Presented at the 1992 ACC, Chicago, Illinois, June 24-26.

ROBUST CONTROL OF SYSTEMS CONSISTING OF SYMMETRICALLY INTERCONNECTED SUBSYSTEMS

M. Hovd S. Skogestad*

Chemical Engineering, University of Trondheim, NTH
N-7034 Trondheim, Norway

Abstract

This paper is concerned with robust control of systems consisting of n similar interacting subsystems. The transfer function matrices for these systems are block circulant matrices. For H_∞ optimal control, we show that we can to simplify controller synthesis by considering two systems of the same dimension as the subsystems instead of the overall system. This leads to a dramatic reduction in system size for system consisting of many subsystems, and we are able to optimize the H_∞ criterion in n directions, as opposed to standard H_∞ synthesis which only optimizes the criterion in the *worst* direction. The structured singular value, μ, is shown to be independent of the structure of the uncertainty for cases with SISO subsystems and only one uncertainty block. For MIMO systems or cases with several uncertainty blocks, we are able to reduce the plant size from $n \cdot n_o \times n \cdot n_i$ to a block diagonal plant with diagonal blocks of size $n_o \times n_i$ (for subsystems of dimension $n_o \times n_i$) if we assume all the uncertainty blocks to be full.

1 Introduction

The paper is concerned with the control of systems consisting of similar subsystems in parallel which interact with each other. This happens quite often in practice, for instance in distribution networks, when there are parallel units (reactors, heat exchangers, etc.) in a chemical plant or for electric power plants operating in parallel. With n nominally identical subsystems in parallel the $n \times n$ transfer matrix of the plant may be written

$$G(s) = \begin{bmatrix} g(s) & i(s) & i(s) & \dots & i(s) \\ i(s) & g(s) & i(s) & \dots & i(s) \\ i(s) & i(s) & g(s) & \dots & i(s) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ i(s) & i(s) & i(s) & \dots & g(s) \end{bmatrix} \quad (1)$$

where the diagonal elements $g(s)$ denote the transfer function of the individual subsystem, and the offdiagonal elements $i(s)$ denote the interactions. For MIMO subsystems, both $g(s)$ and $i(s)$ are matrices. For SISO subsystems, the degree of interaction at a given frequency ω is given by $a(j\omega) = i(j\omega)/g(j\omega)$. We have not found any name for the matrix $G(s)$ in the literature, but we shall refer to it as a *block parallel matrix* in the following, as we believe that transfer function matrices of the form of $G(s)$ in Eq. (1) occur predominantly for nominally identical, interacting processes in parallel. If the individual subsystems have only one input ($n_i = 1$) and one output ($n_o = 1$), $G(s)$ in Eq. (1) will be termed *parallel*.
This type of system is termed a *Symmetrically Interconnected System* by Sundareshan and Elbanna [6]. Sundareshan and Elbanna studies conditions for controllability and observability of the system and solutions to the matrix Riccati and Lyapunov equations. They also give a methodology for the synthesis of a decentralized controller and an alternative "almost decentralized" control structure. No robustness analysis is performed. Lunze [5] studies robust stability of symmetrically interconnected systems. Lunzes model formulation can take account of a variety of different uncertainties and model errors. However, Lunze does not account for the structure of the uncertainty, which can lead to very conservative results.

2 Results from matrix theory

2.1 Systems with SISO subsystems

Consider a circulant matrix C of dimension $n \times n$:

$$C = \begin{bmatrix} c_1 & c_2 & c_3 & \cdots & c_n \\ c_n & c_1 & c_2 & \cdots & c_{n-1} \\ c_{n-1} & c_n & c_1 & \cdots & c_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_2 & c_3 & c_4 & \cdots & c_1 \end{bmatrix} \quad (2)$$

Parallel matrices are a subclass of circulant matrices. This can easily be seen by choosing $c_2 = c_3 = \cdots = c_n$. The results in this section on circulant matrices are from [1] and [2]. Circulant matrices belong to the class known as Toeplitz matrices, as all elements along any one diagonal are identical.

