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Robust Controller Design for Uncertain Time Delay Systems

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Abstract

In this paper we consider robust controller design for uncertain SISO time-delay systems using H_∞ and μ methods. We model the parametric uncertainty with a nominal model plus multiplicative uncertainty and compare a number of different choices for the weight, including real and complex perturbations. Mean values are usually used in the nominal model to get smaller uncertainty. We propose to model also the nominal time-delay as uncertainty. This leads to a larger uncertainty but a delay-free design method for time-delay systems. Surprisingly, though it has larger uncertainty, this new method is not more conservative. The point here is that μ is a worst case measure. Of most importance is not the uncertainty size but the worst uncertainty. This insight is also helpful to the modelling of other uncertain systems.

1 Introduction

H_∞ control, as one of the approaches to robust control, has become a great success in the past decade. With the introduction of the structured singular value[3], μ, structured uncertainty can be handled in the H_∞ framework, and hence the criticized conservatism is substantially reduced. μ can deal with not only robust stability but also *robust performance*, i.e. the required performance specifications are met for a prespecified plant set. The standard 'M-Δ' structure is shown in Figure 1, where M is a stable transfer matrix, comprising plant, controller as well as uncertainty and performance weights, and $\Delta = diag\{\Delta_i\}$ represents the uncertainty structure. In the case of robust performance, Δ also includes a performance block. μ is defined as

$$\mu_\Delta^{-1}(M) = \min_\Delta\{\bar{\sigma}(\Delta)|det(I + M\Delta) = 0\} \tag{1}$$

Robust stability or robust performance is equivalent to

$$\sup_\omega \mu_\Delta(M) < 1 \tag{2}$$

H_∞ and μ methods are now in the practical stage. For a properly formulated problem, we are able to do systematic analysis, design and even synthesis though the μ synthesis is still not fully solved.[1] Several toolboxes exist which make the application much easier. A number of applications has been studied.

[1]The present μ synthesis algorithm, called D-K iteration, is a combination of H_∞ synthesis and the optimal D-scaling. It does not guarantee global convergence.

Although the H_∞ control theory is well-developed, more work on practical applications is needed. Especially, this applies to the selection of performance weights and uncertainty weights. In the H_∞ framework, the performance specifications are required to be given in the form of frequency dependent weights on some input and output signals. If the original specifications are not in this setting, we need to do transformation. The uncertain plant set is modelled by two elements: a nominal model and a norm-bounded uncertainty, i.e., (in SISO case) assuming all the plants in a disk region of the complex plane centered by the nominal model with a radius equal to the uncertainty weight. If μ is used, the uncertainty can be structured in the sense it can have several diagonal blocks, but each block still needs to be norm-bounded. Similarly, we generally need to transform the uncertainty from its original description to this required setting.

Weight selection is, in fact, a reformulation of the original problem. It is important since it may be crucial to the final success. However the problem specific nature makes it difficult to develop general theory for this problem. Hence even some insights and guidelines which are valid only within certain ranges may be very helpful. Some of the work in this area are [8][9][11].

In this paper we study the uncertainty weight selection problem for time-delay systems. An important advantage of the H_∞-framework is that model uncertainty on non-parametric form, including unknown model order, may be handled. However, in this paper we will shown how it can be used *also* when the original description for model uncertainty is on parametric form. Parametric uncertainty can generally be rearranged into the standard 'M-Δ' structure, but this is not the case for time-delay uncertainty unless time-delay uncertainty is approximated by rational term. Moreover, as parametric uncertainty yields real perturbations, the use of complex perturbations often introduces too much conservatism.[2] Usually one lumps the parametric uncertainties into one norm-bounded complex perturbation. A few authors have worked on robust control of time-delay systems, e.g. Laughlin *et al.*[6]. They study systematically the robust performance for Smith predictor, and give a smallest multiplicative uncertainty bound and an approximate weight for the uncertainties in gain, time constant and time delay of the first-order with time-delay model. This is the starting point of this paper. In section 2, we study the approximation of time-delay uncertainty. In section 3, we consider how to model an uncertain time-delay system, i.e. how to choose the nominal model and uncertainty weight pair. In section 4, we consider systems with time-delay uncertainty as well as other parametric uncertainties. In section 5, we make some final remarks. The study here is primarily a numerical one, but it does give some insights, some of which are even not limited to time-delay systems.

2 Approximation of time-delay uncertainty

Many practical systems have time-delays, and the time-delays are often varying. In this paper we consider a set of time-invariant plants with different time-delays. Robust stabilization means that a single controller stabilizes the entire set. However, time-delay is an irrational function, and H_∞ and μ methods are unable to deal with time-delay uncertainty directly. Here we should therefore consider rational approximations, and compare these with the actual uncertainty.

We consider the following SISO plant:

$$\textit{Plant set}: \qquad g_p(s) = g(s)e^{-\delta_\theta \Delta s}, \quad -1 \le \Delta \le 1 \tag{3}$$

$$\textit{Nominal model}: \qquad g(s) = \frac{k}{\tau s+1}e^{-\theta s} \tag{4}$$

[2]There are some progress on computing μ with real perturbation blocks[5], but it is still not satisfactory at present.

