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Robust control of time-delay systems using the Smith predictor

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In this paper we study the robust performance of time-delay systems using the structured singular value (μ). By employing the properties of the Smith predictor, we are able to convert it to a delay-free problem. This not only simplifies the analysis and design but also avoids the problem of the rational time-delay approximation, which is usually needed. A SISO design example is given.

Notation

G	the nominal model of the plant
G_0	the delay-free part of G
G_p	the actual plant
K	the overall controller
K_0	the primary controller in the Smith predictor control scheme
$K_{0\mu}$	μ-optimal primary controller
K_{0r}	order-reduced μ-optimal primary controller
K_μ	μ-optimal controller
NP	nominal performance
NP_0	nominal performance in terms of the delay-free system
NS	nominal stability
RP	robust performance
RP_0	robust performance in terms of the delay-free system
RS	robust stability
S	nominal sensitivity function
S_0	nominal delay-free sensitivity function
S_p	perturbed (actual) sensitivity function
S_{p0}	perturbed delay-free sensitivity function
w_A	additive uncertainty weight
w_I	input multiplicative uncertainty weight
w_o	output multiplicative uncertainty weight
w_P	performance weight
w_{P0}	performance weight on the delay-free sensitivity function
Δ	uncertainty block structure
Δ_A	block structure for additive uncertainty
Δ_I	block structure for input multiplicative uncertainty
Δ_o	block structure for output multiplicative uncertainty
$\bar{\sigma}$	largest singular value
$\underline{\sigma}$	smallest singular value
μ	structured singular value
γ	condition number
θ	time delay

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1. Introduction

The dynamic behaviour of many industrial processes contains inherent time-delays (dead times). Processes with time-delays are inherently difficult to control, i.e. it is difficult to achieve satisfactory performance. The reason is that time-delays limit the achievable bandwidth and the use of high gain feedback. Time-delays also make the design problem more difficult since the presence of a time-delay complicates the analytical and computational aspects of system design. Moreover, some advanced analysis and design methods are incapable of dealing directly with time-delay systems.

A standard feedback control system is shown in Fig. 1, where G_p represents the true ('perturbed') plant with uncertainty and K is the controller. Smith (1957, 1959) proposed a control scheme for time-delay systems, shown in Fig. 2(*a*), in which there is an additional feedback loop, called the Smith predictor, around the 'primary' controller K_0. Here G represents the nominal plant and G_0 the model without time delay. The idea is that this feedback loop predicts the response that would have been made without a delay. In the case of perfect model ($G = G_p$), the Smith predictor minimizes (but not eliminates) the detrimental effect of the time-delay. Palmor (1982) showed that the Smith predictor is an inherent result when minimum variance control is applied for time delay processes. Morari and Zafiriou (1989) related the Smith predictor structure to the IMC structure. The Smith predictor has been extended to multivariable systems with single delay (Alevisakis and Seborg 1973) and with multiple delays (Ogunnaike and Ray 1979).

Marshall (1979) and Palmor and Shinnar (1981), as well as many others, noted that a Smith predictor controller may be very sensitive to model–plant mismatch. The structured singular value (μ) was introduced by Doyle (1982) to analyse the robust stability of multivariable feedback systems with structured uncertainty. It is also used for the robust performance problem since the latter can be transformed to an equivalent robust stability problem with structured perturbations. μ is a worst-case measure, so it provides reliable results. In this paper we will address the problem of the robust performance of the Smith predictor controllers using the structured singular value as our analysis and design tool.

A few authors have worked on robust control of time-delay systems, e.g. Laughlin *et al.* (1987) who systematically studied the robust performance of the SISO Smith predictor within the IMC structure using several design methods. In the present paper, however, we employ the properties of the Smith predictor and develop a delay-free design method. One well-known advantage of the Smith predictor control structure is that it makes the 'primary' controller design problem delay-free if we consider only nominal performance.

In § 2, we give an introduction to μ-analysis and synthesis. The main results are presented in § 3. In § 4, we illustrate our design method by a SISO design

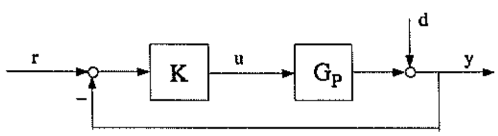


Figure 1. Standard feedback system.

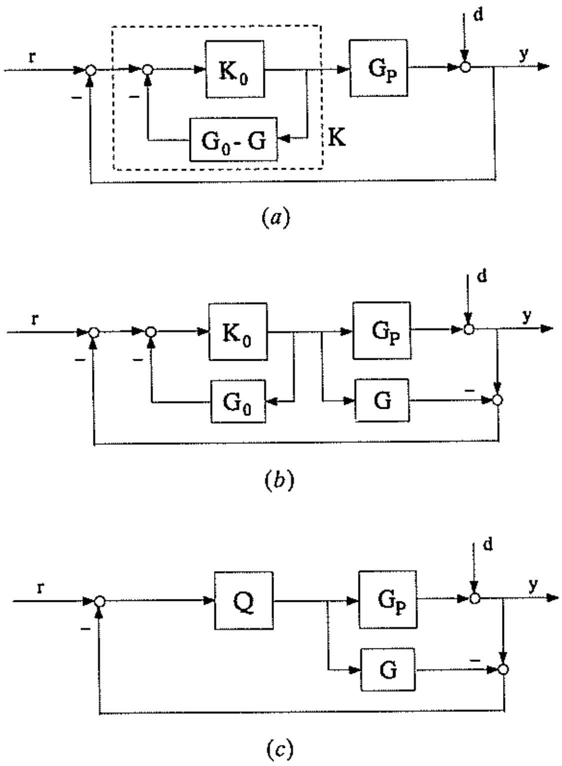


Figure 2. (*a*) Smith predictor control structure; (*b*) rearranged Smith predictor; (*c*) IMC structure.

example which was also studied by Laughlin *et al.* (1987). Results are compared and discussed.

