Performance weight selection for *H*-infinity and μ-control methods

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The paper discusses, from a process control perspective, different approaches to performance weight selection when using H-infinity objectives. Approach A considers direct bounds on important transfer functions such as sensitivity and complementary sensitivity. Approach B considers the output response to sinusoidal disturbances, setpoints and noise. We also give some insight into the practical use of H-infinity and μ methods. μ is the structured singular value (SSV) introduced by Doyle (1982). μ-synthesis is generally not a convex optimisation problem and is presently not straightforward. We will discuss some of the problems we have encountered.

Keywords: Robust control, structured singular value, ill-conditioned, process control

Nomenclature *(also see Fig 3)*

A	steady-state offset specification (Eqn (13))
$C(s)$	controller
D	D-scaling matrix (Eqn (8))
$G(s)$	linear model of process
k	controller gain
$M = \max_\omega \sigma(S(j\omega))$	maximum peak of sensitivity function (Eqn (13))
NP	Nominal Performance
RP	Robust Performance
RGA	Relative Gain Array
RS	Robust Stability
$S(s) = (I + G(s)C(s))^{-1}$	sensitivity function
$T(s) = G(s)C(s)(I + G(s)C(s))^{-1}$	complementary sensitivity function
$x_B = y_2$	output 2 in distillation example
$y_D = y_1$	output 1 in distillation example
w_P	performance weight (Eqn (13))
$\lVert N \rVert_\infty = \max_\omega \sigma(N(j\omega))$	H-infinity norm of N
Δ	complex perturbation matrix
δ	complex perturbation scalar
μ	structured singular value (mu)
$\mu_{RP} = \sup_\omega \mu(N_{RP}(j\omega))$	(peak mu value)
σ	maximum singular value
τ	time constant
τ_{cl}	(maximum) closed-loop time constant
τ_D	controller derivative time constant [min]
τ_I	controller integral time constant [min]
ω	frequency (rad min^{-1})
ω_B	closed-loop bandwidth (where asymptote of $\sigma(S)$ crosses 1)

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1 Introduction

One criticism against the traditional 'optimal' control techniques which use a quadratic cost function (eg, LQG control, see Kwakernaak and Siwan (1972)), has been that the weights (cost matrices and covariance matrices) have a very indirect effect on the behaviour of the closed-loop system. One can generalise the cost function by introducing frequency-dependent weights and use a more general H_2 norm minimisation procedure, but even in this case one cannot specify directly important frequency-domain properties such as bandwidth and magnitude of resonance peaks on certain transfer functions. One natural extension is then to directly specify performance in the frequency domain by employing an H-infinity objective. For a stable transfer function $E(s)$, the H-infinity norm is given by

$$\lVert E \rVert_\infty = \sup_\omega \sigma(E(j\omega)) \qquad \ldots(1)$$

where $\sup_\omega$ denotes the peak value over all frequencies. The H-infinity norm may be viewed as a direct generalisation of the frequency-domain performance specifications used in classical control for single-input single-output (SISO) systems. For example, a very common performance objective is to minimise the weighted sensitivity function, and select $E = w_P S$ where $S = (I + GC)^{-1}$ (see the block diagram of a conventional feedback system in Fig 2).

This paper is based on the paper on ill-conditioned plants by Skogestad *et al* (1988), and the reader is referred to that paper for notation and background material. We shall use the same simplified distillation column (the LV-configuration) as our example. In the previous paper, the effect of various uncertainty descriptions was studied, but here we use the example to discuss alternative choices for the performance weight.

Even though much progress has been made in terms of synthesising H-infinity optimal controllers (eg, Doyle *et al*, 1989), the selection of appropriate performance weights is still very much an art. This may seem somewhat contradictory, as one of the motivations for introducing the H-infinity norm for performance has been to introduce weights which are more meaningful from a physical point of view. There are several reasons why weight selection is difficult.

- One obvious reason is that in most real design cases the specifications are not fixed before the design starts, and the weights are 'knobs' which the engineer adjusts until he obtains a system which performs satisfactorily.