The nominal plant is the popular first order with time-delay model. For simplicity, we assume temporarily only time-delay uncertainty. We want to find out how large δ_θ can be before the closed loop system becomes unstable. The correct answer is

$$\delta_\theta = \frac{PM}{\omega_c} \tag{5}$$

where ω_c is the crossover frequency ($|g(j\omega_c)c(j\omega_c)| = 1$) and PM is the phase margin ($PM = \pi + \angle[g(j\omega_c)c(j\omega_c)]$). This provides a standard for the comparison of different approximations.

2.1 Mathematical approximations of time-delay uncertainty

Firstly we consider some mathematical approximations without any additional constraints. Four different approximations of time delay uncertainty are studied:

A1. This is the simplest approximation from power series expansion. Δ is assumed to be real as original.

$$e^{-\delta_\theta \Delta s} \approx 1 - \delta_\theta s \Delta, \quad -1 \leq \Delta \leq 1 \tag{6}$$

A2. Same as A1 but Δ is relaxed to be complex.

$$e^{-\delta_\theta \Delta s} \approx 1 - \delta_\theta s \Delta, \quad |\Delta| \leq 1 \tag{7}$$

A3. First-order Padé approximation is used. An LFT implementation of the approximation is shown in Figure 2. Δ is real.

$$e^{-\delta_\theta \Delta s} \approx \frac{1 - \frac{\delta_\theta}{2} s\Delta}{1 + \frac{\delta_\theta}{2} s\Delta} = 1 - (1 + \frac{\delta_\theta}{2} s\Delta)^{-1} \delta_\theta s \Delta, \quad -1 \leq \Delta \leq 1 \tag{8}$$

A4. A multiplicative uncertainty approximation with a first-order weight derived from the Padé approximation. Δ is complex.

$$e^{-\delta_\theta \Delta s} \approx 1 - \frac{\delta_\theta s}{1 + \frac{\delta_\theta}{2} s} \Delta, \quad |\Delta| \leq 1 \tag{9}$$

A3 is the real Padé approximation. Although we are able to deal with it when implemented as a LFT, it is a little complicated. In A4, the delay uncertainty is taken as a multiplicative uncertainty, the uncertainty weight is obtained by setting the Δ in the denominator of Padé approximation equal to 1. This is the approximation generally used in robust design[6][14]. Note that A1 and A2 are also multiplicative uncertainties, we can think that the weight is obtained by setting the Δ in the denominator of Padé approximation equal to 0, so it is always larger than the weight in A4.

In each case, we can rearrange the overall system into the '$M - \Delta$' structure. To ensure robust stability, we must have

1. in A1 and A3 where Δ is real

$$|M(j\omega_p)| < 1, \quad \forall\, \omega_p | \angle M(j\omega_p) = i\pi, \quad i \text{ integer} \tag{10}$$

2. in A2 and A4 where Δ is complex

$$\mu_{RS} = \sup_\omega \mu(M) = \sup_\omega |M(j\omega)| < 1 \tag{11}$$

Conditions (10) and (11) can be used to estimate the smallest δ_θ required to destabilize the closed loop system. Note that in A4 the approximation is no longer a function of the product $\delta_\theta \Delta$, and we have to calculate δ_θ iteratively. The Δ in A3 must be real, otherwise we will get meaningless results, since the denominate $1 + \frac{\delta_\theta}{2} j\omega\Delta$ can be zero for $\omega \geq \frac{2}{\delta_\theta}$ and hence infinity uncertainty at high frequencies.

The following PI controller is used in study

$$c(s) = \frac{k_c}{k}\frac{\tau s + 1}{s} \tag{12}$$

where k, τ are the same as in the nominal plant model (4). For different k_c value, the exact δ_θ and its various estimated values from four different approximations are calculated and shown in Table I. We see from Table I:

1. A3 is the best approximation. This is consistent with theoretical analysis since all the others can be thought as further approximation of A3. However, $\delta_{\theta 3}$ is always a bit optimistic. Another possible problem with A3 is that the perturbation should be real, which may make it difficult for applications.

2. $\delta_{\theta 1}$ is always conservative because A1 is an upper bound of A3 (A1 is obtained by setting the Δ in the denominator of Padé approximation equal to 0). A2 is even more conservative since Δ is relaxed to be complex. Note that A1 does not cover the time delay uncertainty though it is always conservative. Neither do A3 and A4.

3. A4 is a lower bound of A3 (A4 is obtained by setting the Δ in the denominator of Padé approximation equal to 1). This introduces optimism. Relaxing the Δ to be complex causes conservativeness. As a result of these two effects, $\delta_{\theta 4}$ is very optimistic when k_c is small (detuned), and is conservative when k_c is large (overtuned). However, it is good in the range where robust stability is reasonable. So A4 might practically be a good approximation. Also note that $\delta_{\theta 4}$ is always larger than $\delta_{\theta 2}$ as expected.

Theoretically, we can get arbitrary high accuracy by using multiple perturbations. We divide time-delay uncertainty $e^{-\delta_\theta \Delta s}$ into n parts $(e^{-\frac{\delta_\theta}{n}\Delta s})^n$ first, then use any approximation for each of the smaller uncertainty $e^{-\frac{\delta_\theta}{n}\Delta s}$. In this case we get n repeated blocks and higher order model. This problem is much more complicated numerically.