2. Review of μ-analysis and synthesis for robust performance

Let us first review some results for the standard feedback system shown in Fig. 1. Robust performance means that the performance objectives are achieved for all possible plants. In the μ-analysis the set of all possible plants $G_p(s)$ is specified by a nominal plant model $G(s)$ and a normalized perturbation block Δ with magnitude $|w(j\omega)|$ (we assume a scalar 'weight' $w(s)$). We shall consider uncertainty to be multiplicative or additive. The uncertainty may also have 'structure', and in this case the specific structure of Δ should be given. Performance requirements are put on sensitivity matrix $S = (I + GK)^{-1}$, i.e. the transfer matrix from disturbance d or command r to control error e. A small sensitivity matrix corresponds to a small control error.

In order to achieve robust performance we also need nominal stability, robust stability and nominal performance.

Nominal stability (NS)

The feedback system shown in Fig. 1 should be internally stable.

Robust stability (RS)

If nominal stability holds for the standard feedback system, then (Doyle *et al.* 1982)

(1) robust stability will be guaranteed for *additive* uncertainty, $G_p = G + w_A\Delta_A$, of magnitude $w_A(s)$ if and only if

$$\mu_{\Delta_A}(w_A K(I + GK)^{-1}) = \mu_{\Delta_A}(w_A KS) < 1, \quad \forall\omega \tag{1}$$

(2) robust stability will be guaranteed for *input multiplicative* uncertainty, $G_p = G(I + w_I\Delta_I)$, of magnitude $w_I(s)$ if and only if

$$\mu_{\Delta_I}(w_I K(I + GK)^{-1}G) = \mu_{\Delta_I}(w_I KSG) < 1, \quad \forall\omega \tag{2}$$

(3) robust stability will be guaranteed for *output multiplicative* uncertainty, $G_p = (I + w_o\Delta_o)G$, of magnitude $w_O(s)$ if and only if

$$\mu_{\Delta_o}(w_o GK(I + GK)^{-1}) = \mu_{\Delta_o}(w_o GKS) < 1, \quad \forall\omega \tag{3}$$

where Δ_A, Δ_I and Δ_o denote uncertainty structures. The weights may easily be generalized to be matrix-valued although we have assumed scalar uncertainty weights.

Nominal performance (NP)

Nominal performance is here defined in terms of the weighted nominal sensitivity matrix $S = (I + GK)^{-1}$. We have

$$\mathrm{NP} \Leftrightarrow \bar{\sigma}(w_P(I + GK)^{-1}) < 1, \quad \forall\omega \tag{4}$$

where w_P is the performance weight. The inverse of $w_P(\omega)$ represents the required performance bound of $\bar{\sigma}(S)$ at each frequency.

Robust performance (RP)

The robust performance requirement can be expressed as

$$\mathrm{RP} \Leftrightarrow \bar{\sigma}(w_P(I + G_pK)^{-1}) < 1, \quad \forall\omega \quad \forall G_p \tag{5}$$

Equation (5) is difficult to test. An equivalent but computationally more useful alternative to (5) is the following μ-condition (Doyle *et al.* 1982)

$$\mathrm{RP} \Leftrightarrow \mu_\Delta(N) < 1, \quad \forall\omega \tag{6}$$

or equivalently

$$\mathrm{RP} \Leftrightarrow \mu_{RP} = \sup_\omega \mu_\Delta(N) < 1 \tag{7}$$

here Δ is the block structure including the true uncertainty structure and the ficticious uncertainty from the performance block.

(1) In the case of additive uncertainty N is given by

$$N_A = \begin{bmatrix} -w_A KS & -w_A KS \\ w_P S & w_P S \end{bmatrix} \tag{8}$$

(2) In the case of input multiplicative uncertainty

$$N_I = \begin{bmatrix} -w_I KSG & -w_I KS \\ w_P SG & w_P S \end{bmatrix} \tag{9}$$

(3) In the case of output multiplicative uncertainty

$$N_o = \begin{bmatrix} -w_o GKS & -w_o GKS \\ w_P S & w_P S \end{bmatrix} \tag{10}$$

From the above definition of N, we see in each case

$$\text{RS} \Leftrightarrow \mu(N_{11}) < 1, \quad \forall \omega \tag{11}$$

$$\text{NP} \Leftrightarrow \mu(N_{22}) = \bar{\sigma}(N_{22}) < 1, \quad \forall \omega \tag{12}$$

Since $\mu(N_{11})$ and $\mu(N_{22})$ are always less or equal to $\mu(N)$, RP will automatically guarantee RS and NP. μ provides a useful tool for robustness analysis. It can also be used for robustness synthesis by minimizing μ_{RP} to obtain the 'μ-optimal' controller.

μ-optimal controller

The $D-K$ iteration is a combination of μ-analysis and H_∞-synthesis, and is supposed to give approximately the μ-optimal controller (Doyle 1982). The objective is to solve

$$\min_{K,D} \|DN(K)D^{-1}\|_\infty \tag{13}$$

by iteratively solving for K and D: for fixed D, we obtain the H_∞-optimal controller. For fixed K, we compute $\mu(N(K))$ and get the optimal D-scaling. The software used in this paper is the μ-toolbox for MATLAB developed by Balas *et al.* (1991).

