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NON-UNIQUENESS OF ROBUST H_∞ DECENTRALIZED PI-CONTROL

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Abstract

In the paper we use as an example the control of identical processes operating in parallel. Such systems are quite common in industry, for example in distribution networks or when parallel units (reactors, heat exchangers, etc.) are used. When performance is measured in terms of the H_∞-norm then the optimal single-loop PI- or PID-tunings are not necessarily equal for the individual loops. The same applies when model uncertainty is included and the structured singluar value μ is used as a performance measure. This is contrary to what one intuitively would expect, and also implies that the optimal solution is non-unique.

1 Introduction

In this paper we use as an example the control of identical processes in parallel which interact with each other. It is quite common in industry that a system is composed of identical subsystems which are symmetrically interconnected. In chemical industries parallel units are used to add flexibility or because one single unit would be too large. Typical examples of parallel units are chemical reactors, heat exchangers or compressors. Another example of symmetrical systems are distribution networks. Lunze [1], [2] studies stability of this type of systems, which he denotes "symmetric composite systems". Skogestad *et al.* [3] use the term "identical parallel process" and study some of the general properties, in particular, the value of the Relative Gain Array (RGA).

In this paper we are not primaly interested in this specific type of system, instead we have choosen this example because of its symmetric properties. With n identical parallel processes the $n \times n$ transfer matrix of the plant at a given frequency may be written

$$G(j\omega) = g(j\omega)\begin{pmatrix} 1 & a(j\omega) & a(j\omega) & \dots & a(j\omega) \\ a(j\omega) & 1 & a(j\omega) & \dots & a(j\omega) \\ a(j\omega) & a(j\omega) & 1 & \dots & a(j\omega) \\ \vdots & \vdots & \vdots & & \\ a(j\omega) & a(j\omega) & a(j\omega) & \dots & 1 \end{pmatrix} \quad (1)$$

where $g(j\omega)$ denotes the diagonal transfer function elements, and $a(j\omega)$ the degree of interaction at a given frequency.

One example of this type of a process is the control of flow in parallel streams from a single source as shown in Fig.1 [4]. Opening valve 1 causes q_1 (flow 1) to increase and q_2 (flow 2) to decrease because of the consequent reduction in pressure head. If there are two parallel steams the steady-state value of a is expected to lie between 0 and -1. The value of 0 would be obtained if the source was a large tank such that the pressure head was unaffected by an increase in flow 1. The value of -1 would be obtained if the source was a pump with constant total flow q. For n parallel streams from a single source similar arguments yield

$$-1/(n-1) \le a \le 0 \quad (2)$$

The lower bound is obtained by considering constant total flow. In this case a change Δq_1 in flow 1 would yield $\Delta q_2 = \Delta q_3 = \dots = \Delta q_n = -\Delta q_1/(n-1)$. A value less than the lower bound $-1/(n-1)$ would imply that the total flow is reduced by opening a valve and does not seem to be possible in a practical situation. For a reactor/cooling example the same lower bound for a is obtained, but the upper bound equals 1 [3].

Our interest in this kind of processes was initiated by results obtained by Skogestad *et al.* [5] for a simplified distillation column model. The model used in that paper is

$$G^{LV}(s) = \frac{1}{75s+1}\begin{pmatrix} 0.878 & -0.864 \\ -1.082 & 1.096 \end{pmatrix} \quad (3)$$

Here the inputs are reflux (L) and boilup ($-V$) and the outputs are product compositions. This is of course not an example of identical parallel processes, but the column model is well approximated by Eq.1 with $a = -0.986$. Skogestad *et al.* [5] studied robust control using Eq.3. They considered uncertainty with respect to the actual value of the inputs L and V, and used the H_∞-norm of the weighted sensitivity as a performance criterion. Note that the uncertainty and performance specifications in this example were identical for the two channels. With the uncertainty and performance weights fixed, the problem of optimizing the worst-case response is solved mathematically by finding the controller C that minimizes the value of the structured singular value μ_{RP} [6]. A value of μ_{RP} less than 1 implies that the worst-case response satisfies the robust performance objective. A multivariable controller with $\mu_{RP} = 1.06$ was obtained for the process in Eq.3 [5] (a later refinement, [7], have pushed μ_{RP} down to 0.978). Skogestad *et al.* [5] also studied the use of single-loop

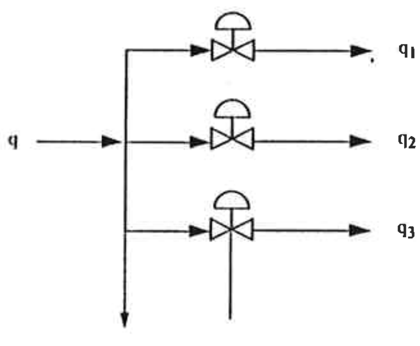

Figure 1. Splitting into parallel streams.