Table I. Stability margin (allowed uncertainty) estimated using different approximations for time-delay uncertainty

k_c	ω_c	PM	Exact δ_θ	A1 $\delta_{\theta 1}$	A2 $\delta_{\theta 2}$	A3 $\delta_{\theta 3}$	A4 $\delta_{\theta 4}$
0.1	0.1	1.4708	14.708	9.5045	9.1434	18.186	∞
0.2	0.2	1.3708	6.8540	4.4868	4.1870	8.1095	8.2375
0.5	0.5	1.0708	2.1416	1.4752	1.2575	2.3640	1.6460
1.0	1.0	0.5708	0.5708	0.4348	0.3196	0.5882	0.3276
1.5	1.5	0.0708	0.0472	0.0428	0.0255	0.0499	0.0255

2.2 Upper bounds for time-delay uncertainty

A mathematical approximation of time-delay uncertainty does not necessarily cover all the possible plants even if it is a very good approximation, such as A3 and A4. In robust design, it is preferable to use a norm-bounded uncertainty which covers all the possible plants. It may be argued that whether it is an upper bound or not is not important as far as it is a good approximation. However, we find that μ-optimal controllers seem to be sensitive to uncertainty unconsidered (we will show this latter in this paper), and this justifies the use of an upper bound. The tightest upper multiplicative uncertainty bound (smallest bound) for time-delay uncertainty is given by[6]

$$\mathbf{Sb}: \quad l(\omega) = \begin{cases} |e^{-j\delta_\theta \omega} - 1|, & \forall \quad \omega < \pi/\delta_\theta \\ 2, & \forall \quad \omega \geq \pi/\delta_\theta \end{cases} \tag{13}$$

We consider the following multiplicative uncertainty weights for the time-delay uncertainty:

$$w_0(s) = \delta_\theta s \tag{14}$$

$$w_1(s) = \frac{\delta_\theta s}{\delta_\theta s/2 + 1} \tag{15}$$

$$w_{1h}(s) = \frac{\delta_\theta s}{\delta_\theta s/3.465 + 1} \tag{16}$$

$$w_{1l}(s) = \frac{3\delta_\theta s}{3\delta_\theta s/2 + 1} \tag{17}$$

$$w_{1a}(s) = \frac{1.2\delta_\theta s}{\delta_\theta s/2 + 1} \tag{18}$$

$$w_2(s) = \frac{\delta_\theta s(2 * 0.2152^2 \delta_\theta s + 1)}{(0.2152\delta_\theta s + 1)^2} \tag{19}$$

$$w_3(s) = \frac{\delta_\theta s}{\delta_\theta s/2 + 1} \frac{(s/\omega_0)^2 + 1.676(s/\omega_0) + 1}{(s/\omega_0)^2 + 1.370(s/\omega_0) + 1}, \quad \omega_0 = 2.363/\delta_\theta \tag{20}$$

All these weights except $w_1(s)$ (and $w_{1l}(s)$) are upper bounds for time-delay uncertainty in the sense they contain all possible plants. They are plotted in Figure 3. $w_0(s)$ is the weight of A2. It approximates the smallest bound **Sb** very well at low frequencies but is much larger at high frequencies. $w_1(s)$ is the weight of A4. It approximates the smallest bound very well at both low and high frequencies with a small error in middle range, where it is a little smaller than the smallest bound. If we want to modify $w_1(s)$ to make it an upper bound, but still limit it to be first-order, we have three choices. 1). Increase $w_1(s)$ at high frequencies without changing the value at low frequencies. This yields $w_{1h}(s)$ which just covers **Sb**. 2). Conversely, increase $w_1(s)$ at low frequencies without changing the value at high frequencies. This yields $w_{1l}(s)$ which almost covers **Sb**. Exact cover is impossible in this case since $l(\omega) = 2$ at finite frequencies. 3). Increase $w_1(s)$ at all frequencies to cover **Sb** which yields $w_{1a}(s)$. The high-order weights $w_2(s)$ and $w_3(s)$ are obtained by fitting the **Sb** with second and third order transfer functions, respectively. We use the constraints that their magnitudes must be the same as **Sb** at both low and high frequencies and must not be less than **Sb** at any frequency. $w_3(s)$ is provided by Lundström[7]. Of course, we can use even higher order weights to get a better fit.

We use the same PI controller and values of k_c as in the last section. In each case, the time-delay uncertainty δ_θ is equal to be the exact uncertainty the system can tolerate before instability, so the 'exact' robust stability (with real time-delay uncertainty) μ_{RSreal} should always be equal to 1. The μ_{RS} computed from **Sb** and different multiplicative uncertainty weights are shown in Table II. We see from Table II:

1. μ_{RSsb} is computed with **Sb**, i.e. using (13). It represents the best results we can get if we limit ourselves using complex multiplicative uncertainty to cover the time-delay uncertainty. The difference between μ_{RSsb} and μ_{RSreal} is the conservatism introduced by the complex multiplicative uncertainty assumption. Somewhat surprisingly, it gets larger as the time delay uncertainty gets smaller.

2. All values, except μ_{RS1}, are larger than 1 as expected since the uncertainty weights are upper bounds (w_{1l} almost).