2.2 Eigenvalues, eigenvectors and singular value decomposition.

Introduce $r_l = exp(\frac{2\pi(l-1)i}{n})$ where $i = \sqrt{-1}$ and $l = 1, \cdots, n$. That is, r_l is a root of the equation $r^n = 1$, and we have

$$1 + r_l + r_l^2 + \cdots + r_l^{n-1} = 0 \text{ for } r_l \neq 1 \quad (3)$$

From the theory of circulant matrices we know that the eigenvalues of the circulant matrix C are given by:

$$\lambda_{Cl} = c_1 + c_2 r_l + c_3 r_l^2 + \cdots + c_n r_l^{n-1} \quad (4)$$

This means that any parallel matrix P has eigenvalues λ_l given by the formula:

$$\lambda_l = c_1 + c_2(r_l + r_l^2 + \cdots + r_l^{n-1}) \quad (5)$$

From Eq. (3) we see that the matrix P will have at most two distinct eigenvalues. These are given by

$$\lambda_1 = (c_1 + (n-1)c_2) \quad (6)$$

$$\lambda_2 = \lambda_3 = \cdots = \lambda_n = c_1 - c_2 \quad (7)$$

The eigenvector corresponding to λ_l is:

$$\underline{m_l} = [1 \; r_l \; r_l^2 \; \cdots \; r_l^{n-1}]^T \quad (8)$$

Since r_l can take n distinct values, P will always have a complete set of eigenvectors, and will thus always be diagonalizable. In fact, all circulant matrices of the same order have the same eigenvectors, and are therefore diagonalized by the same matrix, the Fourier matrix. The Fourier matrix of order n is given by (e. g. [2])

$$F^H = \frac{1}{\sqrt{n}}[m_1 \; m_2 \; \cdots \; m_n] \quad (9)$$

*Author to whom all correspondence should be addressed. E-mail: SKOGE@KJEMI.UNIT.NO, phone: 47-7-594154, fax: 47-7-591410.

F is unitary ($FF^H = F^H F = I$), and we have for any circulant matrix C

$$C = F^H \Lambda_C F; \qquad \Lambda_C = \mathrm{diag}\{\lambda_1, \cdots, \lambda_n\} \tag{10}$$

Furthermore, we have for the singular value decomposition that the singular values $\sigma_i = |\lambda_i|$.

2.2.1 Combinations of parallel matrices

If A and B are parallel matrices of the same dimension and k_i a scalar, then A^T, A^H, $k_1 A + k_2 B$, AB, $\sum_i k_i A^i$ are parallel matrices and A and B commute, that is, $AB = BA$. Note that A^{-1} is also a parallel matrix.

For example, if a process with a parallel transfer function G is controlled by n equal single-loop controllers (ie., $C = cI$), the sensitivity function $S = (I + GC)^{-1}$ and the complementary sensitivity function $H = I - S$ are both parallel matrices.

2.2.2 Matrices consisting of circulant blocks

Consider a matrix M consisting of $m_1 \times m_2$ blocks, each block being a circulant matrix of order n. We shall call the class of such matrices $CB_{m_1,m_2,n}$. The many results from the theory of circulant matrices do not hold for matrices consisting of circulant blocks. However, we can find some results which will prove helpful.

- If M_1 and M_2 both belong to the class $CB_{m_1,m_2,n}$ and α_1 and α_2 are scalars, then $\alpha_1 M_1 + \alpha_2 M_2$ also belongs to the class $CB_{m_1,m_2,n}$ and M_1^H belongs to the class $CB_{m_2,m_1,n}$. If M_1 belongs to the class $CB_{m_1,m_2,n}$ and M_2 belongs to the class $CB_{m_2,m_1,n}$ then $M_1 M_2$ belongs to the class $CB_{m_1,m_1,n}$.
- For "diagonalization" we have

$$\tilde{M}_1 = (I_{m_1} \otimes F_n) M_1 (I_{m_2} \otimes F_n)^H \tag{11}$$

where $\otimes$ denotes the Kronecker product, and $\tilde{M}_1$ is a matrix with the same block structure as M_1, each block in $\tilde{M}_1$ being the (diagonal) eigenvalue matrix of the corresponding block in M_1. This is illustrated by an example. Consider

