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MODELLING OF GAIN AND DELAY UNCERTAINTY IN THE STRUCTURED SINGULAR VALUE FRAMEWORK

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Abstract. Gain and delay uncertainty is commonly used to quantify plant-model mismatch in the process control community. This type of uncertainty description cannot be directly used for robustness analysis and design in the structured singular value (μ) framework. This paper provides analytical expressions for some tight approximations of gain-delay uncertainty based on complex perturbations in a form suitable for *analysis* within the μ-framework. Simple bounds suitable for *synthesis* are also presented. A model that covers the gain uncertainty exactly and closely approximates the delay uncertainty, is derived using real perturbations. Finally these uncertainty models are applied to a distillation example.

Key Words. Control system analysis; control system synthesis; delays; distillation columns; process control; robust control

1 INTRODUCTION

The structured singular value, μ, (Doyle, 1982) has proven to be a very powerful tool for robustness analysis (*e.g.* Skogestad *et al.*, 1988). μ provides a computationally efficient *necessary* and *sufficient* condition for robustness of a set of possible plants defined by an uncertainty description. The condition may be formulated as a Robust Stability (RS) test or as a Robust Performance (RP) test, using the $\mathcal{H}_\infty$ norm to specify the required performance. However, the underlying assumption for necessity and sufficiency of the robustness condition is that the uncertainty description is "tight".

In the process control community it is very common to quantify plant-model mismatch in terms of gain and delay uncertainties. This uncertainty description generates a set Π of possible plants $\hat{g}$

$$\Pi = \{\hat{g}(s)\,|\hat{g}(s) = ke^{-\theta s}\ ; \quad k \in [k_{min},\ k_{max}]\ ,\ \theta \in [\theta_{min},\ \theta_{max}]\}\,. \quad (1)$$

This set cannot be exactly modelled in the μ framework. One may argue that an exact representation of this set is not very important, since the gain-delay uncertainty itself is only an approximation of the actual plant-model mismatch. However, from an engineering point of view, a set of "off the shelf" models for commonly used uncertainty descriptions is very useful and will substantially reduce the effort associated with analysis and synthesis of robust controllers.

The purpose of this paper is then to present some tight approximations of the uncertainty defined in Eq.1. This issue has also been studied by Laughlin *et al.* (1987) and Lundström *et al.* (1991).

2 THE μ-FRAMEWORK

This section only covers aspects of the structured singular value framework with special importance for this paper. A less brief review of μ is given elsewhere in this proceeding by Hovd *et.al.* (1993); the interested reader may also consult Skogestad *et al.* (1988), Stein and Doyle (1991) and Balas *et.al.* (1991).

The general problem formulation in the μ framework is illustrated in Fig.1. The left block diagram consists of an augmented plant P (including nominal plant model and weight functions), a controller K and a (block-diagonal) perturbation matrix $\Delta_U = \text{diag}\{\Delta_1, \cdots, \Delta_n\}$ representing uncertainty. **d** is a vector of external input signals (*e.g.* disturbances and set-points). **e** is a vector of output signals which should be kept small (e.g. manipulated inputs and deviation from set-points). The weights in P are used to specify the performance requirements and to normalize each Δ_i to be less than one in magnitude at each frequency.

The right block diagram in Fig.1 is used for robustness analysis. M is a function of P and K, and Δ_P is a fictitious "performance perturbation" connecting **e** to **d**. Provided that the closed loop system is nominally stable the conditions for RS and RP are:

$$RS \;\Leftrightarrow\; \mu_{RS} = \sup_\omega \mu_{\Delta_U}(M_{11}(j\omega)) < 1 \quad (2)$$

$$RP \;\Leftrightarrow\; \mu_{RP} = \sup_\omega \mu_{\Delta}(M(j\omega)) < 1 \quad (3)$$

Remarks:
1) To formulate a robustness problem as shown in Fig.1 each uncertainty must be represented in terms of a linear fractional transformation (LFT).

$$F_u(\Gamma, \Delta_i) = [\Gamma_{22} + \Gamma_{21}\Delta_i(I - \Gamma_{11}\Delta_i)^{-1}\Gamma_{12}] \quad (4)$$