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PID-controllers and were able to obtain $\mu_{RP} = 1.34$ by adjusting the six controller parameters (k, τ_I, τ_D for each loop). The corresponding controller parameters are:

$$\text{Loop 1}: \quad k = 179, \ \tau_I = 28.9 \text{ min}, \ \tau_D = 0.31 \text{ min} \qquad (4)$$

$$\text{Loop 2}: \quad k = 47, \ \tau_I = 1.38 \text{ min}, \ \tau_D = 0.27 \text{ min} \qquad (5)$$

We note that the integral time for loop 1 is much larger than that of loop 2, and also the gains are very different. Results for other distillation column models which support these findings have also been obtained [8]. Intuitively, one would expect the optimal tunings to be approximately equal for the two loops since the process (including the uncertainty and performance specifications) is almost symmetric. For example, Lunze [2] assumes that decentralized controllers should be identical for identical processes operating in parallel. However, the result above suggests that the optimal PI tunings may be differently tuned loops even if the plant is *completely* symmetric. In this case the optimal tuning is not unique since we may simply interchange the controllers for the two loops and get the same overall performance.

The objective of this paper is to use a completely symmetric plant and examine for some examples when the optimal tuning of a decentralized controller is non-identical. We will also give some explanations to why this is the case.

2 Problem formulation

2.1 Example process

We consider a 2×2 parallel process with transfer function

$$G(s) = \frac{1}{1+\tau s}\begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix} \qquad (6)$$

where a is a real constant. All elements of $G(s)$ share the same dynamics, so the condition number and the RGA are constants and are not functions of frequency. The condition number, $\gamma(G)$, equals $(1+a)/(1-a)$ and the 1,1-element of the RGA is $\lambda_{11} = 1/(1-a^2)$. We will focus on a plant where $a = 0.949$ and $\tau = 100$. This corresponds to a plant with strong interactions, as can be seen from the RGA were $\lambda_{11} = 10$, or from the condition number, $\gamma(G) = 38.2$. In order to study the effect of interaction and of plant dynamics, we shall also consider the following values of a and τ.

- a : 0, 0.827, 0.949, 0.984, and 0.995 (corresponding to $\lambda_{11} \approx$ 1, 3, 10, 30, and 100)
- τ: 10 and 100 min.

The value of τ should be compared to the value of the dead time θ of 1 min allowed by the uncertainty weight (see below). Also note that for the special case of 2×2 plants the sign of a does not matter, that is, the same results, for example with respect to optimal tuning parameters, would be obtained with $a = 0.9$ or $a = -0.9$. The reason is that changing the sign of input 1 and output 1 changes the sign of the off-diagonal elements, but keeps the diagonal elements unchanged. This is not the case for 3×3 plants or higher.

We will use decentralized PI- and PID-controllers on the cascade form with derivative action effective over one decade.

$$C_{PID}(s) = \begin{pmatrix} c_{PID1}(s) & 0 \\ 0 & c_{PID2}(s) \end{pmatrix} \qquad (7)$$

$$c_{PID}(s) = k\frac{1+\tau_I s}{\tau_I s}\frac{1+\tau_D s}{1+0.1\tau_D s} \qquad (8)$$

The controller parameters k, τ_I and τ_D, for each loop are optimized with respect to an objective function based on the structured singular value μ or the H_∞-norm as defined below. For PID controllers we require $\tau_I > \tau_D$. As a comparision to the decentralized controllers, we will sometimes also use a "full" μ-optimal controller, i.e. a multivarilable controller with no restrictions on structure or number of states.

2.2 The μ-objective formulation

A block diagram of the plant is shown in Fig.2. G is the nominal process, as defined in Eq.6. C is the controller. W_P and W_I are frequency dependent weighting matrices. Δ_I is a perturbation block, used to represent uncertainty.