Direct application of μ to time delay systems

The above μ analysis and synthesis procedures also apply to plants with time delays. However, in order to synthesize the μ-optimal controller, we need to approximate the time delay by a rational transfer function, because the H_∞ synthesis software we use cannot deal with irrational terms. We will show in the design example, that this rational time-delay approximation may cause serious performance deterioration. Therefore, we will, in this paper, use the Smith predictor structure where μ can be used in a much simpler way. In the μ-analysis, it is *not* necessary to approximate the time delay.

3. Robust performance of time-delay systems using the Smith predictor

In Fig. 2 (*a*), $G_p(s) = \{\hat{g}_{ij} e^{-\hat{\theta}_{ij}s}\}$ is the actual plant, $G(s) = \{g_{ij} e^{-\theta_{ij}s}\}$ is the nominal model, and $G_0(s) = \{g_{ij}\}$ is the delay-free part of G. The overall controller $K(s)$ includes the 'primary' controller $K_0(s)$ for the delay-free system and the Smith predictor as shown in Fig. 2 (*a*). We have

$$K(s) = K_0(I + (G_0 - G)K_0)^{-1} \tag{14}$$

We also introduce the delay-free sensitivity matrix

$$S_0 = (I + G_0 K_0)^{-1} \tag{15}$$

(The corresponding delay-free system is here defined as the feedback control system consisting of G_0 and the 'primary' controller K_0.)

Nominal stability (NS)

The time delay control system with the Smith predictor shown in Fig. 2 (*a*) should be (internally) stable.

To consider stability, we rearrange the Smith predictor as shown in Fig. 2 (*b*). This is equivalent to the IMC-structure of Morari shown in Fig. 2 (*c*), and we see that the Smith predictor and the IMC control are equivalent when $Q = K_0(I + G_0K_0)^{-1}$. Hence we have nominal stability of the time-delayed control systems if and only if both Q and G are stable, i.e. the corresponding delay-free control system and the plant G must be stable.

Conclusion: Since the Smith predictor can only be applied for stable plants, the nominal stability of the closed loop feedback system shown in Fig. 2 (*a*) is equivalent to the nominal stability of the corresponding delay-free control system. □

Robust stability (RS)

The following robust stability results are obtained from (1)–(3) by noting that (see Fig. 2)

$$Q = K(I + GK)^{-1} = K_0(I + G_0K_0)^{-1} \tag{16}$$

If nominal stability holds, then

(1) robust stability will be guaranteed for *additive* uncertainty bounded by $w_A(s)$ if and only if

$$\mu_{\Delta_A}(w_A K_0 S_0) < 1, \quad \forall\omega \tag{17}$$

(2) robust stability will be guaranteed for *input* multiplicative uncertainty bounded by $w_I(s)$ if and only if

$$\mu_{\Delta_I}(w_I K_0 S_0 G) < 1, \quad \forall\omega \tag{18}$$

If we assume $G = G_0D$, $D = \text{diag}\{e^{-\theta_i s}\}$, i.e. the time delays are on the inputs only, then (18) is equivalent to

$$\mu_{\Delta_I}(w_I K_0 S_0 G_0) < 1, \quad \forall\omega \tag{19}$$

(3) robust stability will be guaranteed for *output* multiplicative uncertainty bounded by $w_O(s)$ if and only if

$$\mu_{\Delta_O}(w_O G K_0 S_0) < 1, \quad \forall\omega \tag{20}$$

If, similarly, we assume $G = DG_0$, $D = \text{diag}\{e^{-\theta_i s}\}$, i.e. the time delays are on the *output* only, then (20) is equivalent to

$$\mu_{\Delta_O}(w_O G_0 K_0 S_0) < 1, \quad \forall\omega \tag{21}$$

Note that (17), (19) and (21) are conditions on the delay-free systems only. (19) and (21) follow from the following property of μ since D is a unitary matrix.

Lemma (Doyle 1982): *If U is a unitary matrix and has the same block structure as Δ, then*

$$\mu_\Delta(UM) = \mu_\Delta(M) = \mu_\Delta(MU) \tag{22}$$

Note that (19) and (21) hold for both the cases when the uncertainty is unstructured (Δ is a full matrix) and structured (Δ is a block-diagonal matrix). This follows since the structure of D is contained in the structure of Δ in both cases.

Conclusion: When the Smith predictor is used, it is interesting to note that not only is the nominal stability of the time-delay control system exactly equivalent to that of the corresponding delay-free control system, but also the robust stability conditions (17), (19) and (21) are the same as those of the delay-free system. □

Nominal performance (NP)

As defined before

$$\mathrm{NP} \Leftrightarrow \bar{\sigma}(w_{\mathrm{P}}S) < 1, \quad \forall \omega \tag{23}$$

However, we want to obtain a delay-free design procedure. We then have to define performance in terms of the corresponding delay-free system

$$\mathrm{NP}_0 \Leftrightarrow \bar{\sigma}(w_{\mathrm{P0}}S_0) < 1, \quad \forall \omega \tag{24}$$

Here the subscript 0 in w_{P0} is used to show explicitly this weight is on the delay free sensitivity function.