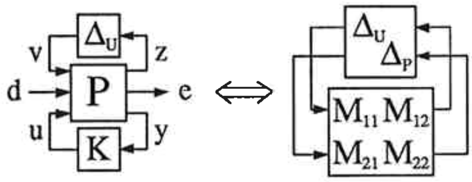

Fig. 1: General problem description

2) The *sufficiency* of the robustness conditions requires that the uncertainty description includes *all* "allowed" uncertainties.
3) The *necessity* of the robustness conditions requires that the uncertainty description includes *only* "allowed" uncertainties.
4) μ analysis is performed on a frequency by frequency basis, while μ synthesis is performed in a state-space setting (Balas *et al.* 1991). For analysis it is therefore sufficient to obtain the frequency response of the interconnection matrix M, which means that even irrational uncertainty models may be used. For μ synthesis, M has to be a finite dimensional state-space model. Because of this tighter bounds may be derived for analysis than for synthesis.
5) At present there is no μ synthesis method for systems where the perturbation matrix Δ includes real entries. Algorithms for analysis of this class of problems are being developed (Young *et al.* 1991), but pure-complex problems are much easier to solve.
6) $\mu_{RP} = 0.8$ means that the uncertainty perturbations in Δ_U could be increased by a factor $1/0.8 = 1.25$ (*i.e.* a larger uncertainty set) and still the performance specifications would be satisfied by a margin of 1.25. The performance margin for the *specified uncertainty set*, i.e. $||\Delta_U||_\infty \leq 1$ may be computed using a "skewed μ" (Packard 1988), here denoted J and defined by

$$J(\omega) = \left\{ J(\omega) \left| \mu \begin{pmatrix} M_{11}(\omega) & M_{12}(\omega) \\ J(\omega)^{-1} M_{21}(\omega) & J(\omega)^{-1} M_{22}(\omega) \end{pmatrix} = 1 \right. \right\} \quad (5)$$

$J = 0.8$ means that the performance is satisfied by a margin of 1.25 for $||\Delta_U||_\infty \leq 1$.

3 UNCERTAINTY MODELLING

The objective of this section is to present approximations of Π (Eq.1). The following notation is used:

$$\bar{k} = \frac{k_{max} + k_{min}}{2} \qquad k_r = \frac{k_{max} - k_{min}}{k_{max} + k_{min}}$$

$$\bar{\theta} = \frac{\theta_{max} + \theta_{min}}{2} \qquad \theta_\delta = \frac{\theta_{max} - \theta_{min}}{2}$$

$$\theta^\star = \max\{|\theta_{min}|, |\theta_{max}|\}$$

3.1 Complex uncertainty, irrational weight

The set Π maps onto a "polygon"-shaped region on the complex plane at each frequency (Fig.2). The simplest way to represent Π within the μ-framework is by a nominal plant model subject to a single complex additive or multiplicative perturbation, which generates a "disk"-shaped region on the complex plane at each frequency. In this section three different choices of nominal models are considered and for each of them analytical expressions for the smallest perturbation needed to *cover* every plant in Π are presented. Δ is complex and $|\Delta(j\omega)| \leq 1 \forall \omega$.

Set Π_1 : Multiplicative uncertainty with nominal model $\bar{k}$.

$$\Pi_1 = \{\hat{g}(j\omega) | \hat{g}(j\omega) = \bar{k}[1 + l_1(\omega)\Delta(j\omega)]\} \quad (6)$$

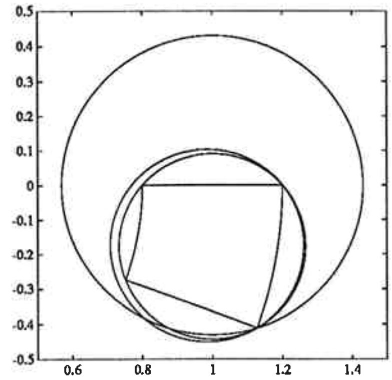

Fig. 2: Comparison of Π (polygon), Π_1 (largest disk), Π_2 (second smallest disk) and Π_3 (smallest disk) on the complex plane at $\omega = 0.35$ [rad/min] for $k \in [0.8, 1.2]$ and $\theta \in [0, 1]$.

$$l_1(\omega) = \begin{cases} |(1 + k_r)e^{-j\theta^\star \omega} - 1| & \text{for } \omega \leq \pi/\theta^\star \\ 2 + k_r & \text{for } \omega > \pi/\theta^\star \end{cases} \quad (7)$$

Set Π_2 : Multiplicative uncertainty with nominal model $\bar{k}e^{-j\omega\bar{\theta}}$.