Any model is subject to some uncertainty, and one sourse of uncertainty which always is present is input uncertainty. The blocks W_I and Δ_I are used to model multiplicative input uncertainty. Let the relative input error be ϵ at steady-state, and assume it increases with frequency such that it reaches 1 (100%) at a frequency of about $1/\theta$. This increase of the error with frequency is used to take care of unmodelled high frequency dynamics, for instance valve dynamics or a neglected time delay.

$$W_I(s) = \begin{pmatrix} w_{I1} & 0 \\ 0 & w_{I2} \end{pmatrix}; \quad w_I(s) = \epsilon\frac{\frac{\theta}{\epsilon}s+1}{\frac{\theta}{2}+1} \qquad (9)$$

We assume 20% uncertainty at low frequencies, and high frequency uncertainty corresponding to a 1 minute delay, i.e. $\epsilon = 0.2$ and $\theta = 1$. Δ_I is a diagonal matrix,

$$\Delta_I(s) = \text{diag}\{\delta_1(s), \cdots, \delta_n(s)\}, \delta_i(s) \in C, \quad \bar{\sigma}(\Delta_I) \leq 1, \forall\omega \quad (10)$$

i.e. the uncertainty is *structured*. G, W_I and Δ_I defines a set of plants $G_p = G(I + \Delta_I W_I)$. The goal of the uncertainty modelling is to make the set G_p as small as possible and still ensure that the "true plant" is within the set.

The objective of the controller is to keep the controlled output vector (y) close to the set-points (y_{sp}) despite the effect of disturbances (d). Assume that the following performance specifications are given: 1) Steady-state offset less than A; 2) Closed-loop bandwidth higher than ω_B; and 3) Amplification of high-frequency noise less than a factor M. These specifications may be reformulated as a bound on the weighted sensitivity function

$$\bar{\sigma}(W_P S(j\omega)) < 1 \ \forall\omega \qquad (11)$$

(which is equivalent to requiring $||W_P S||_\infty < 1$) using the following weight

$$W_P(s) = \begin{pmatrix} w_{P1} & 0 \\ 0 & w_{P2} \end{pmatrix} \qquad (12)$$

where

$$w_P(s) = \frac{1}{M}\frac{\tau_{cl}s + M}{\tau_{cl}s + A}, \quad \text{with} \quad \tau_{cl} = 1/\omega_B \qquad (13)$$

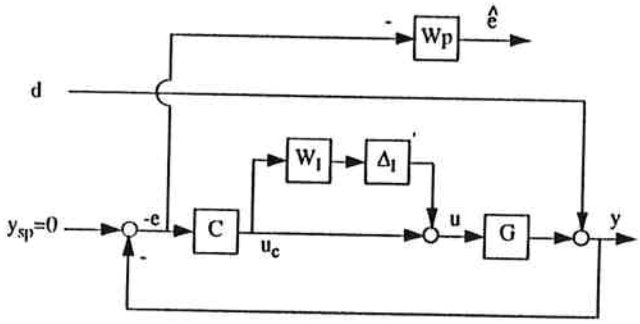

Figure 2. Block diagram of plant with input uncertainty and with disturbances as external inputs.

We shall for both outputs use $A = 0$ (no offset) and $M = 2$. Unless otherwise stated we shall use $\tau_{cl1} = \tau_{cl2} = 20$ min.

Robust Performance (RP) is achieved if the above-mentioned performance criterion is satisfied for all possible plants in the set G_p. Mathematically, this is tested by computing μ of the matrix N_{RP} (see [5]).

$$N_{RP} = \begin{pmatrix} -W_I H_I & -W_I CS \\ W_P SG & W_P S \end{pmatrix} \tag{14}$$

$\mu(N_{RP})$ is computed with respect to the structure diag$\{\Delta_I, \Delta_P\}$, where Δ_P is a full 2×2 matrix. $\mu(N_{RP})$ should be less than one at all frequencies for RP to be satisfied. The peak value of $\mu(N_{RP})$ will be denoted μ_{RP}. The block diagonal elements of N_{RP} are themselves important. We denote them

$$N_{RS} = N_{11} = -W_I H_I \quad \text{and} \quad N_{NP} = N_{22} = W_P S \tag{15}$$

Robust Stability (RS) is achieved if $\mu(N_{RS})$ is less than one at all frequencies (computed with respect to Δ_I). Nominal Performance (NP) is achieved if $\bar{\sigma}(N_{NP})$ is less than one at all frequencies.