$$M_1 = \begin{bmatrix} C_1 & C_2 & C_3 \\ C_4 & C_5 & C_6 \end{bmatrix} \tag{12}$$

where $C_1, C_2, \cdots, C_6$ all are circulant matrices of order n. M_1 then belongs to the class $CB_{2,3,n}$. We then have

$$M_1 = (I_2 \otimes F_n)^H \tilde{M}_1 (I_3 \otimes F_n) \tag{13}$$

$$(I_2 \otimes F_n)^H = \begin{bmatrix} F_n^H & 0 \\ 0 & F_n^H \end{bmatrix} \tag{14}$$

$$(I_3 \otimes F_n) = \begin{bmatrix} F_n & 0 & 0 \\ 0 & F_n & 0 \\ 0 & 0 & F_n \end{bmatrix} \tag{15}$$

$$\tilde{M}_1 = \begin{bmatrix} \Lambda_{C1} & \Lambda_{C2} & \Lambda_{C3} \\ \Lambda_{C4} & \Lambda_{C5} & \Lambda_{C6} \end{bmatrix} \tag{16}$$

where Λ_{Ci} is the (diagonal) eigenvalue matrix of block C_i.

2.3 Systems with MIMO subsystems

2.3.1 Block circulant matrices

The matrix C in Eq. (2) is *block circulant* if $c_1, c_2, \cdots, c_n$ all are blocks of dimension $no \times ni$. For such a block circulant matrix C we have

$$\tilde{C} = (F_n \otimes I_{no}) C (F_n \otimes I_{ni})^H \tag{17}$$

where $\tilde{C} = \mathrm{diag}\{\gamma_1, \gamma_2, \cdots, \gamma_n\}$, and $\gamma_1, \gamma_2, \cdots, \gamma_n$ all have dimension $n_o \times n_i$, and can be calculated from the blocks of C using

$$\begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{bmatrix} = (\sqrt{n} F_n \otimes I_{no})^H \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \tag{18}$$

Proof: Follows from the proof of theorem 5.6.4 in [2], by setting $B_k = I_{no} A_{k+1} I_{ni}$.

If $c_2 = c_3 = \cdots = c_n$ then we term the matrix C *block parallel*, and we have

$$\gamma_1 = c_1 + (n-1)c_2 \tag{19}$$

$$\gamma_2 = \gamma_3 = \cdots = \gamma_n = c_1 - c_2 \tag{20}$$

In this way, a symmetrically interconnected system consisting of n units in parallel can be decomposed into one distinct subprocess γ_1 and $n-1$ equal subprocesses γ_2.

2.3.2 Combinations of block circulant matrices

If A is a block circulant matrix with $n \times n$ blocks, each of size $n_o \times n_i$, then A^H and A^T are block circulant matrices with $n \times n$ blocks of size $ni \times no$. If A^{-1} exists, it is a block circulant matrix. If B is block circulant, consisting of $n \times n$ blocks of size $n_i \times n_b$, then AB is a block circulant matrix with blocks of size $n_o \times n_b$. In general, block circulant matrices do not commute, $AB \neq BA$.

2.3.3 Matrices consisting of blocks which are block circulant

Consider a matrix N consisting of $m_r \times m_c$ blocks, each block being a block circulant matrix with $n \times n$ subblocks. Subblocks belonging to the same column of main blocks must have the same number of columns. Let n_i^c denote the number of columns of the subblocks in column c of main blocks. Likewise, subblocks belonging to the same row of main blocks must have the same number of rows, and we use n_o^r to denote the number of rows in the subblocks of blocks in row r of main blocks. To illustrate, consider

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \tag{21}$$

Let M_{11}, M_{12} M_{21} and M_{22} all be block circulant matrices consisting of $n \times n$ subblocks, and let the subblocks of M_{ij} have dimension $n_o^{ij} \times n_i^{ij}$. Then $n_o^{11} = n_o^{12} = n_o^1$, $n_o^{21} = n_o^{22} = n_o^2$, $n_i^{11} = n_i^{21} = n_i^1$ and $n_i^{12} = n_i^{22} = n_i^2$, but n_o^1 may be different from n_o^2, and n_i^1 may be different from n_i^2. Introduce the matrices

$$\mathcal{F_L} = \mathrm{diag}\{F_n \otimes I_{n_o^r}\} \tag{22}$$

$$\mathcal{F_R} = \mathrm{diag}\{F_n \otimes I_{n_i^c}\} \tag{23}$$

We then have that N can be "diagonalized" by the following transformation

$$\tilde{N} = \mathcal{F_L} N \mathcal{F_R}^H \tag{24}$$

where $\tilde{N}$ is a matrix consisting of block diagonal blocks, where each block of $\tilde{N}$ can be calculated from the corresponding block of N using Eq. (18).