$$\Pi_2 = \{\hat{g}(j\omega) | \hat{g}(j\omega) = \bar{k}e^{-j\omega\bar{\theta}}[1 + l_2(\omega)\Delta(j\omega)]\} \quad (8)$$

$$l_2(\omega) = \begin{cases} |(1 + k_r)e^{-j\omega\theta_\delta} - 1| & \text{for } \omega \leq \pi/\theta_\delta \\ 2 + k_r & \text{for } \omega > \pi/\theta_\delta \end{cases} \quad (9)$$

Set Π_3 : Smallest disk that covers Π. This set may be represented as additive uncertainty with an irrational nominal model. A pure multiplicative description cannot be used because the nominal model is 0 at some frequencies. We use a mixed multiplicative/additive representation.

$$\Pi_3 = \{\hat{g}(j\omega) | \hat{g}(j\omega) = \bar{k}e^{-j\omega\bar{\theta}}[m_3(\omega) + l_3(\omega)\Delta(j\omega)]\} \quad (10)$$

$m_3(\omega)$ and $l_3(\omega)$ are obtained by minimizing $\min_{m_3(\omega)} l_3(\omega)$, s.t. $\Pi \subset \Pi_3$, $\forall \omega$. This constrained optimization may be solved analytically and yields

ω	$m_3(\omega)$	$l_3(\omega)$
ω_A	$\frac{1}{\cos(\theta_\delta \omega)}$	$\sqrt{k_r^2 + \tan^2(\theta_\delta \omega)}$
ω_B	$(1 + k_r)\cos(\theta_\delta \omega)$	$(1 + k_r)\sin(\theta_\delta \omega)$
ω_C	0	$1 + k_r$

for $0 \leq \omega_A < \frac{1}{2\theta_\delta} \arccos\left(\frac{1-k_r}{1+kr}\right) \leq \omega_B < \frac{\pi}{2\theta_\delta} \leq \omega_C$.

Π_1 and Π_2 are special cases of the uncertainty model studied by Laughlin *et al.* (1987). The two set are identical if $\theta_{max} = -\theta_{min}$. Of the three sets above Π_1 generates the largest disk on the complex plane at each frequency and Π_3 the smallest, *i.e.* $l_1(\omega) \geq l_2(\omega) \geq l_3(\omega) \; \forall \omega$. However, this does not mean that Π_3 is always the least conservative approximation of Π. It is not the *size* of the set, but the *worst case plant* within the set that matters. There are plants (possibly "worst-case") within both Π_2 and Π_3 which do not belong to Π_1 as shown in Fig.2.

3.2 Complex uncertainty, rational weight

Rational weights are needed, for example for $\mathcal{H}_\infty$ and μ-synthesis.

Set Π_{1s} : Same as Π_1 but with a rational weight $|w_1(j\omega)| \geq l_1(\omega)\ \forall\omega$.

$$\Pi_{1s} = \{\hat{g}_{1s}(s) | \hat{g}_{1s}(s) = \bar{k}[1 + w_1(s)\Delta(j\omega)]\} \quad (11)$$

$$w_1(s) = \frac{(1 + \frac{k_r}{2})\theta^* s + k_r}{\frac{\theta^*}{2}s + 1} \cdot \frac{\left(\frac{\theta^* s}{c}\right)^2 + 2\zeta_z\left(\frac{\theta^* s}{c}\right) + 1}{\left(\frac{\theta^* s}{c}\right)^2 + 2\zeta_p\left(\frac{\theta^* s}{c}\right) + 1} \quad (12)$$

$$c = 2.363,\ \zeta_z = 0.838 \text{ and } \zeta_p = 0.685 \quad (13)$$

The first part of $w_1(s)$ is derived form a first order Padé approximation, the second part is a correction factor used to obtain $\Pi \subset \Pi_{1s}$. The optimal values of c, ζ_z and ζ_p are dependent of k_r, but independent of $\theta^\star$. Numerical optimization reveals that these parameters do not vary much for different values of k_r and the fixed parameters in Eq.13 may be used without introducing much extra conservativeness.

3.3 Real uncertainty

With real uncertainty it is possible to derive a tight description of Π which avoids covering a polygon by a disk.

Set Π_r : This set is obtained from an nth order Padé approximation (Fig.3).

$$\Gamma_\theta = \left[\begin{array}{c|cc} A & B_{11} & B_{12} \\ \hline C_{11} & D_{11} & D_{21} \\ C_{21} & D_{21} & D_{22} \end{array}\right] = \left[\begin{array}{c|cc} -\frac{2n}{\bar{\theta}} & \frac{2n\theta_\delta}{\bar{\theta}^2} & \frac{2n}{\bar{\theta}} \\ \hline 1 & -\frac{\theta_\delta}{\bar{\theta}} & 0 \\ 2 & -\frac{2\theta_\delta}{\bar{\theta}} & -1 \end{array}\right]$$

$$\Gamma_k = \begin{bmatrix} 0 & k_r \\ 1 & 1 \end{bmatrix} \quad (14)$$

$$-1 \leq \Delta_k \leq 1 \text{ and } -1 \leq \Delta_\theta \leq 1 \quad (15)$$

Note that Δ_θ is a *repeated* perturbation.