3. Although the difference between **Sb** and the magnitude of $w_1(s)$ is very small, the difference between μ_{RSsb} and μ_{RS1} is significant. μ_{RS1} is always less than μ_{RSsb} and hence less conservative. However it is dangerous to reduce the conservatism by releasing the upper bound requirement. In fact, μ_{RS1} is optimistic (< 1, i.e. less than the "exact" μ_{RS} computed with real time-delay uncertainty) for two cases.

4. Increasing the uncertainty at low frequencies (w_{1l}) is more conservative than increasing it at high frequencies (w_{1h}). $w_{1h}(s)$ is the best first-order weight.

5. The differences among μ_{RS1h}, μ_{RS2}, and μ_{RS3} are minor, so increasing the weight order does not improve much. This is because μ_{RS1h} is already quite close to μ_{RSsb}. The remaining conservativeness comes from the complex multiplicative uncertainty assumption, and we can do nothing with this unless we are able to compute with real time-delay uncertainty. μ_{RS3} is even worse than μ_{RS2} and μ_{RS1h} for one case. This is because we have not restricted $w_3(s)$ not larger than $w_2(s)$ at any frequency in the fitting.

6. The conclusion is that $w_{1h}(s)$ is a simple, good and reliable approximation of time-delay uncertainty.

Here the comparison is only based on robust stability. In the next section we will provide further result on robust performance.

Table II. Different complex multiplicative uncertainty approximations for time-delay uncertainty, robust stability μ_{RS}

Weight		Exact	w_{Sb}	w_0	w_{1h}	w_{1l}	w_{1a}	w_1	w_2	w_3
k_c	δ_θ	μ_{RSreal}	μ_{RSsb}	μ_{RS0}	μ_{RS1h}	μ_{RS1l}	μ_{RS1a}	μ_{RS1}	μ_{RS2}	μ_{RS3}
0.1	14.708	1	1.0694	1.6086	1.1148	1.4230	1.1008	0.9023	1.0983	1.0882
0.2	6.8540	1	1.1146	1.6370	1.1660	1.4522	1.1429	0.9368	1.1474	1.1332
0.5	2.1416	1	1.3808	1.7031	1.4254	1.7073	1.4222	1.1657	1.4101	1.4062
1.0	0.5708	1	1.7403	1.7862	1.7417	3.4883	2.0275	1.6619	1.7409	1.7460
1.5	0.0472	1	1.8517	1.8521	1.8517	5.5232	2.2580	1.8509	1.8517	1.8517

3 Modelling of uncertain time-delay systems

The approximation of time-delay uncertainty has been considered in the last section. Another problem is how to model an uncertain time-delay system. For SISO plant

$$Plant\ set: \quad g_p(s) = \frac{k}{\tau s+1} e^{-\theta_p s}, \quad 0 \le \theta_p \le \theta \tag{21}$$

we can model it in two different ways:

Approach 1: Nominal model with time delay.

$$Nominal\ model: \qquad g(s) = \frac{k}{\tau s+1} e^{-\frac{\theta}{2}s} \tag{22}$$

$$Time-delay\ uncertainty: \qquad e^{-\frac{\theta}{2}\Delta s}, \quad -1 \le \Delta \le 1 \tag{23}$$

Approach 2: Nominal model without time delay.

$$Nominal\ model: \qquad g(s) = \frac{k}{\tau s+1} \tag{24}$$

$$Time-delay\ uncertainty: \qquad e^{-\theta\Delta s}, \quad -1 \le \Delta \le 1 \tag{25}$$

Intuitively, one should expect that Approach 1 gives better result than Approach 2. But on the other hand, Approach 2 is much simpler, since the analysis and design is completely delay-free when complex norm-bounded uncertainty is used. As it is well-known, time-delay often greatly complicates the analytical and computational aspects of system analysis and design, and this is the case for H_∞ and μ methods. Moreover, H_∞ synthesis and μ synthesis in both existing toolboxes, μ-Analysis and Synthesis Toolbox[1] and Robust Control Toolbox[2], are even incapable to deal with time-delay. So in Approach 1 we have to approximate the time-delay in nominal model by a rational transfer function before the design. But we found in [15] that rational approximation of time-delay may cause significant deterioration in robust performance because it introduces additional uncertainty. Due to these problems with time delay, we may prefer a delay-free design problem for time-delay systems. This motivates us to evaluate these two approaches.

Note that Approach 2 contains non-causal plants. This may possibly lead to noncausal closed loop systems. However, this does not matter, as we are sure that these non-causal plants do not belong to the possible plant set.

With the specific model parameter $k = 1$, $\tau = 1$ and $\theta = 1$, the two approaches are compared based on robust performance

$$J_{RP} = \sup_{\omega} \sup_{\Delta} |w_P S_p| \tag{26}$$

with performance weight

$$w_P(s) = \frac{s+1}{2s} \tag{27}$$

J_{RP} is very similar to, but not quite as the structured singular value, μ_{RP} for robust performance. μ_{RP} is computationally more efficient, particularly for multivariable systems. Though they are equivalent in the sense that $J_{RP} < 1$ iff $\mu_{RP} < 1$, they are generally different. The reason we use J_{RP} here is that we are able to calculate the "exact" J_{RP} (J_{RP0}) for this simple case. This J_{RP0} provides a standard for the comparison. In the next section, μ_{RP} will be used instead.