3 Realization of full block parallel controllers.

In the following we will consider the use of *block parallel* controllers for the control of plants described by block parallel transfer function matrices. We assume that the controller design has resulted in that the distinct subprocess γ_{G1} of the plant G is controlled by the controller γ_{C1} and the $n-1$ identical subprocesses are controlled by $n-1$ controllers all equal to γ_{C2}, and that γ_{C1} and γ_{C2} both have dimension $n1 \times n2$. The controller C will then be a block parallel matrix with diagonal blocks $c_{ii} = [\gamma_{C1} + (n-1)\gamma_{C2}]/n$ and offdiagonal blocks $c_{ij} = [\gamma_{C1} - \gamma_{C2}]/n$.

In order to not to increase the number of states in the controller unnecessarily, the controller should be realized as

$$C = \mathrm{diag}\{\gamma_{C2}\} + \begin{bmatrix} I_{n1} \\ \vdots \\ I_{n1} \end{bmatrix} \frac{\gamma_{C1} - \gamma_{C2}}{n} \begin{bmatrix} I_{n2} & \cdots & I_{n2} \end{bmatrix} \tag{25}$$

4 Robust control with parallel controllers.

In this section we will use H_∞ theory and the structured singular value μ for analyzing robustness of control systems and for synthesizing controllers meeting predefined robustness criteria. *Throughout section 4 we will assume that the plant can be described by block parallel transfer function matrices.* We will also assume that all performance requirements are the same for all subsystems. The uncertainties are assumed to be the located in the same positions and to have the same magnitudes for the different subsystem, as the subsystems are designed to be nominally identical. *The uncertainty and performance weights are therefore also assumed to be parallel matrices.* Of course, different magnitude bounds for different uncertainties will be allowed.

4.1 H_∞ control.

H_∞ control theory can be used for designing controllers which ensures that the closed loop system satisfies singular value loop shaping specifications. For example, the standard "mixed sensitivity" H_∞ problem is to minimize

$$\|M\|_\infty = \left\| \begin{bmatrix} W_O H \\ W_P S \end{bmatrix} \right\|_\infty \tag{26}$$

where $S = (I + GC)^{-1}$ is the sensitivity function and $H = GC(I + GC)^{-1}$ the complementary sensitivity function. This corresponds to simultaneously trying to optimize robust stability with respect to *output* uncertainty and nominal performance.

Problem: In the following, consider the design of a controller in order to minimize the H_∞ norm of some matrix M (not necessarily of the form used in Eq. (26)). We assume that the plant $G(s)$ is a block parallel matrix consisting of n units in parallel, each block of $G(s)$ having dimension $n_o \times n_i$. Likewise, we assume that all weights are block parallel matrices with blocks of dimension compatible with the dimension of the blocks of $G(s)$. Furthermore, we assume that the matrix M consists of $m_r \times m_c$ blocks, each block of dimension $n \cdot n_o^r \times n \cdot n_i^c$ (e.g. for the matrix M in Eq. (26) we have $m_r = 2$ and $m_c = 1$).

Theorem 1 *The H_∞-optimal controller for this problem can be obtained by designing two H_∞-optimal controllers for the two systems corresponding to the "plants" γ_{G1} and γ_{G2}.*

Proof:

1. Express the matrix $M(s)$, whose H_∞ norm is to be minimized, as a Linear Fractional Transformation (LFT) of the controller $C(s)$ (see Fig 1 a).

   $$M(s) = N_{11}(s) + N_{12}(s)C(s)\left[I - N_{22}(s)C(s)\right]^{-1} N_{21}(s) \tag{27}$$

   N_{11}, N_{12}, N_{21} and N_{22} consist of blocks which are block parallel.