- Another reason is that there are several ways of setting up the problem, and each of these yields different ways of adjusting the weights. There are several physical interpretations of the *H*-infinity norm (Doyle, 1987) which give rise to different procedures for selecting the performance weights. In this paper we will, in the main, discuss two of these procedures:

  Approach A The transfer-function or loop-shaping approach. Here one considers direct bounds on important transfer functions such as S, $T = GCS$ and CS. Often several transfer functions are considered simultaneously and 'stacked' on top of each other when evaluating the *H*-infinity norm. For example, Yue and Postlethwaite (1988) consider the transfer functions S and CS, and use the norm

$$\left\| \begin{matrix} W_1 S W'_1 \\ W_2 CS W'_2 \end{matrix} \right\|_\infty \qquad \ldots(2)$$

  Similarly, Chiang and Safonov (1988; 1990) consider the transfer functions S and T. In this case the first transfer function may be used to specify the bandwidth to achieve acceptable disturbance rejection, whereas the latter is used to avoid amplification of noise at high frequency. McFarlane and Glover (1990) use a direct loop-shaping approach.

  Approach B The signal approach (eg Doyle *et al*, 1987). Here one considers the response to sinusoidal signals. In this approach one cannot directly specify bandwidth etc. However, this approach may be more appropriate for multivariable problems in which a number of objectives must be taken into account simultaneously. Also, in such systems the concept of bandwidth is often difficult to use.

- There are different ways of handling model uncertainty. Above, we discussed nominal performance (NP). The ability to address also robust stability (RS) and robust performance (RP) in a consistent and rigorous manner is probably the most important reason for using the *H*-infinity norm for performance. However, there are at least two approaches for taking model uncertainty into account.

  (1) The mixed NP–RS approach: Add the robust stability condition as an additional *H*-infinity objective to be minimised. One example is to try to optimise simultaneously nominal performance using $w_P S$ and robust stability with respect to relative output uncertainty of magnitude $|w_2(j\omega)|$ using $w_2 T$. These objectives are combined and the controller is designed to minimise the combined objective function

$$\min_C \|N_{mix}\|_\infty; \quad N_{mix} = \begin{pmatrix} w_P S \\ w_2 T \end{pmatrix} \qquad \ldots(3)$$

  Note that this is the same objective as discussed by Chiang and Safonov (1988), but here the bound on T follows as an RS-condition and not as a condition for noise amplification. This follows, since the same transfer function may be given both a performance and a stability interpretation. In practice, these considerations are often combined when selecting the weights, and Approach A for performance selection is usually combined with the mixed NP–RS approach for model uncertainty.

  (2) The RP approach: Use the same *H*-infinity performance specification, but require that it is satisfied (or minimised) not only for the nominal plant, but for all plants as defined by the uncertainty description, that is, require robust performance. For example, when performance is measured in terms of $w_P S$, the robust performance objective with output uncertainty becomes (eg, Skogestad *et al*, 1988)

$$\min_C \sup_\omega \mu(N_{RP}); \quad N_{RP} = \begin{pmatrix} w_2 T & w_2 T \\ w_P S & w_P S \end{pmatrix} \qquad \ldots(4)$$

Comment: For this particular case, with both performance and uncertainty measured at the plant outputs, there is almost no difference between the mixed NP–RS approach and the RP approach. (At each frequency $\mu(N_{RP})$ is by most a factor of $\sqrt{2}$ larger than $\sigma(N_{mix})$). However, for ill-conditioned plants with uncertainty at the plant inputs (which is always present), this is not the case and the mixed approach may yield very poor RP. For example, this applies to the example studied by Skogestad *et al* (1988) which is also studied in this paper.

The RP approach is used in this paper. It is more rigorous than the mixed NP–RS approach, but it requires use of the structured singular value. This makes controller synthesis rather involved, but analysis is straightforward. A good design approach may be to synthesise controllers using the mixed approach, and analyse them using RP and μ. It may be necessary to iterate on the weights in order to obtain acceptable μ-values (Actually, as discussed in the next section, the presently used 'D–K' iteration for μ-synthesis involves solving a series of *H*-infinity problems).

Software to synthesise *H*-infinity controllers has been available for some time, for example, through the Robust Control toolbox in MATLAB (Chiang and Safonov, 1988). Recently, a μ-toolbox for MATLAB has become available (Balas *et al*, 1990). This toolbox includes alternative *H*-infinity software, and μ-analysis and synthesis is included as outlined above. All computations presented in this paper have been done employing this toolbox.

Some important terms:

Nominal stability (NS): the closed-loop system without uncertainty is stable.

Robust stability (RS): the system is stable for all defined uncertainty ('worst case is stable').

Nominal performance (NP): the system satisfies the performance requirements for the case with no uncertainty.

Robust performance (RP): the system satisfies the performance requirements for all defined uncertainty ('worst case satisfies performance requirement').

2. Model uncertainty and the structured singular value

The objective of this section is to give the reader a short introduction to model uncertainty and the structured singular value, μ. A more detailed introduction to μ is given by Doyle (1982; 1987) and Skogestad *et al* (1988).