The peak values J_{RP} for different controllers are shown in Table III. Define J_{RP0} as the exact J_{RP} for the real time-delay uncertainty. J_{RP1} and J_{RP2} are, respectively, computed using Approach 1 and Approach 2 with time-delay uncertainty approximated by a multiplicative uncertainty weight $w_{1h}(s)$, i.e.

$$J_{RP} = \sup_{\omega}\sup_{\Delta} |w_P S(1+\Delta w_{1h} g c S)^{-1}| = \sup_{\omega} \frac{|w_P S|}{1-|w_{1h} g c S|} \tag{28}$$

For this simple example, we are fortunately able to compute J_{RP0} exactly at each frequency.

$$J_{RP0}(\omega) = \sup_{\theta_p} |w_P S_p| = |w_P| \sup_{\theta_p} |(1+c(j\omega)\frac{1}{j\omega+1}e^{-j\omega\theta_p})^{-1}| \tag{29}$$

We can reach the supreme by choosing θ_p within the interval $(0,1)$ to make $c\frac{1}{j\omega+1}e^{-j\omega\theta_p}$ coincide with or closest to the negative real axis of the complex plane. Also introduce the performance J_{RP3} of the plant with the *largest time delay*. J_{RP3} is, however, not a robust performance.

$$J_{RP3} = \sup_{\omega} |w_P(1+c(j\omega)\frac{1}{s+1}e^{-s})^{-1}| \tag{30}$$

The controllers used include a set of PI controllers and 3 μ-optimal controllers. The PI controllers are of the form

$$c(s) = k_c \frac{s+1}{s} \tag{31}$$

with different k_c as shown in Table III. Since H_∞ synthesis and μ synthesis in existing toolboxes[1][2] are not able to handle time-delay, we have to approximate time-delay by a rational transfer function in Approach 1 when we want to get a μ-optimal controller. $C_{\mu 1}$ and $C_{\mu 14}$ are the μ-optimal controllers using Approach 1 with the time-delay in the nominal model approximated by first-order and fourth-order Padé approximations, respectively. $C_{\mu 2}$ is the μ-optimal controller using Approach 2.

The results in Table III are somewhat unanticipated. We see that two approaches give comparable results. Sometimes J_{RP1} is larger and sometimes J_{RP2} is larger. J_{RP2} is not much larger than J_{RP1} in each case, but on the other hand J_{RP1} is much larger for some cases. So we can get a delay-free approach at almost no costs. The reason is that H_∞-norm and μ is a worst case performance measure. Although Approach 2 introduces a lot more uncertainty than necessary, the worst case may not be affected very much. Also note, although Approach 2 has larger uncertainty, the plant set in Approach 2 (g_{pA2}) does not contain the whole plant set in Approach 1 (g_{pA1}) when time-delay uncertainty is approximated by complex multiplicative uncertainty. This is graphically shown in Fig. 4 for frequency $\omega = 2$, and explains why $J_{RP2} < J_{RP1}$ can happen.

Let us look more carefully at the results with μ-optimal controllers.

1. J_{RP2} with $C_{\mu 2}$ is smaller than J_{RP1} with $C_{\mu 1}$ and $C_{\mu 14}$. J_{RP0} with $C_{\mu 2}$ is smaller than J_{RP0} with $C_{\mu 1}$ and $C_{\mu 14}$, too. This shows that from the point of controller design Approach 2 is better.

2. The differences between J_{RP1} and J_{RP2} for two of the μ-optimal controllers are very large, while the differences for the PI controllers are generally small, except for the extreme cases where the controller is poorly tuned. This means that μ-optimal controller can be very sensitive to uncertainty unconsidered.

Table III. Comparison based on robust performance J_{RP} for two different modelling approaches of uncertain time-delay systems

controller	k_c	Exact J_{RP0}	Approach 1 J_{RP1}	Approach 2 J_{RP2}	NP + max delay J_{RP3}
$C_{\mu 1}$		1.2859	**1.3343**	1.3246	1.2859
$C_{\mu 14}$		1.0343	**1.0359**	1.1125	1.0343
$C_{\mu 2}$		0.9759	2.3374	**0.9856**	0.9759
	0.2	2.4982	2.5166	2.5535	2.4982
	0.3	1.6664	1.6982	1.7555	1.6664
	0.4	1.2500	1.3108	1.3880	1.2500
	0.5	1.1390	1.1469	1.2193	1.1390
PI	0.6	1.1948	1.2258	1.2027	1.1948
	0.8	1.4490	1.7841	1.5056	1.4490
	1.0	1.8515	3.5163	2.2196	1.8515
	1.2	2.4344	394.50	4.4034	2.4313
	1.4	5.0678	∞	66.756	5.0354

We see that J_{RP3} is equal to J_{RP0} for most of the cases. This suggests largest time-delay may be the worst case. In fact, we can show that largest time-delay is the worst case up till frequency

$$\omega^* = (\pi - \angle L(j\omega^*))/\theta \tag{32}$$

where L is the open loop transfer function with the largest time-delay. So as long as the peak value frequency for J or μ is below this frequency, the largest time-delay is the worst case. This gives another robust design method for uncertain time-delay system: Design with the largest time-delay and do not consider time-delay uncertainty at all. This method will completely eliminate the conservativeness introduced by multiplicative uncertainty approximation of time-delay uncertainty. But we need to check whether the peak value frequency for J and μ is smaller than ω^* or not after design. This check will not be easy for practical cases in which we also have other uncertainties. Also the time delay may have to be approximated by a rational function.