2. Premultiplication or postmultiplication of $M(s)$ by unitary matrices will not change the singular values, and will thus leave the H_∞ norm unchanged (Fig. 1 b). We use the matrices $\mathcal{F_C}$ and $\mathcal{F_R}$ defined in equations (22) and (23)

   $$\tilde{M} = \mathcal{F_C} M \mathcal{F_R}^H \tag{28}$$

   $$= \tilde{N}_{11} + \tilde{N}_{12}\tilde{C}\left[I - \tilde{N}_{22}\tilde{C}\right]^{-1}\tilde{N}_{21} \tag{29}$$

   $$\tilde{N}_{11} = \mathcal{F_C} N_{11} \mathcal{F_R}^H \tag{30}$$

   $$\tilde{N}_{12} = \mathcal{F_C} N_{12} (F_n \otimes I_{ni})^H \tag{31}$$

   $$\tilde{N}_{21} = (F_n \otimes I_{no}) N_{21} \mathcal{F_C}^H \tag{32}$$

   $$\tilde{N}_{22} = (F_n \otimes I_{no}) N_{22} (F_n \otimes I_{ni})^H \tag{33}$$

   Thus, since N_{11}, N_{12}, N_{21} and N_{22} all consist of blocks which are block parallel, $\tilde{N}_{11}$, $\tilde{N}_{12}$, $\tilde{N}_{21}$ and $\tilde{N}_{22}$ all consist of blocks which are block diagonal, the first subblock in each block being distinct and the other $n-1$ subblocks equal.

This means that the controller design has been decoupled into n non-interacting design subproblems, one of these design subproblems being distinct, and $n-1$ design subproblems being equal. Thus, the controller $\tilde{C}$ can be designed by designing one controller for the distinct subproblem, and one controller for one of the $n-1$ subproblems. The proof is therefore complete.

Comment: The same also holds for the H_2-optimal problem, since the Frobenius norm is also unitarily invariant.
$\tilde{C}$ will be block diagonal, the first block on the diagonal being distinct and the $n-1$ other blocks equal. Consequently,

$$C(s) = (F_n \otimes I_{ni})^H \tilde{C}(s)(F_n \otimes I_{no}) \tag{34}$$

will be a block parallel matrix. We see that instead of designing a $n \cdot n_i \times n \cdot n_o$ dimensional controller for a system with a $n \cdot n_o \times n \cdot n_i$ dimensional plant, $n \cdot \sum_{c=1}^{m_c} n_i^c$ inputs and $n \cdot \sum_{r=1}^{m_r} n_o^r$ outputs, we can design two controllers of dimension $n_i \times n_o$, each controller designed for a system consisting of a plant of dimension $n_o \times n_i$, $\sum_{c=1}^{m_c} n_i^c$ inputs and $\sum_{r=1}^{m_r} n_o^r$ outputs. Note that this corresponds to optimizing the H_∞ objective for both the systems corresponding to γ_{G1} and γ_{G2}. If $m_c = n_i = 1$ or $m_r = n_o = 1$, *all* directions in the H_∞ criterion are optimized. In contrast, standard H_∞ synthesis only optimizes the *worst* direction in the overall H_∞ criterion.

Example 1. Consider the system

$$G(s) = \begin{bmatrix} G_1(s) & G_2(s) & G_2(s) & G_2(s) \\ G_2(s) & G_1(s) & G_2(s) & G_2(s) \\ G_2(s) & G_2(s) & G_1(s) & G_2(s) \\ G_2(s) & G_2(s) & G_2(s) & G_1(s) \end{bmatrix} \tag{35}$$

$$G_1(s) = g(s)\begin{bmatrix} -1.25s^3 + 5.4625s^2 + 1.3063s + 0.0568 & 6.45s^2 + 1.665s + 0.075 & 5.75s^2 + 1.575s + 0.088 \\ 18.9s^2 + 2.85s + 0.102 & -1.25s^3 - 0.8875s^2 - 0.0888s + 0.0774 & -1.25s^3 + 5.2125s^2 + 1.3313s + 0.0838 \end{bmatrix} \tag{36}$$