An important reason for selecting the *H*-infinity norm for performance, is that also model uncertainty may be readily formulated using this norm. In particular, this applies to uncertain or neglected high-frequency dynamics that are always present, and which cannot by modelled by parametric uncertainty in a state–space model with fixed order. In the *H*-infinity framework the model uncertainty is modelled in terms of uncertain perturbations, Δ_i. Using weights they are normalised such that their *H*-infinity norm is less than 1.

$$\|\Delta_i\|_\infty \leq 1 \Longleftrightarrow \sigma(\Delta_i(j\omega)) \leq 1, \quad \forall\omega \qquad \ldots(5)$$

Unstructured uncertainty. In the simplest approach all the uncertainty is lumped into one perturbation matrix, Δ, for example at the output. This is an unstructured uncertainty description, and gives rise to robust stability conditions in terms of the singular value (*H*-infinity norm). For example, for input uncertainty of magnitude w_2, where also 'cross-channel' uncertainty is allowed (Δ is a full matrix) the RS-condition becomes $\|w_2 T_I\| < 1$. However, this approach is generally conservative because it will include a lot of plant cases that cannot occur in practice. If cross-channel uncertainty does not occur in practice, then the correct RS condition is $\mu_\Delta(w_2 T_I) < 1$, $\forall\omega$, where Δ is a diagonal matrix as given in Eqn (6) below.

Structured uncertainty. To model the uncertainty more tightly we must consider structured uncertainty, that is, use several perturbation blocks. Usually each of these blocks is related to a specific physical source of model uncertainty: for instance, a measurement uncertainty or an input uncertainty. For example, for the input uncertainty without cross-channel coupling we need one perturbation block for each input and we get, for a system with n inputs,

$$\Delta = \begin{bmatrix} \delta_1 & & & 0 \\ & \delta_2 & & \\ & & \ddots & \\ 0 & & & \delta_n \end{bmatrix} \qquad \ldots(6)$$

Robust stability. To test for robust stability, the system with the uncertainty blocks is rearranged such that N_{RS} (which includes the uncertainty weights) represents the interconnection matrix from the outputs to the inputs of the uncertainty blocks, Δ. In the following we assume that N_{RS} is stable. Using the small gain theorem, we know that robust stability will be satisfied if $\|N_{RS}\|_\infty < 1$, or equivalently

$$\text{RS} \quad \text{if} \quad \sigma(N_{RS}) < 1; \quad \forall\omega \qquad \ldots(7)$$

However, this bound is generally conservative unless the uncertainty is truly unstructured. First, the issue of stability should be independent of scaling. Thus, an improved robust stability condition is

$$\text{RS} \quad \text{if} \quad \min_{D(\omega)} \sigma(DN_{RS}D^{-1}) < 1; \quad \forall\omega \qquad \ldots(8)$$

where D is a real block-diagonal scaling matrix with structure corresponding to that of Δ, such that $\Delta D = D\Delta$. A further refinement of this idea led to the introduction of the structured singular value (Doyle, 1982; 1987). We have (essentially, this is the definition of μ)

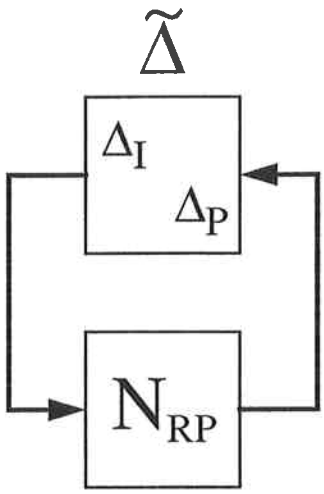

Fig 1 General block structure for μ analysis

$$\text{RS} \quad \text{iff} \quad \mu_\Delta(N_{RS}) < 1; \quad \forall\omega \qquad \ldots(9)$$

Thus $\min_D \sigma(DND^{-1})$ is an upper bound on $\mu(N)$. It is usually very close in magnitude. The largest deviation reported so far is about 10–15% (Doyle, 1982; 1987). Computationally tractable lower bounds for μ also exist and are in common use.