4 Extension to systems with general parametric uncertainty

For simplicity, we considered only time-delay uncertainty in the above studies. In this simple case, we can also get "exact" results which serve as standards for the comparisons of different approximations and of different modelling approaches. The principles, however, apply to general cases as well. As an illustrative example, we consider in this section the first-order with time-delay model, with simultaneous uncertainties in gain, time constant and time-delay.

$$g_p(s) = \frac{k_p}{\tau_p s + 1} e^{-\theta_p s} \tag{33}$$

$$k_p \in [k_{min}, k_{max}], \quad \tau_p \in [\tau_{min}, \tau_{max}], \quad \theta_p \in [\theta_{min}, \theta_{max}]$$

This model is widely-used in modelling industrial processes. The mean values of the parameters are

$$k = \frac{k_{min} + k_{max}}{2}, \quad \tau = \frac{\tau_{min} + \tau_{max}}{2}, \quad \theta = \frac{\theta_{min} + \theta_{max}}{2}$$

The magnitudes of the parametric uncertainties are

$$\delta_k = \frac{k_{max} - k_{min}}{2}, \quad \delta_\tau = \frac{\tau_{max} - \tau_{min}}{2}, \quad \delta_\theta = \frac{\theta_{max} - \theta_{min}}{2}$$

When the mean values are used in the nominal model, Laughlin *et al.*[6] derived the following smallest complex multiplicative uncertainty bound for the parametric uncertainties.

$$\textbf{Sb}: \quad l(\omega) = \begin{cases} \left|\frac{|k|+\delta_k}{|k|}\frac{\tau j\omega+1}{(\tau\mp\delta_\tau)j\omega+1}e^{\pm j\delta_\theta\omega} - 1\right|, & \forall \quad \omega < \omega^* \\ \left|\frac{|k|+\delta_k}{|k|}\frac{\tau j\omega+1}{(\tau\mp\delta_\tau)j\omega+1}\right| + 1, & \forall \quad \omega \geq \omega^* \end{cases} \tag{34}$$

where ω^* is defined implicitly by:

$$\pm\delta_\theta\omega^* + arctan[\frac{\pm\delta_\tau\omega^*}{1+\tau(\tau\mp\delta_\tau)\omega^{*2}}] = \pm\pi, \quad \frac{\pi}{2} \leq \delta_\theta\omega^* \leq \pi$$

This bound applies to both stable and unstable models. The top signs are selected if τ is positive, and the bottom signs are selected if τ is negative indicating an unstable model. They also give an approximate rational weight:

$$w_1(s) = \frac{|k|+\delta_k}{|k|}\frac{\tau s+1}{(\tau\mp\delta_\tau)s+1}\frac{1\pm\delta_\theta s/2}{1\mp\delta_\theta s/2} - 1 \tag{35}$$

This weight is unstable for stable plants, so we need to multiply it with an all-pass to make it stable. We use $w_1(s)$ to denote this weight because it is the same as the $w_1(s)$ in section 2 for the case with time delay uncertainty only. As before, this weight does not cover all the parametric uncertainties and may be optimistic (as well as conservative).

By letting $w_1(s)$ level off at a higher frequency, we can get a weight which just covers the parameter uncertainty. As before we do this as follows

$$w_{1h}(s) = w_1(s)\frac{1+\delta_\theta s/2}{1+\delta_\theta s/k_\omega} \tag{36}$$

where k_ω is larger than 2. The k_ω values which make $w_{1h}(s)$ just cover the parametric uncertainties are different for different systems.

As before we consider the two different modelling approaches: 1) Approach 1 where we use mean values of the gain, time constant and time-delay in the nominal model. 2) Approach 2 where we use only mean values of the gain and time constant in the nominal model, and model *all* of the time-delay as uncertainty. Hence the magnitude of time-delay uncertainty is increased from δ_θ to $\theta + \delta_\theta = \theta_{max}$.

Assume that all mean values are equal to 1. We study two cases: 50% and 20% parameter uncertainty. In Approach 1 (nominal plant has mean time delay), after multiplying by an all-pass, the multiplicative uncertainty weights are

$$w_1^{50}(s) = \frac{2s^2+6.5s+2}{0.5s^2+3s+4} \tag{37}$$

$$w_1^{20}(s) = \frac{2s^2+6.2s+2}{0.8s^2+9s+10} \tag{38}$$

The k_ω values are 2.5 and 3 for 50% and 20% parameter uncertainties, respectively. So we get

$$w_{1h}^{50}(s) = \frac{2s^2+6.5s+2}{(0.5s+1)(4+2s/2.5)} \tag{39}$$

$$w_{1h}^{20}(s) = \frac{2s^2 + 6.2s + 2}{(0.8s+1)(10+2s/3)} \tag{40}$$

In Approach 2 (nominal plant has no time delay), the magnitudes of time-delay uncertainty are increased to 1.5 and 1.2, respectively. We have