In this paper we obtain the optimal controller settings by considering robust performance ("optimize worst case response"). There exists several possible choices for the objective function:

- In the "standard approach" (denoted "Approach 1" in the following) μ_{RP} is minimized with fixed uncertainty and performance weights (i.e., $\tau_{cl} = 20$ min). An optimal μ_{RP}-value different from 1, say 0.7, then means that *both* the performance and the uncertainty weight may be increased by a factor of 1/0.7 and the system will still achieve the robust performance requirement.

- In many cases it seems more reasonable keep the uncertainty *fixed*, and to minimize the "worst-case" peak of $N_{RP_p} = \bar{\sigma}(W_P S_p)$. In this "Fixed uncertainty approach" we define μ' at each frequency as

$$\mu'(j\omega) = \mu \begin{pmatrix} \mu N_{11} & \mu N_{12} \\ N_{21} & N_{22} \end{pmatrix} \tag{16}$$

At any frequency μ' is directly equal to the worst-case $\bar{\sigma}(W_P S)$ Since this peak will be infinite if the system is unstable, we in this case require that there exists a controller which yields RS. Synthesis using this approach is to obtain the controller which minimizes the peak, μ'_{RP}, of $\mu'(j\omega)$. Relative to the standard approach, this tends to penalize systems that are close to instability.

- In the "Achievable performance approach" we adjust the performance weight such that $\mu_{RP} = 1$ (we still keep the uncertainty fixed). This approach is also denoted "Approach 2" [8]. We choose to keep M fixed and adjust τ_{cl} in the weight W_P. That is, the parameters in the controller $C(s)$ are obtained by solving the following nested loop optimization problem:

$$\min_{\tau_{cl}} \left| \left[\min_{C} \mu_{RP}(C, \tau_{cl}) \right] - 1 \right| \tag{17}$$

or the following constrained optimization problem:

$$\min_{\tau_{cl}, C} |\tau_{cl}|; \quad \text{s.t.} \quad \mu_{RP}(C, \tau_{cl}) \leq 1 \tag{18}$$

Here "C" denotes the controller or the adjustable controller parameters. For a solution to exist, we must require also in this case that there exist a controller which achieves RS.

The optimal parameters we present in this paper are obtained using a general optimization routine and since there are local minima there is no guarantee that the settings presented really are the true optimal.

2.3 The H_∞-objective formulation

As a comparision to the μ optimal tunings, we will also obtain optimal tunings by minimizing the following H_∞-norm:

$$\left\| \begin{matrix} W_P S \\ W_I H_I \end{matrix} \right\|_\infty \tag{19}$$

This is a standard "mixed sensitivity" H_∞ problem which corresponds to simultaneously trying to optimize NP, and RS with respect to input uncertainty. We shall use the same weights as defined before (Eq.9 and 13).

3 Analytical expressions

The transfer matrix of a set of identical parallel processes, matrix G in Eq.1, is a circulant matrix. The general form of a circulant matrix C is:

$$C = \begin{bmatrix} c_1 & c_2 & c_3 & \cdots & c_n \\ c_n & c_1 & c_2 & \cdots & c_{n-1} \\ c_{n-1} & c_n & c_1 & \cdots & c_{n-2} \\ \vdots & & & \ddots & \vdots \\ c_2 & c_3 & c_4 & \cdots & c_1 \end{bmatrix} \tag{20}$$

From the theory [9], [10] we know that one property of a circulant matrix is that its eigenvectors are the same as the vectors resulting from a singular value decomposition, and the singular values equals the modulus of the eigenvalues. ($\sigma_i = |\lambda_i|$). Furthermore, if A and B are circulant matrices and k_i a scalar, then A^T, A^H, $k_1 A + k_2 B$, AB, $\sum_i k_i A^i$ are circulant and A and B commute, that is, $AB = BA$. Note that A^{-1} is also a circulant matrix. For example, if a process with a circulant transfer function G is controlled by n equal single-loop controllers (ie., $C = cI$), the sensitivity function $S = (I + GC)^{-1}$ and the complementary sensitivity function $H = I - S$ are both circulant matrices.

