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Brief Paper

Simple Frequency-dependent Tools for Control System Analysis, Structure Selection and Design*

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Key Words—Control system design; decentralized control; frequency response; large-scale systems; robustness; transmission zeros.

Abstract—The paper presents results on frequency-dependent tools for analysis, structure selection and design of control systems. This includes relationships between the relative gain array (RGA) and right half plane zeros, and the use of the RGA as a sensitivity measure with respect to individual element uncertainty and diagonal input uncertainty. It is also shown how frequency-dependent plots of the closely related performance relative gains (PRGA) and a new proposed disturbance measure, the closed-loop disturbance gains (CLDG), can be used to evaluate the achievable performance (controllability) of a plant under decentralized control. These controller-independent measures give constraints on the design of the individual loops, which when satisfied guarantee that the overall system satisfies performance objectives with respect to setpoint tracking and disturbance rejection.

1. *Introduction*

THE RELATIVE GAIN ARRAY (RGA) has found widespread use as a measure of interaction and as a tool for control structure selection for single-loop controllers. It was first introduced by Bristol (1966). It was originally defined at steady-state, but it may easily be extended to higher frequencies (Bristol, 1978). Shinskey (1967, 1984) and McAvoy (1983) have demonstrated practical applications of the RGA. Important advantages with the RGA is that it depends on the plant model only and that it is scaling independent. It is straightforward to generalize the RGA from single-loop controllers to block-diagonal controllers by introducing the block relative gain (BRG) (Manousiouthakis *et al.*, 1986), and most of the results presented in this paper may be generalized in such a manner. However, to simplify the presentation, and because single-loop controllers are most common in practice, we shall consider only the RGA in this paper.

Our interest in the RGA as a frequency-dependent measure was initially focused on its use as a sensitivity measure with respect to model uncertainty (Skogestad and Morari, 1987b, see Theorem 2 and equations (11) below). However, based on its original definition as a steady-state interaction measure for single-loop control, it seemed reasonable that the frequency-dependent RGA should have some use as a performance or stability measure for decentralized control. Some interesting relationships and reports of encouraging applications presented by Nett (1987) led us to investigate this in more detail.

Most authors have confined themselves to use the RGA at steady state, and a thorough review of the use and interpretation of the steady-state RGA is given by Grosdidier (1985). A frequency-dependent interaction measure Y, which is equivalent to the RGA for 2×2 systems, was introduced by Balchen (1958) and Rijnsdorp (1965) and is discussed for $n\times n$ systems in Balchen and Mumme (1988). Balchen also gives some performance interpretation to his measure. Applications of the frequency-dependent RGA are given by McAvoy (1981, 1983).

We use the dynamic RGA as defined by Bristol (1978). Other definitions have also been proposed. Witcher and McAvoy (1977) proposed a time domain definition of the RGA, as did Tung and Edgar (1981). Arkun (1987, 1988) has proposed measures (DBRG and Relative Sensitivity) which include the controller. Balchen and Mumme (1988) generalize the measure Y to include the controller. However, one then loses one of the main advantages of the RGA which is that it depends on the plant model only. These alternative definitions are not considered in this paper.

In the paper we first consider the RGA as a general analysis tool and refer to some of its properties, which we believe are significant for engineering applications. However, the main part of the paper is devoted to decentralized control. We study stability and achievable performance ("controllability") using simple frequency-dependent measures for interactions (PRGA) and disturbances (CLDG).

One of the main criticisms against the use of the RGA has been its "failure" to predict the poor performance one often has when using decentralized control for one-way interactive systems because the RGA matrix is identity in such cases. We propose a new measure, the performance RGA (PRGA) which may be used to address also this performance issue.

2. *Definitions*

2.1. *Relative gain array* (RGA). Consider an $n\times n$ plant $G(s)$.

$$y(s)=G(s)u(s). \quad (1)$$

The open loop gain from input u_j to output y_i is $g_{ij}(s)$ when all other outputs y are uncontrolled. Writing equation (1) as

$$u(s)=G^{-1}(s)y(s), \quad (2)$$

it can be seen that the gain from u_j to y_i is $1/[G^{-1}(s)]_{ji}$ when all other ys are perfectly controlled (e.g. Grosdidier *et al.*, 1985). The relative gain is the ratio of these "open-loop" and "closed-loop" gains. Thus a matrix of relative gains, the RGA matrix, can be computed using the formula

$$\Lambda(s)=G(s)\times(G^{-1}(s))^T, \quad (3)$$

where the $\times$ symbol denotes element by element multiplication (Hadamard or Schur product). The inverse $G^{-1}(s)$ may be non-proper or non-causal, and a physical

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interpretation in terms of perfect control is of course not meaningful except at steady-state. This has caused many authors to discard use of a dynamic RGA, or to restrict its use to plants with no RHP-zeros (Manousiouthakis *et al.*, 1986). This is unfortunate as the dynamic RGA as defined above proves to have a number of useful properties. Furthermore, we shall mainly consider the $\Lambda(s)$ as a function of frequency, $s = j\omega$, and in this case $\Lambda(j\omega)$ may be computed for any plant G except for frequencies corresponding to $j\omega$-axis zeros.

The RGA matrix as defined above has some interesting algebraic properties (e.g. Grosdidier *et al.*, 1985):

(a) It is scaling independent (e.g. independent of units chosen for u and y). Mathematically, $\Lambda(D_1GD_2) = \Lambda(G)$ where D_1 and D_2 are diagonal matrices.
(b) All row and column sums equal one.
(c) Any permutation of rows or columns in G results in the same permutation in the RGA.
(d) If $G(s)$ is triangular (and hence also if it is diagonal), $\Lambda(G) = I$.
(e) Relative perturbations in elements of G and in its inverse are related by $d[G^{-1}]_{ji}/[G^{-1}]_{ji} = -\lambda_{ij} dg_{ij}/g_{ij}$.

These properties are easily proven from the following expression for the individual elements of Λ

$$\lambda_{ij}(s) = (-1)^{i+j} \frac{g_{ij}(s)\det(G^{ij}(s))}{\det(G(s))}. \quad (4)$$

Here G^{ij} denotes the matrix G with subsystem ij removed, that is, row i and column j is deleted.