$$G_2(s) = g(s)\begin{bmatrix} -1.25s^3 - 0.9875s^2 - 0.0887s + 0.0057 & 0.1s^2 - 0.03s - 0.007 & 0.35s^2 + 0.105s + 0.007 \\ -3.05s^2 - 0.215s + 0.007 & -1.25s^3 + 0.8625s^2 - 0.0038s - 0.0092 & -1.25s^3 + 0.0125s^2 + 0.09125s + 0.00775 \end{bmatrix} \tag{37}$$

$$g(s) = 1/(s^3 + 0.35s^2 + 0.035s + 0.001)$$

The design criterion is to minimize the infinity-norm of

$$M = \begin{bmatrix} W_o GC(I + GC)^{-1} \\ W_p (I + GC)^{-1} \\ W_u C(I + GC)^{-1} \end{bmatrix} \tag{38}$$

with weights

$$W_o = 0.2\frac{4s + 1}{0.4s + 1} I_8$$

$$W_p = \mathrm{diag}\left\{ \begin{bmatrix} 0.5\frac{2.5s+1}{2.5s} & 0 \\ 0 & 0.5\frac{0.3s+1}{0.3s} \end{bmatrix} \right\}$$

$$W_u = 0.1 I_{12}$$

We decomposed the plant $G(s)$ into γ_{G1} and $\gamma_{G2} = \gamma_{G3} = \gamma_{G4}$, and designed one controller γ_{C1} for the system corresponding to γ_{G1}, and one controller γ_{C2} for the system corresponding to γ_{G2}. For both these design subproblems a H_∞-norm of 0.91 was achieved, and the same H_∞-norm of 0.91 was achieved for the overall system after calculating the controller C from γ_{C1} and γ_{C2} according to Eq. (25). The best value of the H_∞-norm achieved when attempting H_∞ synthesis on the overall plant was 0.99. This demonstrates weaknesses in the synthesis software we have available[1], and that controller synthesis becomes simpler when the system is decomposed into problems of lower dimension. State space descriptions of the "controllers" γ_{C1} and γ_{C2} are given in Tables 1 and 2.
For the special case when all the following conditions hold

C1. The plant consists of SISO subsystems ($ni = no = 1$).

C2. The plant is of minimum phase.

C3. All weights are scalar times identity matrices.

C4. G and C only appear as products of each other in the problem statement (as in Eq. (26))

the controller design can be reduced further to designing one SISO controller. For this case, if we design a controller λ_{C1} for the plant eigenvalue λ_{G1}, then the H_∞ norm that was obtained for the SISO "system" corresponding to λ_{G1} can be obtained for the overall system by choosing

$$\lambda_{C2} = \cdots = \lambda_{Cn} = \lambda_{C1}\lambda_{G1}/\lambda_{G2} \quad (39)$$

The result will be an inverse-based controller of the type

$$C(s) = k(s)G^{-1}(s) \quad (40)$$

thus effectively transforming the $n \times n$ H_∞ design problem to n identical SISO problems. Whereas condition C3 will normally hold for SISO systems in parallel, condition C4 may well be violated, e.g. if M contains a term like $W_u C(I + GC)^{-1}$ corresponding to a bound on the closed loop transfer function from reference signal to manipulated variables.

4.2 The structured singular value.

We will only briefly state or main result for the structured singular value (μ) [3], because of space limitations. The reader will therefore be assumed to have quite detailed knowledge of μ. We will use the following properties of μ:

$$\rho(M) \leq \mu(M) \leq \bar{\sigma}(M) \quad (41)$$

$$\mu(M) \leq \bar{\sigma}(D_l M D_r^{-1}) \quad (42)$$

D_l and D_r are real positive matrices with a structure such that $D_r^{-1} \Delta D_l = \Delta$.