This set does not quite cover Π, but is a tight approximation. By increasing n, the number of Γ_θ's, an arbitrary close approximation may be obtained. However, in most practical applications a second order approximation would probably suffice. In special cases Π_r is a *subset* of Π: 1) The delay uncertainty includes both prediction and delay, *i.e.* $\theta_{min} < 0$ and $\theta_{max} > 0$, or 2) Either $\theta_{min} = 0$ or $\theta_{max} = 0$.

The parameterization of Γ_θ causes problems if: 1) $\theta_{max} = -\theta_{min} \Rightarrow \bar{\theta} = 0$, and some elements of Γ_θ will be infinite; 2) $\theta_{min} = 0 \Rightarrow (1 - \Gamma_{\theta,11}\Delta)^{-1}$ improper for $\Delta = -1$; 3) $\theta_{max} = 0 \Rightarrow (1 - \Gamma_{\theta,11}\Delta)^{-1}$ improper for $\Delta = 1$. These problems are avoided by adding or subtracting a small quantity to θ_{min} or θ_{max}.

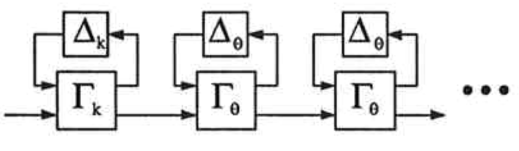

Fig. 3: Uncertainty model with real Δ's

4 DISTILLATION EXAMPLE

The purpose of this section is to show how the uncertainty sets from the previous section may be used for robustness analysis. The example processes is a high-purity distillation column presented in Skogestad *et al.* (1988), however, here the uncertainty is *defined* in terms of gain-delay uncertainty, while Skogestad defined the uncertainty in terms of a proper rational bound on a complex multiplicative perturbation.

4.1 Problem definition

The uncertain plant model is

$$\hat{G}(s) = \frac{1}{75s + 1}\begin{bmatrix} 0.878 & -0.864 \\ 1.082 & -1.096 \end{bmatrix}\begin{bmatrix} k_1 e^{-\theta_1 s} & 0 \\ 0 & k_2 e^{-\theta_2 s} \end{bmatrix} \quad (16)$$

where

$$k_i \in [0.8,\ 1.2]\ ;\ \theta_i \in [0,\ 1]\ ;\ i = 1, 2 \quad (17)$$

i.e. 20% relative gain uncertainty and up to 1 min delay in each input channel.

The required performance is specified in terms of a frequency dependent bound, $W_p(s)$, on the sensitivity function $\hat{S} = (I + \hat{G}K)^{-1}$ for the worst case plant $\hat{G}$.

$$RP \Leftrightarrow \sup_{k,\ \theta} ||W_p(I + \hat{G}K)^{-1}||_\infty < 1 \quad (18)$$

$$W_p(s) = \frac{1}{2}\frac{(20s + 2)}{(20s + 10^{-3})} I_{2\times 2}. \quad (19)$$

4.2 Analysis

The controller, used in this comparison, was synthesised by DK-iteration (Balas *et al.*, 1991) with uncertainty set Π_{1s} representing the gain-delay uncertainty in each input channel. It yields $\mu_{RP} = 1.028$, so RP is almost satisfied.

The complex perturbation sets Π_{1s} and Π_3 are both *outer* approximations of the gain-delay set Π. Because of this we know that J (Eq.5) for these approximations will yield an upper bound of $J(\Pi)$ (denotes J for uncertainty set Π). Similarly, since Π_r is an *inner* approximation (for the uncertainty in this example), $J(\Pi_r)$ yields a lower bound of $J(\Pi)$.