2.2. *Performance relative gain array* (PRGA). One inadequacy of the RGA (e.g. McAvoy, 1983) is that it, because of property *d*, may indicate that interaction is no problem, but significant one-way coupling may exist. To overcome this problem we introduce the performance relative gain array (PRGA). The PRGA-matrix is defined as

$$\Gamma(s) = \tilde{G}(s)G(s)^{-1}, \quad (5)$$

where $\tilde{G}(s)$ is the matrix consisting of only the diagonal elements of $G(s)$, i.e. $\tilde{G} = \text{diag}\{g_{ii}\}$. The matrix Γ was originally introduced at steady-state by Grosdidier (1990) in order to understand the effect of directions under decentralized control. The elements of Γ are given by

$$\gamma_{ij}(s) = g_{ii}(s)[G^{-1}(s)]_{ij} = \frac{g_{ii}(s)}{g_{ji}(s)}\lambda_{ji}(s). \quad (6)$$

Note that the diagonal elements of RGA and PRGA are identical, but otherwise PRGA does not have all the algebraic properties of the RGA. PRGA must be recomputed whenever G is rearranged, whereas RGA only needs to be rearranged in the same way as G. PRGA is independent of *input* scaling, that is, $\Gamma(GD_2) = \Gamma(G)$, but it depends on output scaling. This is reasonable since performance is defined in terms of the magnitude of the outputs.

The measures above may be extended to non-square systems by introducing the pseudoinverse. However, the usefulness of the measures, at least for analyzing decentralized control, then seems limited. In the following $G(s)$ is assumed square.

3. *The RGA as a general analysis tool*

In this section we present some relationships involving the RGA of $G(s)$ which do not assume a decentralized control system. The results are based on the general definition of the RGA given by equation (3), and the physical interpretation preceding equation (3) is of limited interest in this case.

3.1. *The RGA and right half plane zeros.* Consider a transfer matrix $G(s)$. Bristol (1966) claimed in his original paper and later (Bristol, 1978, 1981) that there is a relationship between RHP-zeros and negative values of $\lambda_{ii}(0)$, but Grosdidier *et al.* (1985) showed with a counter-example that this is not true. However, there proves to be a relationship if $\lambda_{ii}(0)$ and $\lambda_{ii}(\infty)$ have different signs.

Theorem 1. Assume $\lim_{s\to\infty}\lambda_{ij}(s)$ is finite and different from zero. Consider a transfer matrix with stable elements and no zeros or poles at $s = 0$. If $\lambda_{ij}(j\infty)$ and $\lambda_{ij}(0)$ have different signs then at least one of the following must be true:
(a) $g_{ij}(s)$ has a RHP zero.
(b) $G(s)$ has a RHP transmission zero.
(c) $G^{ij}(s)$ (i.e. the subsystem with input j and output i removed) has a RHP transmission zero.

Proof. Consider phase changes in $\lambda_{ij}(j\omega)$ as ω goes from 0 to ∞, see details in Skogestad and Hovd (1990).

Thus, different signs of $\lambda_{ij}(j\infty)$ and $\lambda_{ij}(0)$ is a sufficient condition for the existence of RHP zeros or RHP transmission zeros. Any such zeros may be detrimental for control of the system. However, it is not a necessary condition, and there may be RHP-zeros present even if the RGA elements do not change sign. For example, adding a time delay or RHP-zero to an individual input or output channel will not affect the RGA as it may simply be viewed as a kind of scaling.† In most cases the pairings are chosen such that $\lambda_{ii}(\infty)$ is positive and this confirms Bristols claim that negative RGA-elements imply presence of RHP-zeros.

3.2. *The RGA and the optimally scaled condition number.* Consider any complex matrix G. Bristol (1966) pointed out the formal resemblance between the RGA and the condition number $\gamma(G) = \bar{\sigma}(G)/\underline{\sigma}(G) = \bar{\sigma}(G)\bar{\sigma}(G^{-1})$ However, the condition number depends on scaling, whereas the RGA does not. Minimizing the condition number with respect to all input and output scalings yields the optimal condition number

$$\gamma^*(G) = \min_{D_1, D_2} \gamma(D_1GD_2). \quad (7)$$

It is commonly conjectured that there is a close relationship between $\gamma^*(G)$ and the magnitude of the elements in the RGA as is illustrated by the following lower and conjectured upper bounds on $\gamma^*(G)$:

$$\|\Lambda\|_m - \frac{1}{\gamma^*(G)} \leq \gamma^*(G) \leq \|\Lambda\|_1 + k(n). \quad (8)$$

where $\|\Lambda\|_m = 2\max\{\|\Lambda\|_{i1}, \|\Lambda\|_{i\infty}\}$ and $\|\Lambda\|_1 = \Sigma_{ij}|\lambda_{ij}|$, and $k(n)$ is a constant. The lower bound is proven by Nett and Manousiouthakis (1987). The upper bound is proven for 2×2 matrices with $k(2) = 0$ (Grosdidier *et al.*, 1985), but it is only conjectured for the general case with $k(3) = 1$ and $k(4) = 2$ (Skogestad and Morari, 1987b).

3.3. *RGA and individual element uncertainty.*

Theorem 2. The (complex) matrix G becomes singular if we make a relative change $-1/\lambda_{ij}$ in its ijth element, that is, if a single element in G is perturbed from g_{ij} to $g_{pij} = g_{ij}(1 - 1/\lambda_{ij})$.

Proof. Let $G_p(s)$ denote $G(s)$ with g_{pij} substituted for g_{ij}. Using (4), we find by expanding the determinant of $G_p(s)$ by row i or column j that

$$\det(G_p) = \det(G) - \frac{\det(G)}{(-1)^{i+j}\det(G^{ij})}(-1)^{i+j}\det(G^{ij}) = 0. \quad (9)$$

Theorem 2 provides necessary and sufficient condition for singularity of a matrix with element uncertainty. It is actually a quite amazing algebraic property of the RGA which seems to be little known. The theorem was originally presented by Yu and Luyben (1987), but the proof above is much simpler. Theorem 2 has some important control implications.

(1) Element uncertainty. Consider a plant with transfer matrix $G(s)$. If the relative uncertainty in an element at a given frequency is larger than $|1/\lambda_{ij}(j\omega)|$ then the plant may

† Adding a time delay θ_i to each output i yields the plant D_1G where $D_1 = \text{diag}\{e^{-\theta_i s}\}$, but the RGA-matrix is unchanged since $\Lambda(D_1G) = \Lambda(G)$.

have $j\omega$-axis zeros and RHP-zeros at this frequency. However, the assumption of element-by-element uncertainty is often poor from a physical point of view because the elements are usually always coupled in some way.

(2) Process identification. Models of multivariable plants, $G(s)$, are often obtained by identifying one element at the time, for example, by using step or impulse responses. From Theorem 2 it is clear this method will most likely give meaningless results (e.g. wrong sign of $\det(G(0))$ or non-existing RHP-zeros) if there are large RGA-elements within the bandwidth where the model is intended to be used. Consequently, identification must be combined with physical knowledge if a good multivariable model is desired in such cases.

(3) Uncertainty in state matrix. Consider a stable linear system written on state variable form; $dx/dt = Ax + \cdots$. Then changing the ijth element in A from a_{ij} to $a_{ij}(1 - 1/\lambda_{ij}(A))$ yields one eigenvalue of A equal to zero. Thus, we may conclude that systems with large RGA-elements of A, will become unstable for small relative changes in the elements of A.