The structured singular value, μ, of a matrix N is bounded in the following way [6]:

$$\rho(N) \leq \mu(N) \leq \bar{\sigma}(N) \tag{21}$$

$\mu(N) = \rho(N)$ if $\Delta = \delta I$, and $\mu(N) = \bar{\sigma}(N)$ if Δ is unstructured, i.e. if Δ is a full matrix. If N is a circulant matrix, then both equalities hold and we may compute μ exactly. This result implies that for a circulant N there exist a perturbation, $\Delta = \delta I$, which is as bad as any full perturbation matrix. For example, N_{RS} is circulant if we use identical controllers in all loops, so $\rho(N_{RS}) = \mu_{RS} = \bar{\sigma}(N_{RS})$ and μ_{RS} is the same for structured and unstructured uncertainty.

4 Results

4.1 μ-optimal designs

As mentioned earlier, one would expect the optimal controller to be symmetric, since the plant (including the weights) is symmetric (this symmetry argument holds both for a "full" multivariable controller and for a decentralized controller). We used the μ-toolbox for MATLAB [11] to design a full multivariable μ-optimal controller for the case where $\tau = 100$ min, $a = 0.949$

and $\tau_{cl} = 20$ min. The controller design method we used is called "DK-iteration" and it is described by, for instance, Doyle *et al.* [12]. This design method does not impose any restrictions on the structure of the controller, nor does it limit the number of controller parameters. The resulting optimal controller for our problem has 30 states and yields μ_{RP}=0.987, i.e. robust performance is achieved. (The controller is strictly speaking only suboptimal, since convergence to a global optimum is not guaranteed). The most interesting result is that the controller is almost symmetric. In fact, we may adjust it to be perfectly symmetric without increasing μ_{RP}. This result shows that our intuition was right in the case of a full multivariable controller.

The solid curves in Fig.3 shows $\mu(N_{RP}), \mu(N_{RS})$ and $\bar{\sigma}(N_{NP})$ as functions of frequency for optimal tuning of two *identical* PI controllers. The dashed curves are for *non-identical* tuning of the two loops. Optimization approach 1 is used in both cases and optimal tuning parameters are shown in Table 1. All curves are for the case where τ = 100 min, a = 0.949 and τ_{cl} = 20 min. The curves in Fig.3 demonstrates that identical tuning is not optimal. If we adjust the tuning to press down peak "A", then peak "B" will become higher, and vice versa. However, if we allow non-identical tuning it is possible to press down both peaks. μ_{RP} is reduced from 1.28 (identical tuning) to 1.17 (non identical tuning). Note that this result implies that the optimal solution is non unique, we may simply interchange the controllers for the two loops and get the same overall performance.

Table 1 also presents optimal tuning parameters for identical and non-identical PID-controllers. Also in this case non-identical tuning is better than identical.

It is interesting to find out under which circumstanses non-identical PI-tuning is optimal. To do this we consider the following values of a and τ.

- a : 0, 0.827, 0.949, 0.984, and 0.995
- τ: 10 and 100 min.

The results of the parameter optimizations for different combinations of the parameters a and τ are presented in Tables 2 and 3. Optimization "approach 2" is used in both cases. The tables give the best achievable performance (as expressed by the value of τ_{cl} in the performance weight) for the specific plant as well as the corresponding controller settings. Small values of τ_{cl} are good as they imply that fast response may be achieved, even in presence of model uncertainty.

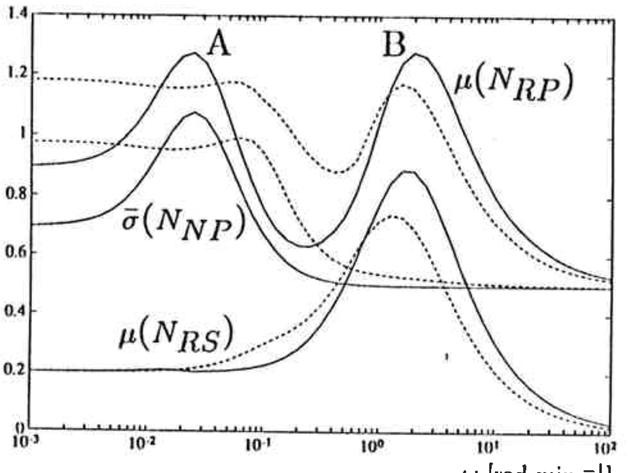

Figure 3. Plots of $\mu(N_{RP})$, $\mu(N_{RS})$ and $\mu(N_{NP})$ for plant with $a = 0.949$, $\tau = 100$ and $\tau_{cl} = 20$. Solid curves: Identical PI-controllers. Dotted curves: Different PI-controllers. Controller tunings are given in Table 1.