Of course, NP is our real objective, and we will try to achieve this by satisfying NP_0. We need to obtain a reasonable weight w_{P0}. Ideally we want to select w_{P0} such that NP and NP_0 are equivalent, that is such that $\bar{\sigma}(w_{\mathrm{P0}}S_0) = \bar{\sigma}(w_{\mathrm{P}}S)$. However, this is not possible before we start the design, because S_0 and S are unknown at this point. Intuitively we expect that we must use tighter performance specifications on S_0 than on S, that is, w_{P0} must be larger in magnitude than w_{P}. This is confirmed by considering the following identity

$$S = I - GK(I + GK)^{-1} = I - GG_0^{-1}(I - S_0) = (I - GG_0^{-1}) + GG_0^{-1}S_0 \tag{25}$$

We want S small. The first term in the right side of (25) is an unavoidable error, and the second term is a 'delayed' error of the delay-free system. We can affect this second term by selecting K_0, and in the limiting case, without model uncertainty, we may even approach perfect control ($S_0 = 0$).

For the case with time-delays on the plant outputs only we have $G = DG_0$, where $D = \mathrm{diag}\{\mathrm{e}^{-\theta_i}\}$ and $GG_0^{-1} = D$. It then follows that

$$S = (I - D) + DS_0 \tag{26}$$

and

$$|\bar{\sigma}(S) - \bar{\sigma}(S_0)| \leqslant \bar{\sigma}(I - D) \tag{27}$$

Here $\bar{\sigma}(I - D) = |1 - \mathrm{e}^{-\mathrm{j}\theta_{\mathrm{m}}\omega}|$, where θ_{m} is the largest time-delay in the output channels. This term is approximately $|\theta_{\mathrm{m}}s|$ at low frequencies, reaches 1 at $\omega \approx 1/\theta_{\mathrm{m}}$, and oscillates between 0 and 2 at high frequencies. It then follows for the case of output time-delays, that the difference between $\bar{\sigma}(S)$ and $\bar{\sigma}(S_0)$ is relatively small. Hence, specifying NP_0 is a reasonable and often simpler alternative. Obviously this always holds for SISO systems where we have $GG_0^{-1} = D = \mathrm{e}^{-\theta_s}$.

For the general MIMO case (i.e. the time-delays may not be on the output), we can still try to specify performance in terms of NP_0. From (25) we have

$$\bar{\sigma}(S) \leq \bar{\sigma}(I - GG_0^{-1}) + \bar{\sigma}(GG_0^{-1}S_0) \leq \bar{\sigma}(I - GG_0^{-1}) + \bar{\sigma}(GG_0^{-1})\bar{\sigma}(S_0) \quad (28)$$

By choosing

$$|w_{P0}^{-1}| \leq (|w_P^{-1}| - \bar{\sigma}(I - GG_0^{-1}))/\bar{\sigma}(GG_0^{-1}) \quad (29)$$

NP will be guaranteed if NP_0 is satisfied. Note that: (1) this does not mean that we choose w_{P0} such that $|w_P S| = |w_{P0} S_0|$; (2) we may have to modify w_{P0}^{-1} at high frequencies since we must have $|w_{P0}^{-1}| \geq 1$ at high frequencies for real systems; (3) with this weight NP_0 is only a sufficient condition for NP, and it may result in a too tight performance weight if $\bar{\sigma}(GG_0^{-1})$ is large ($\gg 1$). The conservativeness comes primarily from the second inequality in (28).

Robust performance (RP)

Robust performance is defined as

$$\mathrm{RP} \Leftrightarrow \bar{\sigma}(w_P S_p) < 1, \quad \forall\omega, \quad \forall\Delta_A, \Delta_I \text{ or } \Delta_0 \quad (30)$$

The corresponding μ-tests in terms of the system with time-delay are given in (6)–(10).

For the same reason as mentioned for nominal performance, we would like to have μ-tests on the delay-free system which guarantee RP. One approach is first to consider RP_0 of the delay-free system

$$\mathrm{RP}_0 \Leftrightarrow \bar{\sigma}(w_{P0} S_{p0}) < 1, \quad \forall\omega, \quad \forall\Delta_A, \Delta_I \text{ or } \Delta_0 \quad (31)$$

We get the following equivalent μ-condition

$$\mathrm{RP}_0 \Leftrightarrow \mu_{\mathrm{RP}_0} = \sup_{\omega} \mu_\Delta(N_0) < 1 \quad (32)$$

where N_0 is the same as N in (8)–(10) except that all K and G should be replaced by K_0 and G_0. Detailed expressions for S_p, S_{p0} and their relationship are given in Appendix A. For example, we have for additive uncertainty

$$S_p = (I - GG_0^{-1})(I + w_A\Delta_A K_0(I + G_0K_0)^{-1})^{-1} + GG_0^{-1}S_{p0} \quad (33)$$

This is the same as (25) except that the first term on the right-hand side is changed with a factor $(I + w_A\Delta_A K_0(I + G_0K_0)^{-1})^{-1}$. By choosing

$$|w_{P0}^{-1}| \leq (|w_P^{-1}| - \bar{\sigma}(I - GG_0^{-1}) \sup_{\Delta_A} \bar{\sigma}((I + w_A\Delta_A K_0(I + G_0K_0)^{-1})^{-1}))/\bar{\sigma}(GG_0^{-1}) \quad (34)$$

RP will be guaranteed if RP_0 is satisfied. Comparing (34) with (29), we see that it requires tighter performance weight w_{P0} to satisfy RP than to satisfy NP, since $\bar{\sigma}((I + w_A\Delta_A K_0(I + G_0K_0)^{-1})^{-1}$ is always larger than 1. Although this term involves the controller K_0, it is generally independent of K_0 at low frequencies where $K_0S_0 \approx G_0^{-1}$, such that we are able to obtain w_{P0} before starting the design. If the additional term $\bar{\sigma}((I + w_A\Delta_A K_0S_0)^{-1})$ is not too large (relative to 1), or equivalently if $\underline{\sigma}(I + w_A\Delta_A K_0S_0)$ is not too small (relative to 1), then the weight w_{P0} for RP will not be much different from that for NP. The term $\bar{\sigma}((I + w_A\Delta_A K_0S_0)^{-1})$ is indeed close to 1 for SISO systems with a reasonable margin to RS, but it may be very large for some MIMO systems, particularly when Δ is structured. However, for most MIMO systems our design approach in terms of RP_0 is expected to work well.