3.4. *RGA and diagonal input uncertainty.* One kind of uncertainty that is *always* present is input uncertainty. Let the nominal plant model be $G(s)$, and the true (perturbed) plant be $G_p = G(I + \Delta)$. $\Delta = \text{diag}\{\Delta_i\}$ is a diagonal matrix consisting of the relative uncertainty (error) in the gain of each input channel. If an "inverse-based" controller (decoupler) is used, $C(s) = G^{-1}(s)K(s)$, where $K(s)$ is a diagonal matrix, then the true open loop gain G_pC is

$$G_pC = (I + G\Delta G^{-1})K. \qquad (10)$$

Result. The diagonal elements of $G\Delta G^{-1}$ are directly given by the RGA (Skogestad and Morari, 1987b)

$$(G\Delta G^{-1})_{ii} = \Sigma_{j=1}^{n} \lambda_{ij}(G)\Delta_j. \qquad (11)$$

Thus, if the plant has large RGA elements and an inverse-based controller is used, the overall system will be extremely sensitive to input uncertainty.

3.5. *RGA and decentralized integral controllability* (DIC).

Definition of DIC. A plant $G(s)$ (corresponding to a given pairing) is DIC if there exists a stabilizing decentralized controller with integral action such that each individual loop may be detuned independently by a factor ϵ_i $(0 \le \epsilon_i \le 1)$ without introducing instability. DIC is a property of the plant and the chosen pairings. Unstable plants are not DIC.

Theorem 3. Assume $C(s)$ is diagonal and that $G(s)C(s)$ is stable and proper. Then $\lambda_{ii}(0) < 0$ for any $i \Rightarrow$ not *DIC*.

Proof. Follows from Theorem 6 in Grosdidier *et al.* (1985). This condition is tight for 2×2 systems since in this case (Skogestad and Morari, 1988) $DIC \Leftrightarrow \lambda_{11}(0) = \lambda_{22}(0) > 0$. For 3×3 systems with $\lambda_{ii} > 0$ the necessary and sufficient condition is $DIC \Leftrightarrow \sqrt{\lambda_{11}(0)} + \sqrt{\lambda_{22}(0)} + \sqrt{\lambda_{33}(0)} > 1$ (Yu and Fan, 1900).

3.6. *RGA and stability of decentralized control systems.* Apart from the relationships between RGA and DIC presented above, we have not found any strong relationships between the RGA and overall nominal stability (NS) of a decentralized control system. However the following theorem holds.

Theorem 4. *If* $\Lambda(G) = I$ $\forall\omega$ then stability of the individual loops imply stability of the entire system.

The proof is straightforward. We find from equation (3) that $\Lambda(G) = I$ can only arise from triangular $G(s)$ (with diagonal $G(s)$ as a special case) or from $G(s)$-matrices that can be made triangular by interchanging rows and columns in such a way that the diagonal elements remain the same but in a different order (the pairings remain the same). A plant with a triangularizable transfer matrix (as described above) controlled by a diagonal controller has only "one-way coupling" and will always be stable provided the individual loops are stable.

For plants that cannot be made triangular by row and column interchanges Theorem 4 is of little use as it does not tell what deviations from $\Lambda(G) = I$ can be tolerated without impairing stability. Care should be taken to distinguish Theorem 4 from what may be termed the conventional pairing rule.

Conventional pairing "rule". Prefer pairings ij with $\lambda_{ij}(j\omega)$ close to 1 (e.g. Bristol, 1966; Seborg *et al.*, 1989)

We emphasize that the conventional pairing rule is an engineering rule of thumb, and is not based on any proof. Indeed, pairing in accordance with the conventional pairing rule may result in unstable systems even if the individual loops are tuned to be stable (for systems of dimension larger than 2×2).

Example 1. Counterexample to the conventional pairing rule.

Consider the plant

$$G(s) = \frac{(1-s)}{(1+5s)^2}\begin{pmatrix} 1 & -4.19 & -25.96 \\ 6.19 & 1 & -25.96 \\ 1 & 1 & 1 \end{pmatrix}. \qquad (12)$$

The corresponding RGA matrix is at all frequencies

$$\Lambda(G) = \begin{pmatrix} 1 & 5 & -5 \\ -5 & 1 & 5 \\ 5 & -5 & 1 \end{pmatrix}. \qquad (13)$$

If we use the pairing indicated by equation (12) and tune individual PI controllers according to the Ziegler–Nichols tuning rules we obtain controllers $c_i(s) = 4.46((7.58s + 1)/7.58s)$. However, the overall system becomes unstable even though the individual loops are stable. In order to obtain overall stability we have to detune the controller gains by a factor of 125.

4. *Performance relationships for decentralized control*

In this section we consider the implications of overall performance requirements (nominal performance—NP) on the single-loop designs. We derive bounds on the designs of the individual loops

$$|g_{ii}c_i(j\omega)| > b_i(\omega); \quad \omega < \omega_B, \qquad (14)$$

which when satisfied yield performance (NP) of the overall system (with all loops closed). Note that the relationships for performance derived below require stability of the overall system (NS) as a prerequisite, that is, NS must be tested separately.

4.1. *Notation.* The controller $C(s)$ is diagonal with entries $c_i(s)$ (see Fig. 1). This implies that after the variable pairing has been determined, the order of the elements in y and u has been arranged so that the plant transfer matrix $G(s)$ has the elements corresponding to the paired variables on the main diagonal. Let $y(s)$ denote the output response for the overall system when all loops are closed and let $e(s) = y(s) - r(s)$ denote the output error. Then

$$e(s) = -S(s)r(s) + S(s)G_d(s)z(s); \quad S = (I + GC)^{-1}. \qquad (15)$$

Here z denotes the disturbances. G is assumed to be a $n \times n$ square matrix, but G_d may be nonsquare. The bandwidth of

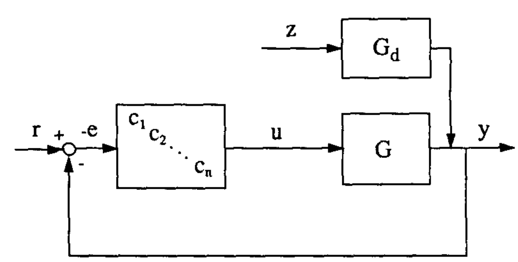

FIG. 1. Block diagram of decentralized control structure.

the system, ω_B, is defined as the frequency where $\sigma(GC(j\omega))$ (or the asymptote of $\bar{\sigma}(S(j\omega))$) crosses one. This frequency range is also called called the "crossover region".

