n_o} \cdots I_{n_o}] \quad (61)$$

which is the same K as in equation (49). One disadvantage with a distributed implementation is that it increases the number of states in the overall controller K. If the controller in equation (49) has kn states, a realization according to equation (61) will result in a controller with $k(n+1)$ states.

Advantages with the distributed implementation in Fig. 9 are:

(1) It is transparent and makes implementation, tuning and correction easier.
(2) There are cases, e.g. for electric power generators in a network, where the individual units are separated by large distances, and it may become impractical to calculate all control moves centrally, or one may want to update the central control moves less frequently.
(3) If the connection to the central control system is lost, then some local control is maintained since the plant is controlled by the controller $K' = \operatorname{diag}\{\gamma_{K2}\}$, and the

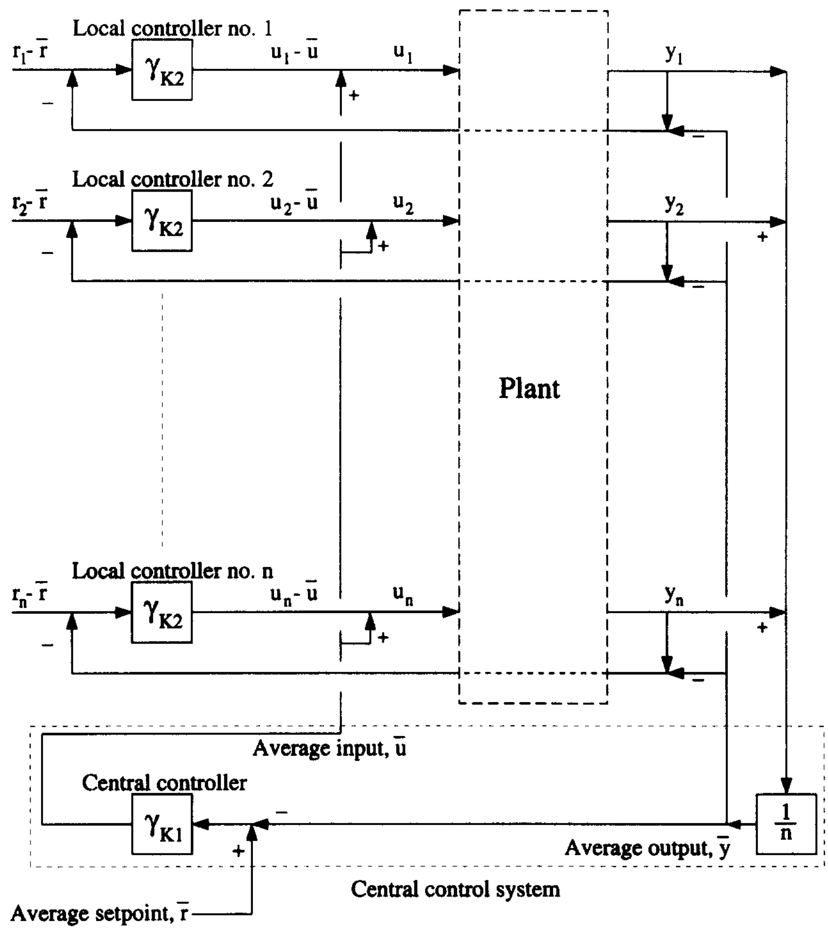


FIG. 9. Distributed implementation of a full block parallel controller K.

closed-loop system will remain stable provided $\gamma_{K2}(s)$ stabilizes $\gamma_{G1}(s)$ in addition to $\gamma_{G2}(s)$.

Remark to advantage 3. It will not always be possible to find one controller $\gamma_{K2}(s)$ which stabilizes both $\gamma_{G1}(s)$ and $\gamma_{G2}(s)$. For instance, if integral action is required, a necessary requirement for the existence of a controller $\gamma_{K2}(s)$ which stabilizes both $\gamma_{G1}(s)$ and $\gamma_{G2}(s)$ is that the determinants of $\gamma_{G1}(0)$ and $\gamma_{G2}(0)$ have the same sign (e.g. Grosdidier *et al.*, 1985). For the physically motivated examples of parallel processes in Section 3, the bounds in equation (10) imply that $\lambda_{G1}(0)$ and $\lambda_{G2}(0)$ always have the same sign. There are also cases when $\gamma_{G1}(s)$ has a severe RHP zero that is not present in $\gamma_{G2}(s)$. In such cases, requiring $\gamma_{K2}(s)$ to stabilize both $\gamma_{G1}(s)$ and $\gamma_{G2}(s)$ will imply that the control of the 'distance from average' will have to be made much slower than would otherwise be the case.