Table 2 present the optimal PI-settings when we require the two single-loop controllers to be identical and Table 3 present the results when the tunings are allowed to be different. For the case with τ=10 min there is ***no improvement*** by allowing the controllers to be different, so these results are omitted from Table 3. For the case with τ=100 min we note that the achievable closed-loop time constant, τ_{cl}, may be significantly reduced for "intermediate" values of a corresponding to RGA-values between 10 and 30. For example, with τ=100 min and a=0.949 (λ_{11}=10) the value of τ_{cl} may be reduced from 51.4 to 44.2 min.

We also note that for the cases where the improvement is largest, the integral time for one loop is at about the open-loop time constant τ while the integral time for the other loop is much smaller. This means that one loop is tuned tighter than the other, and despite this *both* loops achieve shorter closed-loop time constants than for identical tuning.

We now want to see how much speed we have to give up in one channel in order to get fast response in the other. Fixing $\tau_{cl2} = 20$ min and using approach 2 to minimize τ_{cl1} gives $\tau_{cl1} = 45.6$ min. (The tuning parameters are presented as the last entry in Table 3.) Thus, for this specific problem we are able to achieve fast response in one channel at almost no cost.

Table 1: Optimal tuning parameters for plant with a = 0.949, τ = 100 and τ_{cl} = 20. Tuning approach 1.

Controller	μ_{RP}	k_1	k_2	τ_{I1} min.	τ_{I2} min.	τ_{D1} min.	τ_{D2} min.
PI	1.28	81.0	81.0	57.5	57.5		
PI	1.23	73.7	35.7	103	2.72		
PID	1.17	92.1	92.1	72.6	72.6	0.318	0.318
PID	1.09	83.7	39.7	100	2.08	0.285	0.836

Table 2: Identical PI-controllers. Tuning approach 2.

τ min.	a	τ_{cl} min.	k	τ_I min.
10	0.000	2.59	5.57	11.5
10	0.827	12.1	3.86	5.27
10	0.949	26.1	2.95	3.06
10	0.984	81.7	2.90	3.03
10	0.995	263	2.88	3.03
100	0.000	2.66	54.2	115
100	0.827	16.3	42.2	58.9
100	0.949	51.4	39.7	65.1
100	0.984	119	38.7	39.2
100	0.995	203	37.8	19.5

Table 3: Different PI-controllers. Tuning approach 2.

τ min.	a	τ_{cl} min.	k_1	k_2	τ_{I1} min.	τ_{I2} min.
100	0.000	2.66	54.2	54.2	115	115
100	0.827	15.3	40.1	35.5	112	4.20
100	0.949	44.2	42.3	25.0	101	5.00
100	0.984	119	38.7	38.7	39.2	39.2
100	0.995	203	37.8	37.8	19.5	19.5
100	0.949	45.6/20	41.0	22.9	100	3.18

Summing up: The closed-loop time constant with equal tuning in both loops is 51.4 min. Different tunings makes it possible to improve the time constant to 44.2 min. By allowing one loop to react slightly slower, $\tau_{cl} = 45.6$ min we may improve the response in the other channel to $\tau_{cl} = 20$ min.

The findings above are illustrated in Fig.4 by simulations of the response to a unit step disturbance acting on output 1 at time $t = 0$. In the simulation we use a $\pm$ 20% gain error and a 1 minute delay ($\theta = 1$).

$$u(s) = \frac{\theta^2 s^2 - 6\theta s + 12}{\theta^2 s^2 + 6\theta s + 12}\begin{pmatrix} 1.2 & 0 \\ 0 & 0.8 \end{pmatrix} u_c(s) \qquad (22)$$

The solid curves shows the response for identical controllers ($\tau_{cl} =$ 51.4), and the dotted curves are for non-identical controllers ($\tau_{cl} =$ 44.2). The simulation shows that the differently tuned loops reject the disturbance better than identical controllers, in particular as it approaches steady-state. One should keep in mind that simulations can only be performed for specific choices of disturbances and model errors, and one should not necessarily expect a good correlation between a single simulation and the worst-case response for which μ is a measure. However, in this case the correlation is good.