Similar results exist for input multiplicative uncertainty with input time-delays and output multiplicative uncertainty with output delays (see Appendix A). For example, for the case with input multiplicative uncertainty and input time-delays, the equivalent condition to (34) is

$$|w_{P0}^{-1}| \leqslant (|w_P^{-1}| - \bar{\sigma}(I - GG_0^{-1}) \sup_{\Delta_I} \bar{\sigma}((I + w_I \Delta_I G_0 K_0 (I + G_0 K_0)^{-1})^{-1}))/\bar{\sigma}(GG_0^{-1}) \tag{35}$$

Robustness of the Smith predictor controller

For SISO systems, the NS and RS conditions of the time-delay systems with the Smith predictor controller are exactly the same as those of a corresponding delay-free system; the differences between NP and NP_0, and between RP and RP_0 are small; so we would expect the same robustness. However, it is commonly accepted that the Smith predictor controller may lead to poor robustness. Possible explanations are the following.

(1) Although the conditions are the same, the uncertainty weights $w_A(s)$, $w_I(s)$ and $w_O(s)$ are generally larger for time-delay systems because the time delay itself is uncertain.

(2) In design as well as implementation analysis, time delays are often approximated by rational transfer functions. This introduces additional uncertainty. Indeed, we have found that time-delay systems may be very sensitive to this approximation error.

(3) The use of a Smith predictor enables the use of a high gain controller. Hence, we may use too high a gain if we consider only nominal performance.

4. Design example

We consider an SISO example from Laughlin *et al.* (1987). The plant model is a first-order process with time delay:

$$G(s) = \frac{k}{\tau s + 1} e^{-\theta s}, \quad G_0(s) = \frac{k}{\tau s + 1} \tag{36}$$

The nominal values of k, τ and θ are all equal to 1. We use the same multiplicative uncertainty weight as given by Laughlin *et al.*:

$$1 \cdot 5 \left(\frac{s + 1}{0 \cdot 5s + 1}\right)\left(\frac{1 + 0 \cdot 25s}{1 - 0 \cdot 25s}\right) - 1 \tag{37}$$

which is derived from a simultaneous 50% parameter uncertainty in k, τ and θ. This weight is unstable and we multiply it by an all-pass to get the following equivalent weight:

$$w_I(s) = \frac{4s^2 + 13s + 4}{s^2 + 6s + 8} \tag{38}$$

The performance weight on the sensitivity function S is

$$w_P(s) = \frac{1}{M}\frac{\tau_P s + 1}{\tau_P s} = \frac{1}{2}\frac{3{\cdot}397s + 1}{3{\cdot}397s} \tag{39}$$

The RP-condition (7) becomes for this special case

$$\mu_{RP} = \sup_{\omega}(|w_P S| + |w_I GKS|) < 1 \tag{40}$$

and the objective of the controller design is to minimize μ_{RP}.

Performance weight for delay-free system. From (35) we get that the corresponding performance weight for the delay-free system must satisfy

$$|w_{P0}^{-1}| \leq (|w_P^{-1}| - |1 - e^{-\theta s}|\alpha) \tag{41}$$

where

$$\alpha = \sup_{\Delta_I} \bar{\sigma}((I + w_I \Delta_I G_0 K_0 S_0)^{-1}) \tag{42}$$

Note that if we were to satisfy NP then $\alpha = 1$. For RP the exact value of α is difficult to evaluate. However, *at low frequencies* $G_0 K_0 S_0 \approx I$ so

$$\alpha \approx (1 - |w_I|)^{-1} = 2 \tag{43}$$

and $|1 - e^{-j\theta\omega}| \approx \theta\omega$ so condition (41) reduces to

$$|w_{P0}^{-1}| \leq |w_P^{-1}| - 2\theta\omega \tag{44}$$

where $\theta = 1$ in our case. At high frequencies the upper bound on $|w_{P0}^{-1}|$ given by condition (41) is smaller than one, which is meaningless. We thus disregard this bound and instead select some reasonable value no less than $M = 2$. We assume that $w_{P0}(s)$ is of the following form:

$$w_{P0}(s) = \frac{1}{M_0}\frac{\tau_{P0}s + 1}{\tau_{P0}s} \tag{45}$$

and choose as a 'reasonable' value $M_0 = 1{\cdot}5$. To make the inequality (44) hold exactly at low frequencies, we get $\tau_{P0} = 3{\cdot}196$ and hence

$$w_{P0}(s) = \frac{1}{1{\cdot}5}\frac{3{\cdot}196s + 1}{3{\cdot}196s} \tag{46}$$

The performance weights w_P^{-1}, w_{P0}^{-1} and the uncertainty weight $|w_I^{-1}|$ are shown in Fig. 3.