4.2.1 Special case: SISO subsystems and only one uncertainty block for each subsystem.

In this section we consider the case with SISO subsystems and only one source of uncertainty in each subsystem, and the perturbation block Δ and the interconnection matrix M are both square matrices of dimension $n \times n$.
μ analysis: The interconnection matrix M will be a parallel matrix. We know that for parallel matrices the magnitude of the eigenvalues equal the magnitude of the singular values, and from Eq. (41) we see that the value of μ will equal the spectral radius. Thus for this problem the value of μ is completely insensitive to the structure of the uncertainty for any complex uncertainty, we get the same value for μ regardless of whether the uncertainty block is full, diagonal or is a repeated scalar block.
μ synthesis: The observation that the value of μ is insensitive to the structure of the uncertainty can be used to simplify the design of a robust controller. The controller design can be performed by designing controllers for two SISO systems, one SISO system for λ_1 of the plant, and another SISO system for $\lambda_2, \cdots, \lambda_n$.[2]

[1] The μ-tools and Robust Control toolboxes for MATLAB^{TM}.

4.2.2 μ analysis and synthesis for systems with more than one uncertainty block.

For systems with more than one uncertainty block, or for systems consisting of MIMO subsystems, the interconnection matrix M will in general not be parallel (or circulant), but it will be a matrix consisting of blocks which are block parallel. For matrices consisting of blocks which are block parallel, the magnitudes of the eigenvalues and the singular values will in general differ, and the upper and lower limits in Eq. (41) will not be particularly helpful.
However, for the special case of *full* uncertainty blocks (blocks of dimension $n \cdot \sum_{c=k1}^{k2} n_i^c \times n \cdot \sum_{r=l1}^{l2} n_o^r$), we can use Eq. (42) to simplify the calculation of the upper bound. Analyzing $D_l M D_r^{-1}$ in the same way as M was analyzed in the proof of Theorem 1, we see that we can reduce the system to that corresponding to the "plant" $\tilde{G} = \mathrm{diag}\{\gamma_{G1}, \gamma_{G2}\}$. However, unlike the H_∞ case, we cannot consider γ_{G1} and γ_{G2} independently, because of the assumption of full uncertainty blocks.
μ analysis. We can calculate the upper bound in Eq. (42) by considering a block diagonal plant $\tilde{G} = \mathrm{diag}\{\gamma_{G1}, \gamma_{G2}\}$ with full uncertainty blocks instead of considering the full plant G with full uncertainty blocks.
μ synthesis. We can synthesize a block diagonal controller for a block diagonal plant $\tilde{G} = \mathrm{diag}\{\gamma_{G1}, \gamma_{G2}\}$ with full uncertainty blocks instead of synthesizing a full controller for the full plant G and full uncertainty blocks.
Note that the conventional $D - K$ iteration method of μ-synthesis is not likely to be sucessful in this case. The assumption of full uncertainty blocks means that it is only *nominally* that the plant is block parallel. The *worst case* plant is likely not to be block diagonal, and $D - K$ iteration will hence produce a controller which is not block diagonal. We do not know how to handle terms outside the blocks along the main diagonal, and performance is likely to suffer severely if they are ignored. Thus, a sysnthesis procedure which guarantees a block diagonal controller should be used.
When the subsystems of the plant have multiple inputs and multiple outputs, the uncertainty weights must be chosen carefully to try to minimize the conservativeness introduced by assuming full uncertainty blocks, as scaling problems must otherwise be expected.

5 Conclusions

Robust control. The paper demonstrates that for symmetrically interconnected systems the plant can be reduced from dimension $n \cdot no \times n \cdot ni$ to an equivalent block diagonal plant with two blocks of dimension $no \times ni$ along the diagonal for the purposes of robustness analysis and robust controller synthesis using the structured singular value. For H_∞ synthesis and analysis the two blocks in this equivalent plant can be considered separately.
Extension to circulant processes. The results in this paper will apply also to processes with circulant transfer function matrices. Such processes occur for example in the cross-directional control in paper manufacturing if edge effects are neglected [4], [7]). For circulant processes the plant can be reduced from dimension $n \cdot no \times n \cdot ni$ to an equivalent block diagonal plant with k blocks of dimension $no \times ni$ along the diagonal, where k is the number of distinct blocks in Eq. (18). However, for block circulant plants one may have to restrict the dynamics of the controller blocks in order to obtain a realizable controller.

References

[1] Bellman, R., 1970, *Introduction to Matrix Analysis*, McGraw-Hill, New York.