Comment on uniqueness of decentralized controllers: The fact that the value of τ in the model makes a difference demonstrates that it is not only the decentralized structure in itself that leads to different tunings in this case, but also the limited number of degrees of freedom in the PI and PID controllers. For example, consider the optimal identical PI controller c_{10} for the plant G_{10} with $\tau = 10$ min. Then the identical decentralized controller $c_{100} = c_{10}(1 + 100s)/(1 + 10s)$ [1] would give the same τ_{cl}-values when applied to G_{100}, as for c_{10} applied to G_{10}. For example, we see from Table 2 that it should be possible for the case $\tau = 100$ min, $a = 0.949$ to obtain identical decentralized controller which achieve $\tau_{cl} = 26.2$ min (whereas the best identical PI's in Table 2 give 51.4 min).

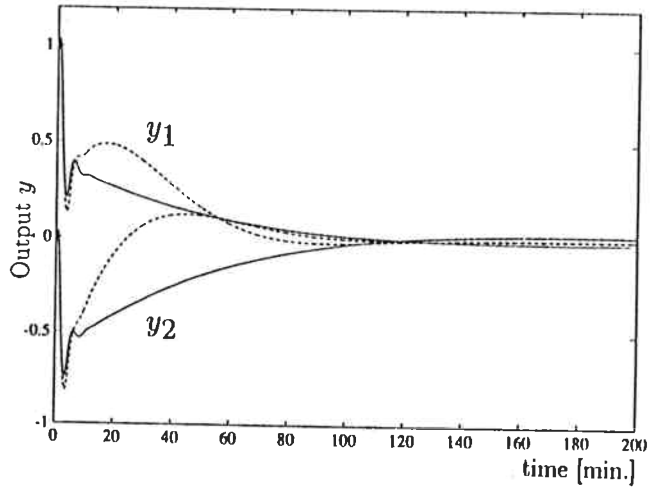

Figure 4. Response to a unit step disturbanse at $t = 0$ acting on output 1. Solid curves: Identical PI controllers ($\tau_{cl} =$ 51.4). Dotted curves: Different PI-controllers ($\tau_{cl} = 44.2$). Controller tunings are given in Table 2 and 3, respectively. Uncertainty (Eq.22) is used in both simulations.

[1]This is actually a controller of the PID-form in Eq.8, but with no restrictions on the parameters τ_D and $0.1\tau_D$

4.2 H_∞-optimal designs

The H_∞ optimal PI tuning for $\tau = 100$ min is non-identical, just like the μ optimal tunings. The H_∞ results are presented in Table 4 and 5.

5 Discussion

Our results demonstrate that the μ-optimal single-loop controllers do not have to be equal for identical parallel processes when we use decentralized (single-loop) PI- or PID-controllers. However, for some of these cases it became optimal to have identical controllers if we did not limit the structure of the decentralized controllers to PI or PID. The reason for this is probably as follows: Consider e.g. μ_{RP} as a performance objective. There always seems to be a local minimum around the point of identical controllers. In some cases there also exists local minimas corresponding to having the controllers differently tuned. The location of the global minimum may be either of these, and the optimal decentralized control structure (identical or not) may therefore easily be altered, for example, by going from a PID to a more general controller strucure as discussed earlier.

The reason why it is optimal to have different tunings in some cases may be understood intuitively as follows: With identically tuned controllers the resonance peaks of the individual loops are at the same frequency and may be strongly amplified by the interactions (when the parameter a is close to 1). When the loops are tuned differently the interactions between the loops is much less (as an extreme consider the case when the controller gain in loop 1 is zero). Balchen [13] discuss this effect for the case of nominal stability. In our case of RP, we see that the peak "A" in Fig.3 is mainly caused by a peak in $\bar{\sigma}(W_P S)$ for NP. By use of non-identical tuning the interactions between the loops is less and the "A"-peak is decreased and flattend out (the fact that it is flattened out means that the performance at other frequencies is worse).