Smith predictor design for $M_0 = 1{\cdot}5$

We now design a μ-optimal Smith-predictor controller based on the delay free plant. Using DK-iteration, we obtain a six-order 'μ-optimal primary controller', $K_{0\mu}$, for the delay-free plant $G_0 = 1/(s+1)$ using the uncertainty weight w_I (38), and the performance weight w_{P0} (46). The corresponding μ-plot is very flat at all frequencies (confirming that the DK-iteration has converged) with a peak value of $\mu_{RP_0} = 1{\cdot}0557$. We then apply this controller in a Smith predictor scheme to get the 'real' controller $K = K_0(I + (G_0 - G)K_0)^{-1}$. The corresponding real and delay-free sensitivity functions for the nominal case, S

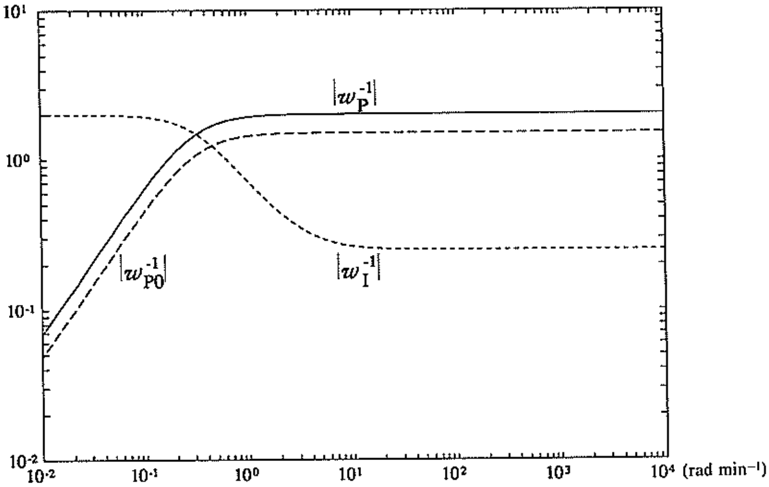


Figure 3. Uncertainty and performance weights.

and S_0, are shown in Fig. 4. When we analyse for robust performance of the real system with delay using w_P (38) we obtain a peak value $\mu_{RP} = 1{\cdot}0802$.

Improved design by adjusting M_0

Note that with our simple performance weight, w_{P0}, we still have one degree of freedom, M_0, which can be used to adjust the high frequency performance. Above, we selected more or less arbitrarily $M_0 = 1{\cdot}5$. In the Table, we show results with different values of M_0. For each (M_0, τ_{P0}) pair, the performance weight w_{P0} is the same at low frequencies, but it becomes tighter and thus more difficult to satisfy at high frequencies as M_0 decreases. Hence μ_{RP_0} increases monotonically as M_0 decreases, while μ_{RP} initially decreases before it starts

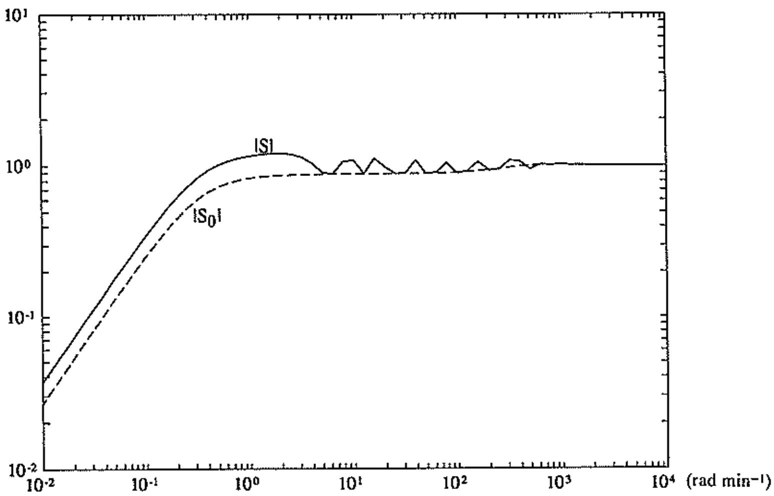


Figure 4. Sensitivity functions S and S_0.

M_0	τ_{P0}	μ_{RP_0}	μ_{RP}
2·00	2·397	0·9727	1·1620
1·70	2·820	1·0142	1·1114
1·50	3·196	1·0557	1·0802
1·45	3·306	1·0686	1·0703
1·40	3·424	1·0823	1·0593
1·35	3·551	1·0979	1·0639
1·30	3·688	1·1157	1·0816

Effect of M_0 on controller design.

increasing. In our example, the lowest value is $\mu_{RP} = 1{\cdot}0593$, which is obtained for $M_0 = 1{\cdot}4$. The corresponding 'μ-optimal primary controller' can be reduced from order six to order five without affecting μ_{RP_0} and μ_{RP}. The state space realization of this reduced controller, denoted K_{0r}, is given in Appendix B. The corresponding μ-plot when applied to the original problem using a Smith predictor is given by the dashed line in Fig. 5. We see that the μ-curve is flat, just above one at low frequencies, and drops off at higher frequencies.

Other designs

(*a*) *Smith predictor*. Laughlin *et al.* (1987) used a Smith predictor with K_0 as a simple PI-controller, and obtained by parameter optimization a controller with $\mu_{RP} = 1{\cdot}122$. Note that Laughlin *et al.* minimized μ_{RP} for the real system with time delay, while we minimize μ_{RP_0} for the delay-free system.