The matrix consisting of only the diagonal elements of $G(s)$ is denoted $\tilde{G}(s)$, $\tilde{y}(s)$ denotes the response of the individual subsystems, that is, $\tilde{y}_i(s)$ is the response when loop i is closed and the other loops are open. The closed-loop sensitivity functions for the individual loops may be collected in the diagonal matrix $\tilde{S}$:

$$\tilde{e}(s) = -\tilde{S}(s)r(s) + \tilde{S}(s)G_d(s)z(s). \quad (16)$$

$$\tilde{S} = (I + \tilde{G}C)^{-1} = \text{diag}\{\tilde{s}_{ii}\}; \quad \tilde{s}_{ii} = (1 + g_{ii}c_i)^{-1}. \quad (17)$$

Note that the elements in $\tilde{S}$ are not equal to the diagonal elements of S.

4.2. *Performance requirements* (*definition of NP*). Assume that G and G_d have been scaled such that at each frequency (1) the expected disturbances, $|z_k(j\omega)|$, are less than one, and (2) the outputs, y_i, are such that the expected setpoint changes, $|r_j(j\omega)|$ are less than one. As a NP performance specification we shall require for any setpoint change, r_j, that the offset e_i is bounded:

$$|e_i(j\omega)/r_j(j\omega)| = |S_{ij}(j\omega)| < 1/|w_{ri}(j\omega)|; \quad \forall\omega, \forall i, \forall j. \quad (18)$$

Here $w_{ri}(s)$ is a scalar performance weight. For any disturbance z_k we require that

$$|e_i(j\omega)/z_k(j\omega)| = |[SG_d]_{ik}(j\omega)| < 1/|w_{di}(j\omega)|; \quad \forall\omega, \forall i, \forall k. \quad (19)$$

Typically, both weights $|w_{di}(j\omega)|$ and $|w_{ri}(j\omega)|$ are large at low frequencies where small offset is desired. $|w_{ri}|$ is often about 0.5 at high frequencies to guarantee an amplification of high-frequency noise of 2 or less. Thus we have a number of performance specifications we want satisfied simultaneously.

4.3. *Bounds on single-loop designs.* In this section we shall use the above definition of performance to obtain bounds on the individual transfer functions $g_{ii}c_i$ at low frequencies. The Laplace variable s is omitted to simplify notation. For $\omega < \omega_B$ we may usually assume

$$S = (I + GC)^{-1} \approx (GC)^{-1}, \quad (20)$$

We thus have:

$$e = -Sr + SG_dz \approx -C^{-1}G^{-1}r + C^{-1}G^{-1}G_dz. \quad (21)$$

$$= -(\tilde{G}C)^{-1}\tilde{G}G^{-1}r + (\tilde{G}C)^{-1}\tilde{G}G^{-1}G_dz; \quad \omega < \omega_B, \quad (22)$$

where $\Gamma = \tilde{G}G^{-1}$ is the PRGA matrix, and $\tilde{G}G^{-1}G_d = \Gamma G_d$ is known as the closed loop disturbance gain (CLDG) matrix (Skogestad and Hovd, 1990). The elements of Γ are denoted by γ_{ij} and those of ΓG_d are denoted by δ_{ij}. The step from (21) to (22) requires that the diagonal elements of G are nonzero. We have proven the following theorem.

Theorem 5. For plants with nonzero diagonal elements in $G(s)$, and at frequencies $\omega < \omega_B$ where (20) holds, the *NP* specifications (18) and (19) are satisfied iff

$$|g_{ii}c_i(j\omega)| > |\gamma_{ij}w_{ri}(j\omega)|; \quad \forall\omega < \omega_B, \forall i, \forall j, \quad (23)$$

$$|g_{ii}c_i(j\omega)| > |\delta_{ik}w_{di}(j\omega)|; \quad \forall\omega < \omega_B, \forall i, \forall k. \quad (24)$$

For a given choice of pairings Theorem 5 provides lower bounds on the individual loop gains to achieve NP. We get one bound on the loop gain $g_{ii}c_i$ for each setpoint j and each disturbance k. The bounds may be difficult to satisfy if γ_{ij} or δ_{ik} are large. A plot of $|\gamma_{ij}(j\omega)|$ as a function of frequency will give useful information about for which input–output pairs ij we may expect interactions. A plot of $|\delta_{ik}(j\omega)|$ will give useful information about which disturbances k are difficult to reject.

Comparison with all loops open. To get a better physical interpretation of the PRGA and CLDG consider the response e_i to a setpoint change r_i and a disturbance z_k when all the other loops are *open*. We get

$$e_i = -(1 + g_{ii}c_i)^{-1}r_i + (1 + g_{ii}c_i)^{-1}g_{dik}z_k. \quad (25)$$

When all loops are *closed* simultaneously and we assume $\tilde{S} \approx (\tilde{G}C)^{-1}$ we get from (22)

$$e \approx -\tilde{S}\Gamma r + \tilde{S}\Gamma G_dz; \quad \omega < \omega_B, \quad (26)$$

or

$$e_i \approx -(1 + g_{ii}c_i)^{-1}\gamma_{ij}r_j + (1 + g_{ii}c_i)^{-1}\delta_{ik}z_k; \quad \omega < \omega_B. \quad (27)$$

Comparing (25) and (27) we see for a setpoint change r_i in loop i that the performance relative gain, γ_{ii}, gives the approximate change in offset caused by closing all the loops. In addition, γ_{ij} gives the effect of setpoint change r_j on output ϵ_i when the other loops are closed. That is, for $\omega < \omega_B$ we have $s_{ij}/\tilde{s}_{ii} \approx \gamma_{ij}$, and we see that γ_{ii} is a measure of performance degradation at low and intermediate frequencies. Similarly, for loop i and disturbance z_k we see that g_{dik} in (25) is replaced by δ_{ik} in (27), which explains why the name closed loop disturbance gain is chosen for δ_{ik}. Also note that the ratio δ_{ik}/g_{dik} is the Relative Disturbance Gain (RDG) introduced by Stanley *et al.* (1985).

4.4. *Limitations of Theorem* 5.

(1) The main limitation with the bounds in Theorem 5 is that they apply only to lower and intermediate frequencies.

(2) Furthermore, they only address performance, and stability must be considered separately. For example, for input disturbances, i.e. $G_d = G$, we get the closed-loop disturbance gains $\delta_{ik}(s) = g_{ii}(s)$. Thus, it seems from performance considerations with respect to input disturbances that large diagonal elements in G (when appropriately scaled for disturbances) should be avoided. This is opposite of the conventional pairing rule of selecting inputs that have large effects on the controlled variables (i.e. $|g_{ii}(j\omega)|$ should be large), e.g. Balchen and Mumme (1988) and Seborg *et al.* (1989). The reason for the apparent discrepancy is stability issues and even more importantly input constraints which generally favor pairing on large elements.