Although the problem statement in this paper is identical as seen from any loop, the uncertainty perturbation block, Δ_I, does actually allow the parallel processes *not* to be identical when the μ_{RP} criterion is used. At a first glance, one may believe the

Table 4: Identical PI-controllers. H_∞-norm objective function, Eq.19. Tuning approach 1.

τ min.	a	$\|\cdot\|_\infty$	k	τ_I min.
100	0.827	0.706	41.2	74.4
100	0.949	1.011	89.5	53.2
100	0.984	1.358	194	37.0
100	0.995	1.667	468	28.8

Table 5: Different PI-controllers. H_∞-norm objective function, Eq.19. Tuning approach 1.

τ min.	a	$\|\cdot\|_\infty$	k_1	k_2	τ_{I1} min.	τ_{I2} min.
100	0.827	0.7022	41.3	39.4	133	24.0
100	0.949	0.9380	78.9	46.7	104	2.42
100	0.984	1.273	175	92.6	92.4	3.14
100	0.995	1.624	414	317	78.6	3.76

results with different tunings to be related to this fact. However, this is not the case, since we have seen that also the H_∞ criterion may lead to different tunings. An other way to make sure that the different tunings are not related to the perturbation block is to restrict Δ_I to be a repeated, diagonal, scalar perturbation. This restriction guarantees the *perturbed* parallel processes to be identical. Our numerical findings show the optimal PI tunings are not affected when Δ_I is restricted in this way (this is obvious if we consider N_{RS}, which is a circulant matrix, but not obvious for N_{RP}).

The fact that the optimal tunings for the loops are not identical is of course an interesting result from a theoretical point of view, but it has practical implications only if there is a real improvement by using different controllers. Even if the objective function is improved by different tuning, we may still prefer identical tuning of practical reasons, such as easier maintainence and tuning. However, our results do show that in *some* cases we may make one of the loops significantly faster (by a factor of 2) with no deterioration in the other loop (compared to the performance with identical tunings). Similar observations have been made for distillation columns, and is probably one of the reasons why distillation columns often are tuned with one loop fast and one loop slow.

Our numerical results for higher-order systems ($n > 2$) indicate that the advantage of using different tunings is less than for the 2×2 case.

6 Conclusions

The results above demonstrated that when robust performance (μ_{RP}) is used as a performance measure, the optimal PI- and PID-tunings for single-loop controllers are not necessarily equal even though the problem statement is completely symmetric. The same result is obtained when an H_∞ criterion is used. We have not obtained similar results with an H_2 objective function. The result imply that the solution is not unique since we may simply interchange controller 1 and 2 without changing the value of the objective functuion.

NOMENCLATURE (also see Fig.2)

A - steady-state offset specification (Eq.13)
a - degree of interaction (Eq.1 and 6)
$C(s)$ - controller
$G(s)$ - linear model of process
$H_I(s) = C(s)G(s)(I + C(s)G(s))^{-1}$ - input complementary sensitivity function
k - controller gain
$M = \max_\omega \bar{\sigma}(S)(j\omega)$ - maximum peak of sensitivity function (Eq.13)
N_{RP}, N_{RS}, N_{NP} - see Eq. 14 and 15
NP - Nominal Performance
RGA - Relative Gain Array, elements are λ_{ij}
RP - Robust Performance
RS - Robust Stability
$S(s) = (I + G(s)C(s))^{-1}$ - sensitivity function
w_I - input uncertainty weight (Eq.9)
w_P - performance weight (Eq.13)

Greek symbols

$\|A\|_\infty = \max_\omega \bar{\sigma}(A(j\omega))$ - H_∞-norm of A
$\gamma = \bar{\sigma}/\underline{\sigma}$ - condition number
Δ - block diagonal perturbation matrix
ϵ - relative input error at steady-state
θ - time delay
λ_i - eigenvalue
$\lambda_{11}(j\omega) = (1 - \frac{g_{12}(j\omega)g_{21}(j\omega)}{g_{11}(j\omega)g_{22}(j\omega)})^{-1}$ - 1,1-element in RGA.
μ - structured singular value
μ' - see Eq.16
ρ - spectral radius
$\bar{\sigma}$ - maximum singular value
$\underline{\sigma}$ - minimum singular value
τ - plant time constant
τ_{cl} - (maximum) closed-loop time constant
τ_D - controller derivative time constant
τ_I - controller integral time constant
ω - frequency (rad min^{-1})

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