(*b*) *μ-Optimal controller*. The μ-optimal controller is the controller that minimizes μ_{RP}. Laughlin *et al.* obtained a μ-optimal controller with μ_{RP} about 1·08 (observed from their plot), but using newer software (Balas *et al.*, 1991), we were able to get a 14th-order μ-optimal controller, K_μ, with $\mu_{RP} = 1{\cdot}0525$.

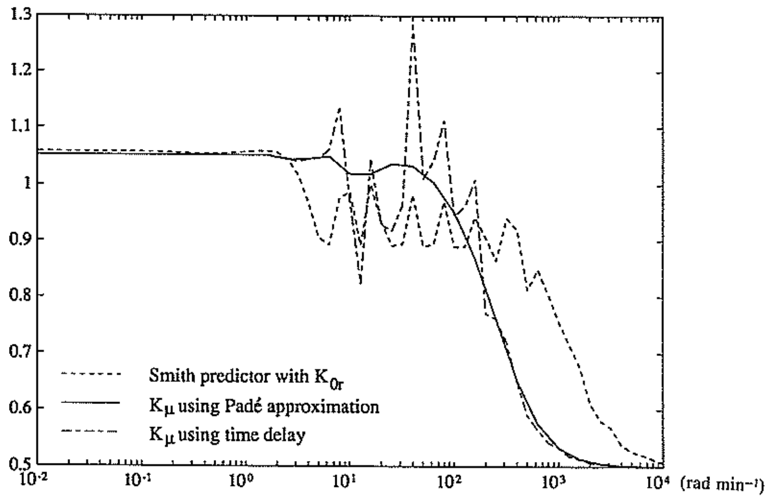
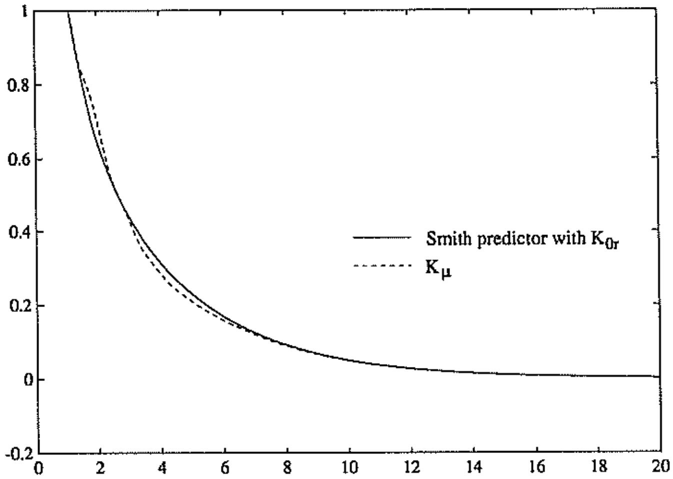


Figure 5. μ_{RP} curves with $|w_P|$ on sensitivity function S.

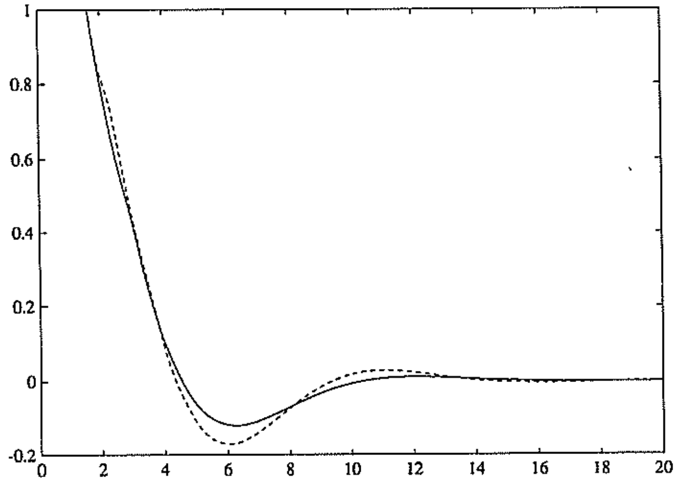
(*c*) *Comparison*. We note that our Smith predictor design with $M_0 = 1{\cdot}4$ yields a value $\mu_{RP} = 1{\cdot}0593$ which is only 0·6% higher than for the μ-optimal controller. Responses for the two controllers are compared in Fig. 6 for (*a*) the nominal plant, and (*b*) one perturbed plant (which is the worst case among the extreme parameter values). We see that in the nominal case the responses are very similar, while in the perturbed case the Smith-predictor controller yields slightly less overshoot.

The problem with rational time-delay approximation

In order to synthesize the μ-optimal controller, one needs to approximate the time-delay by a rational function. Laughlin *et al*. (1987) used a fourth-order



(*a*)



(*b*)

Figure 6. Response to unit step disturbance at the output: (*a*) nominal case with $k = \tau = \theta = 1$; (*b*) Perturbed case with $k = \tau = \theta = 1{\cdot}5$.

Padé approximation, and so did we. Generally, one would believe that a fourth-order Padé approximation should be sufficiently accurate, but this appears not to be true. Figure 5 also shows the μ-plots with the μ-optimal controller, K_μ, applied to a plant with the Padé approximation and with the real time-delay, respectively. In the latter case, the μ-plot is seen to contain several large peaks, whose locations are identical with the peaks of Padé approximation error. The bandwidth of our system is about 0·2, while the fourth Padé approximation is quite accurate for frequencies less than 5. Hence one may think that it is sufficiently accurate. However, if we consider the μ-plot, we find that μ for robust performance is very flat up to a frequency of about 100; this requires the approximation to be good at least up to frequency 100. This means that the robust performance problem may put a much more severe restriction on the approximation of the time-delay. To overcome this, one may add a filter to the μ-optimal controller or combine the approximation error explicitly into the model uncertainty. One simple method is to let the uncertainty weight approach infinity at high frequency (whereas $w_I(s)$ levels off at 4). Since the method proposed in this paper is completely delay-free, it does not suffer from those problems. This is another advantage of our method.