6.4. *A note on decentralized control*

Note from above that if the central controller fails we are left with a block-diagonal controller K', that is, a block-decentralized control system with identical blocks $\gamma_{K2}(s)$. A natural question to ask is whether the optimal block-decentralized controller for block parallel processes should have identical blocks. Intuitively, this seems reasonable (e.g. Lunze, 1989), since all the local plants are identical and it seems logical that their optimal local controllers should be identical. However, we show with a simple example that decentralized control with identical local controllers cannot be optimal for all cases.

Example 6. Consider the plant

$$G(s) = \frac{1}{100s+1}\begin{bmatrix} 1 & -0.6 & -0.6 \\ -0.6 & 1 & -0.6 \\ -0.6 & -0.6 & 1 \end{bmatrix}. \quad (62)$$

The corresponding subplants are:

$$\lambda_{G1}(s) = \frac{-0.2}{100s+1}; \quad \lambda_{G2}(s) = \lambda_{G3}(s) = \frac{1.6}{100s+1}.$$

The objective is to design a stabilizing decentralized controller with integral action. We assume that the local input is used to control the local output, that is, a pairing corresponding to the diagonal elements of $G(s)$ is used. Since the sign of $\lambda_{G1}(0)$ differs from the sign of $\lambda_{G2}(0) = \lambda_{G3}(0)$, we know that there does not exist a decentralized controller with identical controllers in all channels and integral action that will stabilize $G(s)$. On the other hand, the controller

$$C(s) = \text{diag}\{c_1(s), c_2(s), c_3(s)\}$$
$$c_1(s) = c_2(s) = 100\frac{100s+1}{100s};$$
$$c_3(s) = -10\frac{100s+1}{100s} \quad (63)$$

which has different controller elements, results in a stable closed-loop system.

Remark. In this example the interaction parameter $a = \beta(0)/\alpha(0) = -0.6$ is outside the limits $-0.5 \le a \le 1$ in equation (10), which we derived in Section 3 for some physically motivated systems. Nevertheless, this example demonstrates that identical local controllers do not in general give the optimal decentralized controller for parallel plants, and numerical evidence (Lundström *et al.*, 1991) suggests that the same holds also when a is within the bounds of equation (10) such that $\lambda_{G1}(0)$ and $\lambda_{G2}(0)$ have the same sign.

7. CONTROLLER DESIGN FOR BLOCK SYMMETRIC CIRCULANT SYSTEMS

The results in the previous section on H_2- and H_∞-optimal control of block parallel systems are easily generalized to processes described by block symmetric circulant transfer function matrices of the form in equation (1). If the individual blocks α and β_i have dimension $n_i \times n_o$, G will have dimension $n \cdot n_i \times n \cdot n_o$. Note that it is only the block structure of G that needs to be symmetric, the individual blocks α, $\beta_1, \beta_2, \ldots$, and thus G itself, need not be symmetric. If α, β_1, $\beta_2, \ldots$ are of dimension 1×1, $G(s)$ is termed a symmetric circulant matrix (recall Section 4.2).

Let k be the number of independent blocks γ_{Gi} in equation (34). In general, if n is an even number, $k = n/2 + 1$, and if n is an odd number, $k = (n-1)/2 + 1$.*

Theorem 2. Consider the design of a controller in order to minimize $\|M\|_\infty$ where the interconnection matrix M is a function of the plant $G(s)$, the controller $K(s)$ and some weights $W_i(s)$, and may be written as an *LFT* of the controller as given in (42). Assume that:

(1) $G(s)$ is described by a block symmetric circulant transfer function matrix [equation (1)], consisting of $n \times n$ blocks, each block [$\alpha(s)$ or $\beta_i(s)$] of dimension $n_o \times n_i$.
(2) All weights $W_i(s)$ are block symmetric circulant matrices with blocks with dimensions compatible with the dimension of the blocks of $G(s)$.
(3) $M(s)$ has overall dimension $n \cdot r \times n \cdot c$.

Then the $n \cdot n_i \times n \cdot n_o$ optimal controller K is block symmetric circulant and is obtained by solving k independent H_∞-optimal controller problems, each involving minimization of the H_∞-norm of a $r \times c$ interconnection matrix to obtain a $n_i \times n_o$ controller.

Proof. Similar to the proof of Theorem 1, using the same diagonalizing transformation $\tilde{N} = \mathcal{F}_{\mathcal{L}} N \mathcal{F}_{\mathcal{R}}^H$ (40) where $\mathcal{F}_{\mathcal{L}}$ and $\mathcal{F}_{\mathcal{R}}$ have the unitary Fourier matrix F_n as blocks, and making use of the fact that the H_∞-norm is unitary invariant.