Fig.4 shows J for the distillation example where the uncertainty is modelled by sets Π_{1s} (solid), Π_3 (dash) and Π_r (using $n = 2$) (dash-dot). The "true" $J(\Pi)$ is bounded from above by the solid and the dashed curve, and from below by the dash-dot curve. An interesting observation is that the smallest upper bound and the lower bound are quite close to each other, *i.e.* $J(\Pi)$ is determined by rather tight bounds. At some frequencies between 0.01 and 0.1 [rad/min] $J(\Pi_3) > J(\Pi_{1s})$, which shows that at these frequencies the smaller set Π_3 includes plants which are worse than any plant within the larger set Π_{1s}. However, at most frequencies the smallest set $J(\Pi_3)$ yields the tightest upper bound on $J(\Pi)$. At most frequencies above π [rad/min] $J(\Pi_3)$ and $J(\Pi_r)$ are identical (if the computation of $J(\Pi_r)$ had converged they would be identical for all frequencies above π [rad/min]).

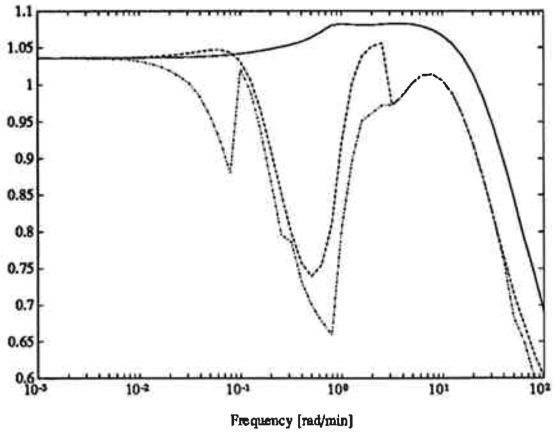

Fig. 4: The "skewed μ" J as a function of frequency for different uncertainty sets; $J(\Pi_{1s})$ (solid), $J(\Pi_3)$ (dash) and $J(\Pi_r)$ (using $n = 2$) (dash-dot).

5 DISCUSSION

The only uncertainty model suitable for synthesis, presented in this paper, is Π_{1s} which has the advantage of a very simple nominal model with no delay. The reason for not including any delay in the nominal model is to keep the order of $P(s)$ as low as possible, since a controller obtained by DK-iteration has the same number of states as the augmented plant $P(s)$ (Fig.1) including the D-scales. Of the same reason, the correction factor in $w_1(s)$ (Eq.12) may be omitted when it is not necessary to cover every plant in Π.

Set Π_2, where the average delay is included in the nominal model, could of course also be approximated by a rational model suitable for synthesis. This is done by Laughlin *et al.* (1987), however, the result is often more conservative. The explanation to this lies in the high frequency behavior of Π_1 and Π_2. At high frequencies the sets have equal disk radii (on the complex plane), but while the disk center of Π_1 is fixed at $1+0j$ the center of Π_2 moves along the unit circle and thereby Π_2 effectively covers a larger region, resulting in extra conservativeness.

The analysis in section 4.2 is based on J (Eq.5) instead of μ, since a comparison based on μ may be misleading when different representations of a given uncertainty set are studied. Consider a gain-delay uncertain plant with $k_r = 0.2$ and $\theta_\delta = 1$ and represent this uncertainty by sets Π_3 and Π_r. At a frequency $\omega > \frac{\pi}{2\theta_\delta}$, Π_3 covers all plants within a disk with center on the origin and radius $1+k_r$, Π_r covers all plants within an annular region with outer radius $1 + k_r$. It can be shown that the worst case plant is at a maximum distance from the origin, so Π_1 and Π_r covers the same worst case plant. Assume that a controller K_1 yields $\mu = 1.1$ for uncertainty set Π_3, *i.e.* a performance margin of 1/1.1 for all plants within a radius $\Delta_U(1 + k_r) = \frac{1}{1.1}(1 + 0.2) \approx 1.09$. Consider another controller K_2 which yields the same μ but for Π_r, *i.e.* the same performance margin but for a *larger* radius $1 + \Delta_U k_r = 1 + \frac{1}{1.1}0.2 \approx 1.18$. This shows that μ for the two cases cannot be directly compared.

6 CONCLUSIONS

Single-input-single-output gain-delay uncertainty cannot be *exactly* represented in the structured singular value framework, but has to be approximated into a linear fractional form.

The smallest single complex perturbation that covers a gain-delay uncertainty may be derived analytically (Set Π_3). This uncertainty set may be used for analysis only.

A delay free nominal model subject to a low order multiplicative perturbation (Set Π_{1s}) is recommended for μ-synthesis.

An arbitrary tight approximation of the gain-delay uncertainty may be derived using real structured uncertainty (Set Π_r). This uncertainty set may in principle be used both for analysis and synthesis, but no synthesis method for real perturbations is available at present.

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