5. Conclusions

In this paper we have used the Smith predictor structure to convert the robust performance problem of a time delay system into a delay-free problem. This not only makes the analysis and synthesis easier but also avoids the problem resulting from rational approximation of time-delays. Rational approximation of time-delays may sometimes cause problems even if we think that we have used a very good approximation. The design example shows that our delay-free approach works well at least for SISO systems, and we were able to obtain a less complex controller with the same performance as the true μ-optimal controller.

Acknowledgment

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Appendix A

Relationships between S_p and S_{p0}

The perturbed (actual) sensitivity function for the time-delay system with the Smith predictor controller is

$$\begin{aligned} S_p &= (I + G_pK)^{-1} \\ &= S(I + (G_p - G)K(I + GK)^{-1})^{-1} \\ &= S(I + (G_p - G)K_0(I + G_0K_0)^{-1})^{-1} \\ &= S(I + w_A\Delta_A K_0(I + G_0K_0)^{-1})^{-1} \\ &= S(I + w_I G\Delta_I K_0(I + G_0K_0)^{-1})^{-1} \\ &= S(I + w_0\Delta_0 GK_0(I + G_0K_0)^{-1})^{-1} \end{aligned} \quad (47)$$

The perturbed sensitivity function for the delay-free system is

$$\begin{aligned} S_{p0} &= S_0(I + w_A \Delta_A K_0 (I + G_0 K_0)^{-1})^{-1} \\ &= S_0(I + w_I G_0 \Delta_I K_0 (I + G_0 K_0)^{-1})^{-1} \\ &= S_0(I + w_0 \Delta_0 G_0 K_0 (I + G_0 K_0)^{-1})^{-1} \end{aligned} \tag{48}$$

It is easy to see from the expressions that we have for *additive* uncertainty

$$S_p = S S_0^{-1} S_{p0} \tag{49}$$

For *input multiplicative* uncertainty and *input* delays, note

$$G \Delta_I = G_0 D \Delta_I = G_0 \Delta_I^* \tag{50}$$

where $D = \operatorname{diag} e^{-\theta_i s}$. Hence, Δ_I^* has the same structure and the same bound as Δ_I. In this sense, (49) still holds for the case of *input multiplicative* uncertainty and *input* delays. Similarly, (49) also holds for the case of *output multiplicative* uncertainty and *output* delays.

By straightforward derivation, we can also develop the following relationships between S_p and S_{p0}. For *additive* uncertainty

$$S_p = (I - G G_0^{-1})(I + w_A \Delta_A K_0 (I + G_0 K_0)^{-1})^{-1} + G G_0^{-1} S_{p0} \tag{51}$$

For *input multiplicative* uncertainty and *input* delays

$$S_p = (I - G G_0^{-1})(I + w_I G_0 \Delta_I K_0 (I + G_0 K_0)^{-1})^{-1} + G G_0^{-1} S_{p0} \tag{52}$$

For *output multiplicative* uncertainty and *output* delays

$$S_p = (I - G G_0^{-1})(I + w_0 \Delta_0 G_0 K_0 (I + G_0 K_0)^{-1})^{-1} + G G_0^{-1} S_{p0} \tag{53}$$

Appendix B

The μ-optimal primary controller for $M_0 = 1{\cdot}4$

The reduced fifth-order 'μ-optimal primary controller' for $M_0 = 1{\cdot}4$ is

$$K_{0r} = C(sI - A)^{-1} B + D \tag{54}$$

where

$$A = \begin{bmatrix} -2{\cdot}9206 \times 10^{-7} & 4{\cdot}3856 \times 10^{-4} & 2{\cdot}9463 \times 10^{-6} & -3{\cdot}6367 \times 10^{-5} & 2{\cdot}7749 \times 10^{-7} \\ 4{\cdot}3856 \times 10^{-4} & -3{\cdot}2245 \times 10^{3} & -1{\cdot}3259 \times 10^{3} & 5{\cdot}4805 \times 10^{2} & -4{\cdot}0820 \\ -2{\cdot}9466 \times 10^{-6} & 1{\cdot}3259 \times 10^{3} & -1{\cdot}5042 \times 10^{-1} & 3{\cdot}6225 & -2{\cdot}8323 \times 10^{2} \\ 3{\cdot}6367 \times 10^{-5} & -5{\cdot}4805 \times 10^{2} & 3{\cdot}6225 & -9{\cdot}1324 \times 10^{2} & 1{\cdot}3704 \times 10 \\ -2{\cdot}7749 \times 10^{-7} & 4{\cdot}0820 & -2{\cdot}8323 \times 10^{-2} & 1{\cdot}3704 \times 10 & -3{\cdot}1279 \end{bmatrix}$$

$$B^{T} = (5.9856 \times 10^{-1} \quad -4.4943 \times 10^{2} \quad 3.0191 \quad -3.7266 \times 10 \quad 2.8435 \times 10^{-1})$$

$$C = (5.9856 \times 10^{-1} \quad -4.4943 \times 10^{2} \quad -3.0191 \quad 3.7266 \times 10 \quad -2.8435 \times 10^{-1})$$

$$D = 1.1708 \times 10^{-4}$$

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