[2] We have here only one uncertainty block in the problem statement, and no explicitly stated performance requirement. It will therefore not be meaningful to try to find the controller which minimizes μ, as the optimum will then be no feedback, $C = 0$. Instead, one could maximize the system bandwidth subject to $\mu < 1$.

[2] Davis, P.J., 1979, *Circulant Matrices*, Wiley, New York.

[3] Doyle, J. C., Wall, J. E. and Stein, G. (1982). Performance and Robustness Analysis for Structured Uncertainty. *Proc. IEEE Conf. Decision Contr.*, Orlando, FL.

[4] Laughlin, D. L., Braatz, R. D., and Morari, M. (1992). Robust Performance of Cross-Directional Basis-Weight Control in Paper Machines. Submitted to *Automatica*, preprint.

[5] Lunze, J. (1989). Stability Analysis of Large-scale Systems Composed of Strongly Coupled Similar Subsystems, *Automatica*, **25**, 4, pp. 561-570.

[6] Sundareshan, M. K. and Elbanna, R. M. (1991). Qualitative Analysis and Decentralized Controller Synthesis for a Class of Large-scale Systems with Symmetrically Interconnected Subsystems, *Automatica* **27**, 2, pp. 383-388.

[7] Wilhelm, R. G. Jr. and Fjeld, M. (1983). Control Algorithms for Cross Directional Control: The State of the Art. *Preprints 5th IFAC PRP Conference*, Antwerp, Belgium.

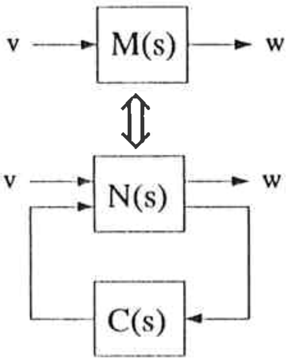

Figure 1(a). Expressing M(s) as a LFT of C(s)

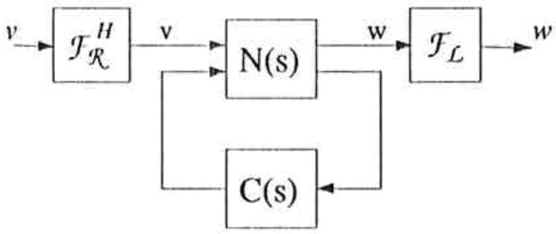

Figure 1(b). Pre- and postmultiplication by unitary matrices.

Tables.

State space descriptions of the controllers found in section 4. The state space descriptions are given as the A,B,C,D matrices in $\gamma_{Ci} = C(sI - A)^{-1}B + D$.

Table 1: γ_{C1} for the controller found in example 1.

diag{ A }	B	
-231.439	0.0134	2.9209
-2.470	1.2090	-2.1960
-0.823	-0.3021	0.5568
-1.064	-0.3085	0.0152
-0.176	-0.1196	-3.4066
-1E-08	1.82E-04	4.6785
-2e-09	0.7891	3.1e-09
C^T		
0.0565	-41.2565	37.2301
0.0455	-0.4676	0.3644
0.1665	-0.1291	-0.0588
0.0720	-0.1922	0.0877
-0.0221	0.0114	-0.0050
-0.0331	-0.0253	0.0350
-0.0395	-0.0237	0.0410
D		
	-0.0125	0
	0	-6.25E-03
	0	-6.25E-03

Table 2: γ_{C2} for the controller found in example 1.

diag{ A }	B	
-3591.5	0.0107	-20.3237
-266.7	0.0330	-24.2771
-130.4	0.0257	15.3908
-1.120	0.0663	1.031E-03
-0.0521	0.0734	-0.3711
-3.9E-08	-3.0E-10	-0.6377
-4.7E-09	-0.2745	-6.5E-11
C^T		
-1241	316.2	-171.4
53.76	41.55	38.59
-18.46	46.87	21.23
-9.744E-03	0.2968	0.1639
1.219E-02	-2.804E-03	4.267E-03
-3.593E-02	2.020E-02	2.295E-03
6.100E-03	-1.288E-02	-4.801E-03
D		
	0	0
	0	0
	0	0