(3) Theorem 5 requires that the approximation $S \approx (GC)^{-1}$ holds for individual elements in S. It may appear that this approximation is poor for elements in S corresponding to elements in G^{-1} equal to zero. However, we show in Appendix 1 that if I^{ji} and G^{ji} have zero gain in the same direction, the approximation in (20) holds also for this element. Thus, in most cases Theorem 5 will hold (structurally) also for the zero elements in G^{-1} provided $g_{ii} \neq 0$. For example, it holds for all elements when G is triangular.

(4) Another limitation with Theorem 5 is the assumption that $g_{ii} \neq 0$, $\forall i$. However, we may derive alternative bounds when $g_{ii} = 0$ as shown below for the 2×2 case.

2×2 *plant with diagonal element zero.* Without loss of generality assume $g_{11} = 0$. In this case the previously derived performance bounds (23) apply neither to loop 1 nor to loop 2. To derive appropriate bounds consider the elements s_{ij} of $S = (I + GC)^{-1}$ directly, and assume

$$|g_{22}c_2| \gg 1; \quad |c_1| \gg \left|\frac{g_{22}}{g_{12}g_{21}}\right|, \quad (28)$$

such that $\det(I + GC) \approx \det(GC)$. The performance requirements for setpoint tracking to replace (23) then become

$$|c_1| > \left|\frac{g_{22}}{g_{12}g_{21}}w_{r1}\right|; \quad |c_1| > \left|\frac{1}{g_{21}}w_{r1}\right|;$$
$$|g_{22}c_2| > \left|\frac{g_{22}}{g_{12}}w_{r2}\right|; \quad |c_1c_2| > \left|\frac{1}{g_{12}g_{21}}w_{r2}\right|. \quad (29)$$

The last bound puts a requirement on the product of the controller gains. This is reasonable since with $g_{11} = 0$ input u_1 can only affect output y_1 by the indirect action of control loop 2. For disturbance rejection, closed loop disturbance gains can be calculated for loop 2 as if g_{11} is non-zero. For loop 1, bounds on the controller gain c_1 for disturbance rejection are found from the elements in the first row of $[G^{-1}G_d]$.

4.5. *Comparison with previous work.* Mathematically, the performance specification (18) and (19) used above may be written

$$\|[W_r S \ W_d S G_d](j\omega)\|_e < 1, \quad \forall \omega, \tag{30}$$

where the e-norm used spatially (channels) is the spatial ∞-norm, which is the largest modulus of the elements in the matrix. $W_r = \text{diag}\{w_{ri}\}$ and $W_d = \text{diag}\{w_{di}\}$ are diagonal matrices specifying the desired performance in each output. This performance specification is very similar to the H_∞-norm, but in the latter case the induced 2-norm is used spatially. Consider the special case where $W_r = W_d = W_P$ and we have for the H_∞-performance specification

$$\bar{\sigma}(W_P[S \ SG_d](j\omega)) < 1, \quad \forall \omega \Leftrightarrow \|W_p[S \ SG_d]\|_\infty < 1. \tag{31}$$

Skogestad and Morari (1989) have shown how one from the NP-condition (31) may derive the tightest possible bounds on the individual loops, for example, in terms of bounds on $|h_i|$, $|s_i|$ or $|g_{ii}c_i|$. These results are very powerful, but unfortunately the same bound is used for all loops, and this may be conservative. It is possible to derive less conservative bounds by introducing additional adjustable parameters ("weights"), but it is not all obvious how this should be done *a priori* (see Nett and Uhtgenannt, 1988, for an example on how difficult it is even for a very simple case). However, using the spatial ∞-norm for the matrix as in (18), (19) and (30) makes it is much simpler to derive tight bounds on the individual loops.

In the paper we have shown that $\gamma_{ii} = \lambda_{ii}$ is a measure of performance degradation in terms of the diagonal elements in the sensitivity function, S. These results apply at small frequencies below crossover, but for control purposes the most important frequency region is close to crossover. Nett (e.g. Minto and Nett, 1989) has presented results which relate $(\Lambda - I)$ and performance degradation in terms of $H = I - S$. These bounds are most useful at frequencies *beyond* crossover, but this frequency region by itself is not too interesting. However, our results complement each other and indicate that we should have $\Lambda \approx I$ at crossover in order to avoid degradation in performance when other loops are opened or closed. This provides a performance justification for the conventional pairing "rule". However, this justification only applies if we have a design objective to maintain the same performance for the overall system as for the individual loops. However, we may want to sacrifice the latter in order to meet some other design objective, as demonstrated in example 3 below.

5. Examples

5.1. *Example* 1 *continued.* We return to Example 1 in Section 3.6 to illustrate that pairing according to $\lambda_{ii} = 1 \forall i$ may be undesirable from the point of view of performance. We consider the two alternative pairings corresponding to positive RGA values. For the pairing corresponding to $\lambda_{ii} = 1$ the transfer function matrix $G(s)$ is given by (12), whereas for the pairing corresponding to $\lambda_{ii} = 5$ the transfer function matrix is rearranged to give

$$G'(s) = \frac{(1-s)}{(1+5s)^2}\begin{pmatrix} -4.19 & -25.96 & 1 \\ 1 & -25.96 & 6.19 \\ 1 & 1 & 1 \end{pmatrix}. \tag{32}$$

The control problem is formulated as follows: With a diagonal performance weight $Wp(s)$ with all diagonal elements equal

$$Wp(s)_{i,i} = 0.5\frac{\tau_{CL}s + 1}{\tau_{CL}s}, \tag{33}$$

minimize τ_{CL} subject to an upper bound on the weighted sensitivity

$$\|WpS\|_\infty \leq 1. \tag{34}$$

This means that we allow a maximum peak in the sensitivity function at high frequencies of 2, and seek the controller which minimizes the closed loop time constant in the slowest direction. Only PI controllers were considered, and the tunings were obtained by a numerical search. The results demonstrate that it is advantageous to choose the pairing corresponding to $\lambda_{ii} = 5$ rather than $\lambda_{ii} = 1$. For the pairing corresponding to $\lambda_{ii} = 5$ we were able to fulfill (34) with $\tau_{CL} = 220$ whereas for the pairing corresponding to $\lambda_{ii} = 1$ we had to increase τ_{CL} to 1160 in order to be able to fulfill (34). Although the resulting closed loop systems are quite slow for both pairings (relative to the RHP zero at $s = 1$) the pairing corresponding to $\lambda_{ii} = 5$ is significantly better. The controller parameters are given in Table 1.