$$w_1^{50}(s) = \frac{6s^2 + 11.5s + 2}{1.5s^2 + 5s + 4} \tag{41}$$

$$w_1^{20}(s) = \frac{6s^2 + 8.6s + 1}{2.4s^2 + 7s + 5} \tag{42}$$

$$w_{1h}^{50}(s) = \frac{6s^2 + 11.5s + 2}{(0.5s+1)(4+6s/2.58)} \tag{43}$$

$$w_{1h}^{20}(s) = \frac{6s^2 + 8.6s + 1}{(0.8s+1)(5+6s/2.95)} \tag{44}$$

μ-optimal controllers are designed using the two different approaches, $C_{\mu 1}$ with Approach 1 and $C_{\mu 2}$ with Approach 2. Uncertainty weights used are the above $w_{1h}(s)$. The performance weight for case with 50% parameter uncertainty is

$$w_P^{50}(s) = \frac{1}{2}\frac{3.397s + 1}{3.397s} \tag{45}$$

and for 20% parameter uncertainty case

$$w_P^{20}(s) = \frac{1}{2}\frac{1.5s + 1}{1.5s} \tag{46}$$

In Approach 1, the time-delay in the nominal model is approximated by fourth-order Padé approximation. The robust performance in μ_{RP} is shown in Table IV. The values in parathesis are computed with fourth-order Padé approximation of the time-delay in nominal model as required when designing $C_{\mu 1}$, the others are computed with real time-delay.

From Table IV, we see:

1. The two approaches still give comparable results. Note that the ratio of time-delay uncertainty magnitudes in Approach 1 and Approach 2 has decreased from 1:2 in Section 3 to 1:3 in the 50% parameter uncertainty case, and to 1:6 in the 20% parameter uncertainty case. Smaller the ratio, more uncertainty is introduced in Approach 2. Intuitively we might think that Approach 2 becomes more conservative. But this result demonstrates that Approach 2 does not depend on the uncertainty ratio very much and hence apply to a wide range. For the 20% parameter uncertainty case, the original plant set g_p and the plant sets in A1 and A2 are shown in Fig. 5 for frequency $\omega = 2$. We clearly see that g_{pA2} yields a much larger plant set, but g_{pA1} does include a few plants not contained in g_{pA2}.

Table IV. Comparison based on robust performance μ_{RP} for two different modelling approaches of uncertain time-delay systems, general case

	controller	μ_{RP1}	μ_{RP2}
50% parameter uncertainty	$C_{\mu 1}$	1.2620(1.0599)	1.1899
	$C_{\mu 2}$	1.2952	1.0566
20% parameter uncertainty	$C_{\mu 1}$	1.1511(0.9756)	1.5582
	$C_{\mu 2}$	1.4104	1.0210

2. Though fourth-order Padé approximation is used, we still find significant deterioration in robust performance when we calculate it with real time-delay. Generally we should think that fourth-order Padé approximation is good enough, however, μ-optimal controllers seem to be very sensitive to uncertainty unconsidered. The large differences between μ_{RP1} and μ_{RP2} of same controllers reveal the same problem of μ-optimal controllers.

For comparison, we also design a μ-optimal controller using uncertainty weight $w_1^{50}(s)$ in (41) instead of the corresponding $w_{1h}^{50}(s)$ for the case of 50% parameter uncertainty and Approach 2. We get a $\mu_{RP} = 1.0327$. Compared with $\mu_{RP} = 1.0566$ which is designed using $w_{1h}^{50}(s)$ in (43), we see the increase in μ_{RP} is very small. This means that using an upper bound in controller design introduce little conservatism. On the other hand, when we calculate μ_{RP} of this μ-optimal controller using uncertainty weight $w_{1h}^{50}(s)$ in (43), the value is 1.2035, nearly 20% increase. This means that the robust performance might deteriorate significantly when this μ-optimal controller is applied to the original problem.

5 Conclusions

In this paper, the robust controller design for uncertain time-delay systems has been studied. From the above results and discussions, we are ready to make the following conclusions:

1. μ-optimal controllers seem to be very sensitive to uncertainty unconsidered. Hence we should be careful with the rational approximations of time-delay uncertainties and time-delays in the nominal model. It is preferable to use an upper bound to approximate the original uncertainties.

2. For the multiplicative uncertainty approximation of time-delay uncertainty, the upper bound $w_{1h}(s)$ adjusted at high frequencies is a simple and good weight.

3. For the modelling of uncertain time-delay systems, the approach of using a nominal model *without* time delay (Approach 2) is better since it gives comparable results but leads to a delay-free design problem for the time-delay systems.

4. The transformation from original uncertainty description to the norm-bounded complex perturbation inevitably introduces conservatism. In order to reduce conservatism, usually one chooses a nominal model which minimizes uncertainty weight. However, μ is a worst case measure. Of most importance is not the size of uncertainty but the worst uncertainty. So the correct way to reduce conservatism is not to minimize the uncertainty size but to minimize the worst uncertainty introduced in transformation. Of course, minimizing the uncertainty size is easy and direct. However, which is the worst uncertainty is generally not obvious. Research work on identifying worst uncertainty may be worthwhile.

5. Note that all results are for SISO plants, and then MIMO systems may behave entirely differently.

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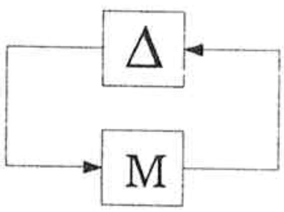

Figure 1: Standard "M-Δ" structure.