The same theorem holds for H_2-optimal control since also the Frobenius norm is unitary invariant. For H_∞-optimal control we get super-optimality where the H_∞-norm is optimized in n directions. The resulting controller K can be found from equation (49). Note, however, that one has to exercise some care when finding the real matrix R used for controller realization in equation (49). Recall from Section 4 that only linear combinations of eigenvectors corresponding to identical eigenvalues may be used to find real eigenvectors. The matrix R found from equation (16)–(19) fulfills this requirement.

8. CONCLUSIONS

For parallel systems [equation (3)], instead of considering a plant of dimension $n \cdot n_o \times n \cdot n_i$ we can consider two subplants, each of dimension $n_o \times n_i$ when designing a H_∞-optimal or H_2-optimal controller.

For plants described by block symmetric circulant processes [equation (1)], we have to consider k subplants, where $k = n/2 + 1$ if n is even and $k = (n-1)/2 + 1$ if n is odd.

For both cases the optimal controller has the same structure as that of the plant. For H_∞-synthesis the resulting block parallel controller optimizes the H_∞ criterion in n directions.

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* Also note that if n is an even number, $k = v + 1$, if n is odd $k = v$, see Section 4.2.

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APPENDIX: STATE SPACE DESCRIPTIONS OF CONTROLLERS FOUND IN THE EXAMPLES

Table A.1. State space description λ_{k1} for the controller found in example 3

A			
$-4.993E-07$	$-1.633E-04$	$2.858E-02$	$3.357E-06$
$-1.853E-04$	-0.1051	$-7.644E+02$	$3.392E-02$
$2.929E-02$	$7.995E+02$	$-2.908E+03$	-0.5641
$-3.220E-06$	$-6.248E-02$	0.2264	-2.000
B	**C^T**	**D**	
0.6859	0.6972	0	
$1.436E+02$	$1.454E+02$		
$-2.420E+04$	$-2.419E+04$		
1.891	-2.346		

Table A.2. State space description of λ_{K2} for the controller found in example 3

A			
$-5.009E-07$	$-2.046E-02$	$-1.183E-05$	$3.129E-05$
$-2.001E-02$	$1.294E+03$	$-6.275E+02$	3.182
$1.352E-05$	$6.276E+02$	$-1.430E-04$	$1.056E-03$
$-2.966E-05$	-3.182	$1.276E-03$	-1.611
B	**C^T**	**D**	
0.5429	0.5314	0	
$1.353E+04$	$1.353E+04$		
-5.061	4.495		
16.61	-16.61		

Table A.3. State space description of γ_{K1} for the controller found in example 4

diag $\{A\}$	B	
-231.439	0.0134	2.9209
-2.470	1.2090	-2.1960
-0.823	-0.3021	0.5568
-1.064	-0.3085	0.0152
-0.176	-0.1196	-3.4066
$-1E-08$	$1.82E-04$	4.6785
$-2E-09$	0.7891	$3.1E-09$
	C^T	
0.0565	-41.2565	37.2301
0.0455	-0.4676	0.3644
0.1665	-0.1291	-0.0588
0.0720	-0.1922	0.0877
-0.0221	0.0114	-0.0050
-0.0331	-0.0253	0.0350
-0.0395	-0.0237	0.0410
	D	
	-0.0125	0
	0	$-6.25E-03$
	0	$-6.25E-03$

TABLE A.4. STATE SPACE DESCRIPTION OF γ_{K2} FOR THE CONTROLLER FOUND IN EXAMPLE 4

diag {A}	B	
−3591.5	0.0107	−20.3237
−266.7	0.0330	−24.2771
−130.4	0.0257	15.3908
−1.120	0.0663	1.031E−03
−0.0521	0.0734	−0.3711
−3.9E−08	−3.0E−10	−0.6377
−4.7E−09	−0.2745	−6.5E−11
	C^T	
−1241	316.2	−171.4
53.76	41.55	38.59
−18.46	46.87	21.23
−9.744E−03	0.2968	0.1639
1.219E−02	−2.804E−03	4.267E−03
−3.593E−02	2.020E−02	2.295E−03
6.100E−03	−1.288E−02	−4.801E−03
	D	
	0	0
	0	0
	0	0

TABLE A.5. STATE SPACE DESCRIPTION OF λ_{K1} FOR THE CONTROLLER FOUND IN EXAMPLE 5

	A		B
−1.00E−07	−1.73E−03	1.29E−06	0.187
−1.73E−03	−4.44E+05	1.11E+03	1.63E+0.3
−1.30E−06	−1.11E+03	−0.60	1.20
	C		D
0.188	1.62E+03	−1.21	0