TABLE 1. PI CONTROLLER PARAMETERS FOR EXAMPLE 1 ($c_i(s) = k_i((\tau_i s + 1)/\tau_i s)$)

	$\lambda_{ii} = 5$	$\lambda_{ii} = 1$
k_1	−0.6840	0.1230
τ_1	24.15	32.40
k_2	−0.02425	0.1443
τ_2	7.270	34.54
k_3	0.007685	0.002940
τ_3	0.3688	3.988
τ_{CL}	220	1160

5.2. *Example* 2: *distillation column control.* In order to demonstrate the use of the frequency dependent PRGA and CLDG for evaluation of expected control performance and control structure selection, a binary distillation column with 40 theoretical trays plus a total condenser is considered. This is the same example as studied by Skogestad *et al.* (1988), but we use a more rigorous model which includes liquid dynamics in addition to the composition dynamics. Using model reduction, the number of states in the model was reduced from 82 to 5. Disturbances in feed flowrate $F(z_1)$ and feed composition $z_F(z_2)$, are included in the model. The LV configuration is used, that is, the manipulated inputs are reflux $L(u_1)$ and boilup $V(u_2)$. Outputs are the product compositions $y_D(y_1)$ and $x_B(y_2)$. The model then becomes

$$\begin{pmatrix} dy_1 \\ dy_2 \end{pmatrix} = G(s)\begin{pmatrix} du_1 \\ du_2 \end{pmatrix} + G_d(s)\begin{pmatrix} dz_1 \\ dz_2 \end{pmatrix}. \tag{35}$$

A state space description is given in Appendix 2. The disturbances and outputs have been scaled such that a magnitude of 1 corresponds to a change in F of 30%, a change in z_F of 20%, and a change in x_B and y_D of 0.01 mole fraction units.

Pairings. We choose u_1 to control y_1 and u_2 to control y_2, as indicated by (35), in order to have positive steady state relative gains. This is in agreement with industrial practice.

Analysis of the model. Figure 2 shows the open-loop disturbance gains, g_{dik}, as a function of frequency. These gains are quite similar in magnitude and rejecting disturbances z_1 and z_2 seems to be equally difficult. However, this conclusion is incorrect. The reason is that the *direction* of these two disturbances is quite different, that is, disturbance 2 is well aligned with G and is easy to reject, while disturbance 1 is not (Skogestad and Morari, 1987a). This is seen from Fig. 3 where the closed-loop disturbance gains, δ_{i2}, for z_2 are seen to be much smaller than δ_{i1} for z_1. The diagonal relative gains for the loops are also included in Fig. 3 (note that $\gamma_{11} = \lambda_{11}$ and $\gamma_{11} = \gamma_{22}$ for 2×2 plants). We see that rejection of disturbance 1 (as indicated by $|\delta_{i1}|$) and setpoint following (as indicated by $|\gamma_{ii}|$) put similar bounds

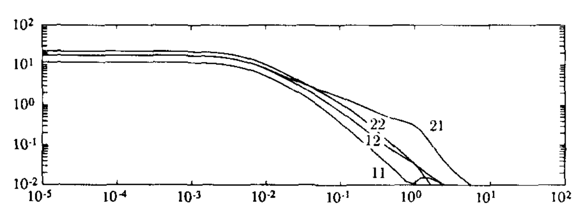

FIG. 2. Open loop disturbance gains, $|g_{dik}|$, for example 2.

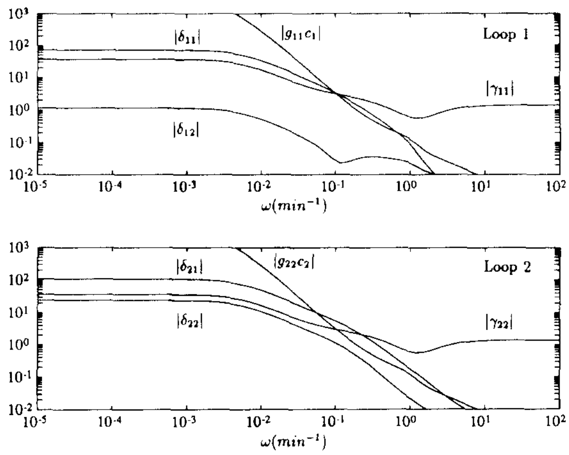

FIG. 3. Bounds on loop gains for example 2. NP at low frequencies ($\omega < \omega_B$) requires $|g_{ii}c_i|/|\gamma_{ii}| > |w_{ri}|$ and $|g_{ii}c_i|/|\delta_{ik}| > |w_{di}|$. Performance weights w_{ri} and w_{di} are not shown, but these are generally large at low frequencies and approach 1 at $\omega \approx \omega_B$.

on the loop gain $|g_{ii}c_i|$. Assuming that the performance requirement around crossover corresponds to performance weights $|w_d(j\omega_B)| \approx |w_r(j\omega_B)| \approx 1$ we find that the minimum bandwidth requirement for both loops is about 0.5 rad min^{-1}. Note that interactions become severe and performance will deteriorate drastically if the loops are detuned much below this value.

Observed control performance. To check the validity of the above results we designed single-loop PI controllers by optimizing robust performance with a one minute time delay using (31) as the performance specification. The controllers obtained are:

$$c_1(s) = 0.261\frac{1+3.76s}{3.76s}; \quad c_2(s) = -0.375\frac{1+3.31s}{3.31s}. \quad (36)$$

The loop gains, $|g_{ii}c_i|$, with these controllers are also shown in Fig. 3. The loop gains are seen to be larger than the closed-loop disturbance gains, $|\delta_{ik}|$, at all frequencies up to crossover. Closed-loop simulations with these controllers are shown in Fig. 4. The simulations confirm that disturbance 2 is much easier rejected than disturbance 1. In summary, there is an excellent correlation between the analysis based on $|\delta_{ik}|$ in Fig. 3 and the simulations. This is not surprising when one considers Fig. 5 which shows the accuracy of the approximation $[S(s)G_d(s)]_{ik} \approx [\tilde{S}\Gamma G_d]_{ik}$ which formed the basis for the analysis in Fig. 3. The approximation is very good at low frequencies, but as expected is poorer at frequencies around the closed loop bandwidth. The most

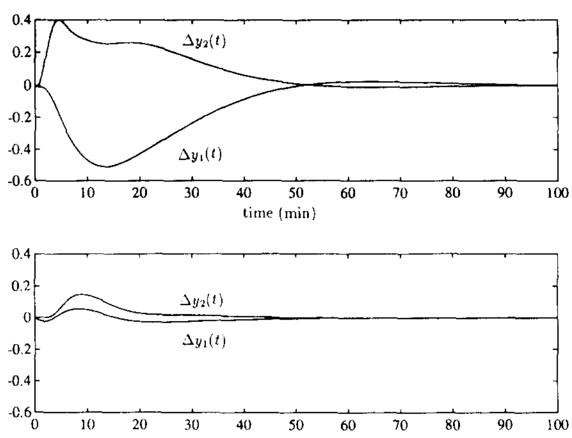

FIG. 4. Responses for example 2 to a unit step disturbance in z_1 (top) and in z_2 (bottom).