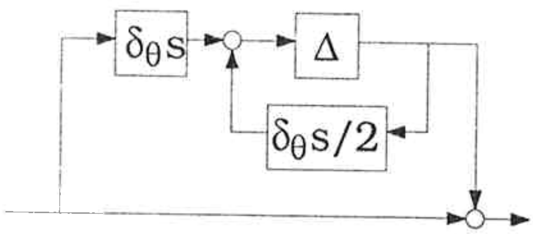

Figure 2: An LFT implementation of Padé approximation.

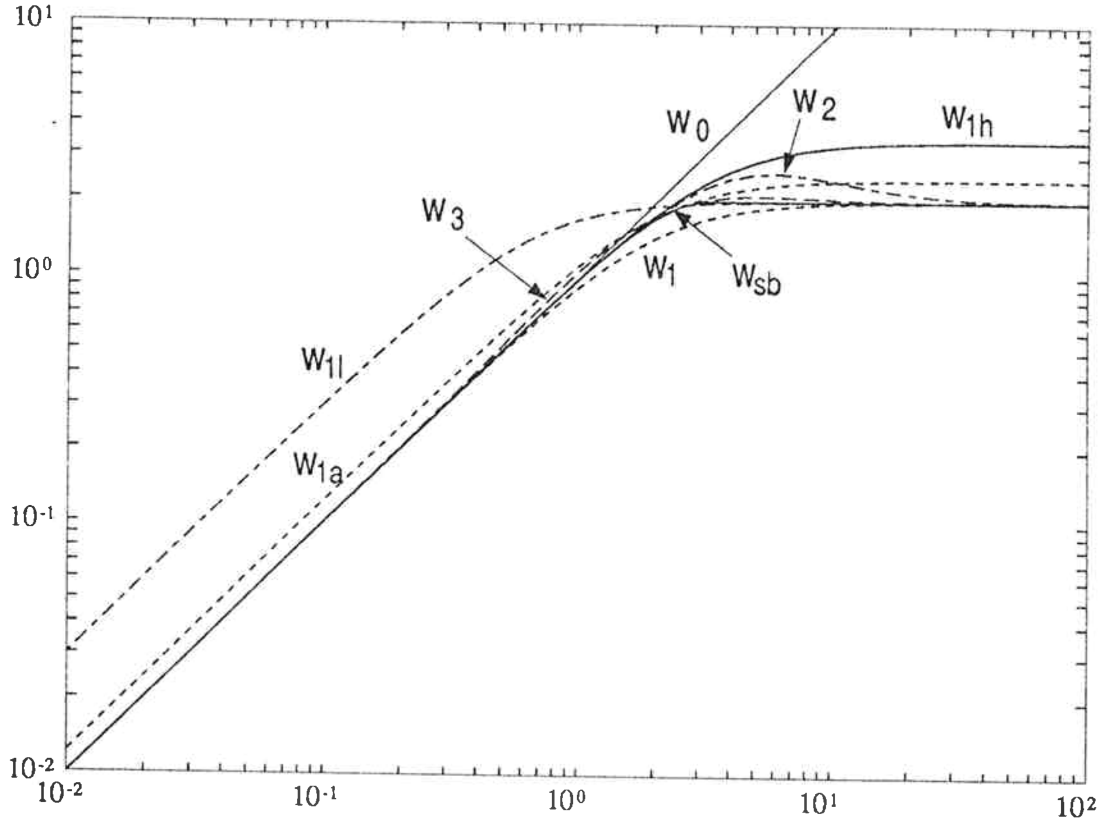

Figure 3: Different uncertainty weights for time delay uncertainty.

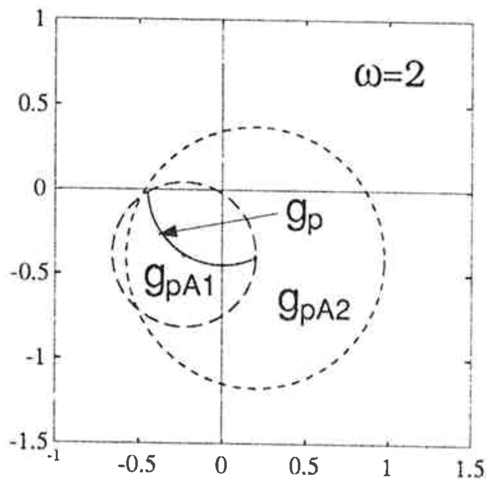

Figure 4: Plant set g_p in eq. (21) and its disc approximations in A1 and A2.

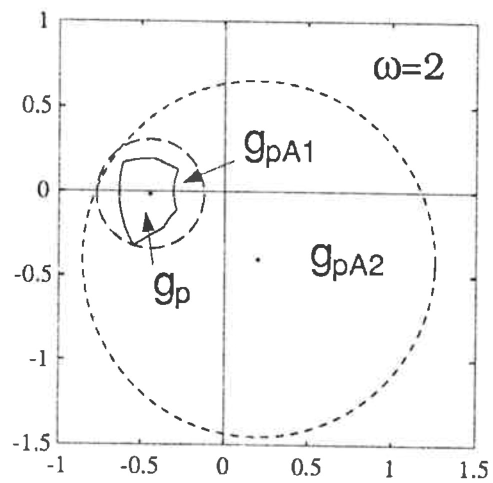

Figure 5: Plant set g_p in eq. (33) and its disc approximations in A1 and A2.