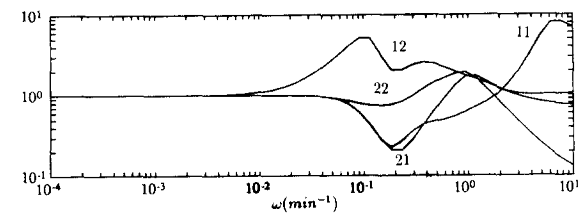

FIG. 5. Check of approximation $SG_d \approx \tilde{S}\Gamma G_d$ for example 2. The figure shows the magnitude of $[SG_d]_{ik}/[\tilde{S}\Gamma G_d]_{ik}$.

significant deviation occurs for $i = 1$, $k = 2$ at frequencies around 0.1 rad min^{-1}, where we see that the actual disturbance rejection is poorer than the approximation. This explains why the effect of z_2 on y_1 in Fig. 4 is somewhat poorer than might be expected from Fig. 3.

5.3. *Example* 3: *pairing corresponding to* $\lambda_{ii} = 0$. In order to demonstrate that acceptable performance may be achieved even with pairings corresponding to $\lambda_{ii} = 0$, consider control of the top part of a distillation column. It is desired to control the top product composition (y_1) and the level in the condenser (y_2). The manipulated inputs are the distillate flowrate (u_1) and the reflux flowrate (u_2). The vapor flowrate entering the top of the column is considered to be the only disturbance (z_1). The achievable bandwidth is limited by unmodeled measurement delay in y_1 of one minute and valve dynamics in u_2 equivalent to a time delay of 0.1 minute. After scaling, the resulting transfer functions are

$$\begin{pmatrix} dy_1 \\ dy_2 \end{pmatrix} = \begin{pmatrix} 0 & \dfrac{100}{1+100s} \\ \dfrac{-1}{s} & \dfrac{-1}{s} \end{pmatrix} \begin{pmatrix} du_1 \\ du_2 \end{pmatrix} + \begin{pmatrix} \dfrac{-100}{1+100s} \\ \dfrac{1}{s} \end{pmatrix} dz_1. \quad (37)$$

This pairing corresponds to $\lambda_{11} = \lambda_{22} = 0$ at all frequencies. This pairing may be preferred in some cases, for example, if the reflux is large such that *constraints* on the distillate flowrate make level control with this input difficult. The chosen controllers are

$$c_1(s) = -0.5\frac{1+10s}{10s}; \quad c_2(s) = -5. \quad (38)$$

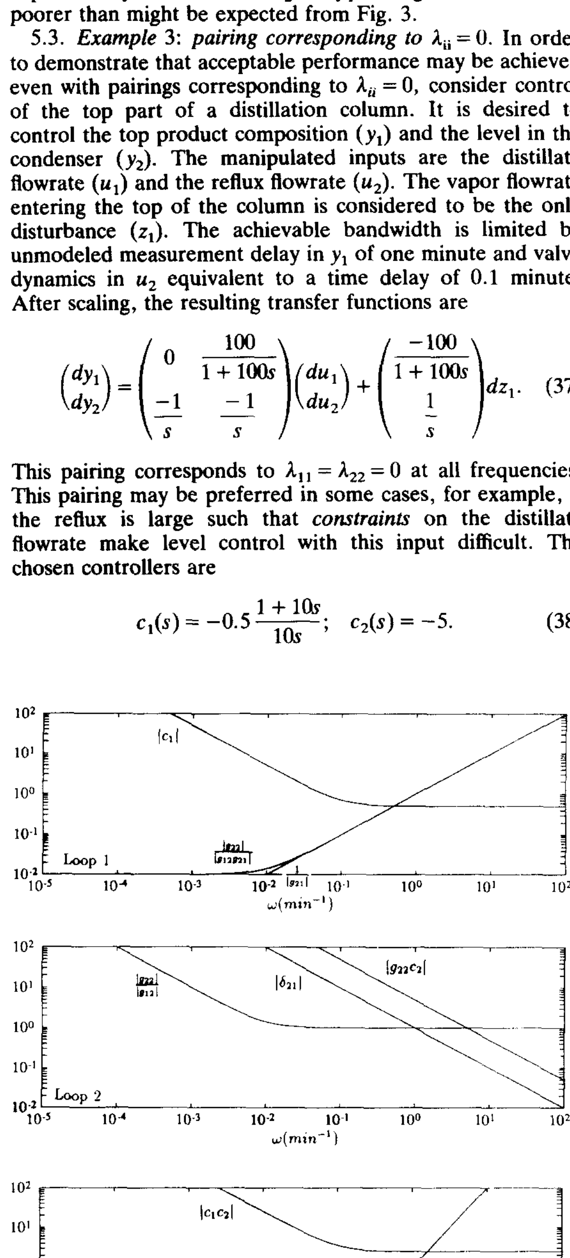

FIG. 6. Bounds on controller and loop gains for example 3.

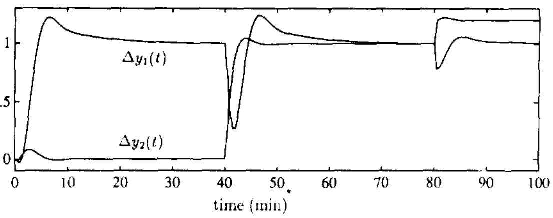

FIG. 7. Responses for example 3 to unit step changes in r_1 at $t = 0$, in r_2 at $t = 40$, and in z_1 at $t = 80$.

To check NP, the controllers and the bounds (29) for the case with zero diagonal elements are shown in Fig. 6. The bounds in Fig. 6 indicate that interactions put no serious limitations on achievable performance. In Fig. 7 we show responses to changes in setpoints r_1 and r_2 and in disturbance z_1. In the simulations a first order filter with a time constant of one minute is used for both setpoints, and a one minute time delay in the measurement of y_1 and a 0.1 minute time delay in manipulated variable u_2 are approximated with first order Padé approximations. The observed control performance is satisfactory, although there is an undesirable interaction from setpoint r_2 to output y_1. This interaction cannot be predicted from Fig. 6, as equation (20) does not hold in the crossover region where the interactions occur.

6. *Conclusions on decentralized control*

In the paper we have derived bounds on the designs of the individual loops which when satisfied yield performance (NP) of the overall system (with all loops closed). For setpoint tracking the bounds are given by the performance relative gains, $|\gamma_{ii}|$ (equation (23), and for disturbance rejection by the closed-loop disturbance gains, $|\delta_{ik}|$ (equation (24)). The bounds are tight (necessary and sufficient) at low frequencies where $S \approx (GC)^{-1}$. It is desirable that the bounds are as small as possible because a large bound requires a large bandwidth in loop i. Since stability of the individual loops is desired this may be impossible if $g_{ii}(s)$ contains time delays, neglected or uncertain dynamics, or RHP-zeros.

Importantly, these bounds depend on the model of the process only, that is, are independent of the controller. This means that frequency-dependent plots of γ_{ii} and δ_{ik} may be used to evaluate the achievable closed-loop performance (controllability) under decentralized control. Plants with small values of these measures are preferred. Furthermore, the values of δ_{ik} may tell the engineer which disturbance k will be most difficult to handle using feedback control. This may pinpoint the need for using feedforward control, or for modifying the process. For example, in process control adding a feed buffer tank will dampen the effect of disturbances in flowrate, temperature or composition. Plots of δ_{ik} may be used to tell if a tank is necessary and what holdup (residence time) would be needed.

The bounds may also be used to obtain a first guess of the controller parameters. However, as the derivation of the bounds depends on approximations which are valid at low frequencies only, undesirable effects may occur at frequencies around the closed loop bandwidth. Thus the behavior of the closed-loop system must be checked using other methods, and the controllers possibly redesigned.

Acknowledgements—Support from NTNF is gratefully acknowledged.

Nomenclature. (See also Section 4.1.)

D_1, D_2—Diagonal matrices
$e = y - r$—Vector of offsets
$g_{ij} = [G]_{ij}$—ijth element of G
$g_{dik} = [G_d]_{ik}$—ikth element of G_d
G—Plant transfer matrix
$\tilde{G}$—Matrix consisting of diagonal elements of G
G^{ij}—G with row i and column j removed
r—Vector of reference outputs (setpoints)
u—Vector of manipulated inputs
w_{di}—Performance weight for disturbance rejection in loop i.
w_{ri}—Performance weight for setpoint following in loop i.
y—Vector of outputs
$Y = g_{12}g_{21}/g_{11}g_{22}$—Rijnsdorp or Balchen interaction measure for 2×2 system
z—Vector of disturbances

Greek letters

$\delta_{ik} = g_{ii}[G^{-1}G_d]_{ik} = [\tilde{G}G^{-1}G_d]_{ik}$—Closed loop disturbance gain (CLDG)
$\gamma(G) = \bar{\sigma}(G)/\underline{\sigma}(G)$—Condition number
$\gamma^*(G) = \min_{D_1 D_2} \gamma(D_1 G D_2)$—Optimal (minimized) condition number
$\gamma_{ij} = g_{ii}[G^{-1}]_{ij} = [\tilde{G}G^{-1}]_{ij}$—Performance relative gain
Γ—Matrix of performance relative gains (PRGA)
$\lambda_i(G)$—ith eigenvalue of matrix G
$\lambda_{ij}(G) = g_{ij}[G^{-1}]_{ji}$—$ij$th element in RGA-matrix Λ
Λ—RGA matrix
ω—frequency
ω_B—closed loop bandwidth

Norms

$\rho(A) = \max_i |\lambda_i(A)|$—Spectral radius
$\bar{\sigma}(A)$—Maximum singular value or spectral norm ($= \|A\|_{i2}$—induced 2-norm)
$\underline{\sigma}(A) = 1/\bar{\sigma}(A^{-1})$—minimum singular value
$\|A\|_1 = \Sigma_{i,j} |a_{ij}|$—1-norm
$\|A\|_2 = (\Sigma_{ij} |a_{ij}|^2)^{0.5}$—2-norm (Euclidean norm)
$\|A\|_e = \max_{ij} |a_{ij}|$—$e$-norm (magnitude of largest element in A)
$\|G\|_\infty = \sup_\omega \bar{\sigma}(G)$—$H^\infty$-norm of $G(j\omega)$
$\|A\|_{i1} = \max_j \Sigma_i |a_{ij}|$—induced 1-norm ("largest column sum")
$\|A\|_{i\infty} = \max_i \Sigma_j |a_{ij}|$—induced ∞-norm ("largest row sum")
$\|A\|_m = 2\max\{\|A\|_{i1}, \|A\|_{i\infty}\}$

Subscripts

i—Index for outputs or loops
j—Index for manipulated inputs or setpoints
k—Index for disturbances

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Appendix 1

Consider a non-singular plant transfer function matrix G, and assume that neither G nor the diagonal controller C has a pole on the imaginary axis at the frequency in question. We have $[G^{-1}]_{ij} = (-1)^{i+j}\det(G^{ji})/\det(G)$ where G^{ji} denotes the matrix G with row j and column i removed. Correspondingly, $S_{ji} = (-1)^{i+j}\det(I + GC)^{ji}/\det(I + GC)$.

Proposition.

$[G^{-1}]_{ij} = 0$ and (I^{ji} and G^{ji} have zero gain in the same input direction)) $\Rightarrow S_{ij} = [(I + GC)^{-1}]_{ij} = 0$.

Proof.

If I^{ji} and G^{ji} have zero gain in the same input direction, $(GC)^{ji}$ will have zero gain in the same direction, as C is diagonal. Thus, $\det(I^{ji} + (GC)^{ji}) = \det(I + GC)^{ji} = 0 \Rightarrow S_{ij} = 0$.

Appendix 2

Transfer function matrices for distillation column in Example 2. The transfer function matrices $G(s)$ and $G_d(s)$ can be calculated from the formulae

$$G(s) = C(sI - A)^{-1}B + D,$$

and

$$G_d(s) = C(sI - A)^{-1}B_d + D_d.$$

$$A = \begin{pmatrix} -5.161e-3 & 0 & 0 & 0 & 0 \\ 0 & -7.366e-2 & 0 & 0 & 0 \\ 0 & 0 & -1.829e-1 & 0 & 0 \\ 0 & 0 & 0 & -4.620e-1 & 9.895e-1 \\ 0 & 0 & 0 & -9.895e-1 & -4.620e-1 \end{pmatrix}, \quad (39)$$

$$B = \begin{pmatrix} -6.296e-2 & 6.236e-2 \\ 5.481e-3 & -1.719e-2 \\ 3.041e-3 & -1.078e-2 \\ -1.856e-2 & -1.393e-2 \\ -1.229e-1 & -5.608e-3 \end{pmatrix}, \quad (40)$$

$$C = \begin{pmatrix} -7.223 & -5.170 & 3.836 & -1.633e-1 & 1.121 \\ -8.913 & 4.728 & 9.876 & 8.425 & 2.186 \end{pmatrix}, \quad (41)$$

$$D = D_d = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad (42)$$

$$B_d = \begin{pmatrix} -9.364e-3 & -1.333e-2 \\ 1.960e-2 & 8.018e-3 \\ 3.266e-3 & -2.116e-2 \\ -2.827e-2 & 5.319e-3 \\ -6.784e-3 & 2.719e-3 \end{pmatrix}, \quad (43)$$