Robust performance. An additional bonus of using the *H*-infinity norm both for performance and uncertainty is that the robust-performance problem may be recast as a robust-stability problem (Doyle, 1982), with the performance specification represented as a fake uncertainty block. To test for robust performance, one considers the interconnection matrix N_{RP} from the outputs to the inputs of *all* the Δ-blocks, including the 'full' Δ_P-block for performance. N_{RP} depends on the plant G, the controller C and on the weights used to define uncertainty and performance. The condition for robust performance within the *H*-infinity framework is (see Fig 1)

$$\text{RP} \quad \text{iff} \quad \mu_{\tilde{\Delta}}(N_{RP}) < 1, \quad \forall\omega, \quad \tilde{\Delta} = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_P \end{bmatrix} \qquad \ldots(10)$$

Analysis of robust performance for a given controller using μ is straightforward, but controller design using μ-synthesis is still rather involved. The present 'D–K iteration' uses the upper bound on μ, and involves solving a number of 'scaled' *H*-infinity problems. We will discuss this further in section 6.

Uncertainty weights. Since uncertainty modelling using the *H*-infinity framework is a worst-case approach, generally one should not include too many sources of uncertainty, since otherwise it becomes very unlikely for the worst case to occur in practice. One should, therefore, lump various sources of uncertainty into a single perturbation whenever this may be done in a non-conservative manner. On the other hand, one should be careful about excluding physically meaningful sources of uncertainty that limit achievable performance. From this it follows that selecting appropriate uncertainty weights is very problem-dependent, and it is important that guidelines for specific classes of problems be developed.

Sometimes one might use a smaller uncertainty set for robust performance than for robust stability. The

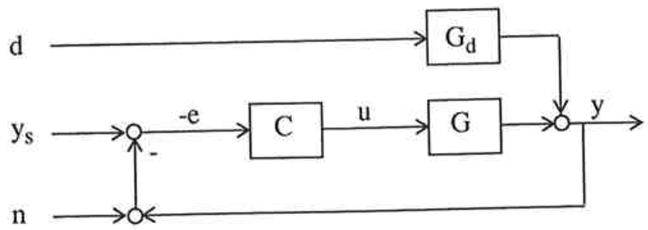

Fig 2 Block diagram of conventional feedback system

idea is to guarantee stability for a large set of possible plants, but require performance only for a subset. This is to avoid very conservative designs with poor nominal performance.

For the example in this paper we consider only input uncertainty. The effect of output uncertainty, time-constant uncertainty and correlated-gain uncertainty was studied by Skogestad *et al* (1988). They found that these sources of uncertainty were less important than the input uncertainty for this particular ill-conditioned plant.

3. Performance weights

There are several different physical interpretations of the H-infinity norm of E (Doyle, 1987, Zhou *et al*, 1990) and, as mentioned in the introduction, this gives rise to different methods for weight selection.

Approach A. Consider E as a transfer function. Since $\|E\|_\infty = \sup_\omega \sigma(E(j\omega))$ the H-infinity norm may be viewed as a direct generalisation of classical frequency-domain bounds on transfer functions (loop-shaping) to the multivariable case.

Approach B. Alternatively, consider $E(j\omega)$ as the frequency-by-frequency sinusoidal response. That is, for a unit sinusoidal input to channel j with frequency ω, the steady-state output in channel i is equal to $E_{ij}(j\omega)$. To consider all the channels combined, we use the maximum singular value, $\sigma(E(j\omega))$, which gives the worst-case (with respect to choice of direction) amplification of a unit sinusoidal input of frequency ω through the system.

Approach C. The induced norm from bounded-power-spectrum inputs to bounded-power-spectrum outputs in the time domain is equal to the H-infinity norm.

There are also other interpretations of the H-infinity norm: it is equal to the induced 2-norm (energy) in the time domain; it is equal to the induced power norm; and it is also equal to induced norm in the time domain from signals of bounded magnitude to outputs of bounded power.

The following discussion is mostly relevant to approaches B and C. A general way to define performance within the H-infinity framework is to consider the H-infinity norm of the closed-loop transfer function E between the external weighted input vector $\mathbf{w}$ (disturbances d, setpoints y_s and noise n) and the weighted output vector $\mathbf{z}$ (may include $y-y_s$, manipulated inputs u which should be kept small etc). Weights are chosen such that the magnitude (in terms of the 2-norm) of the normalised external input vector is less than one at all frequencies, ie, $\|\mathbf{w}(j\omega)\|_\infty \leq 1$, and such that for acceptable performance the normalised output vector is less than 1 at all frequencies, ie, $\|\mathbf{z}(j\omega)\|_\infty < 1$. With $\mathbf{z} = E\mathbf{w}$ the performance requirement becomes

$$\|E\|_\infty = \sup_\omega \sigma(E(j\omega)) < 1 \qquad \ldots(11)$$

Introducing the weights W_d, W_s, W_n, W_e and W_u into Fig 2 yields the block diagram in Fig 3 where E is given as shown by the dotted box. We also have introduced an 'ideal response' W_f from y_s to y, and use a 'two-degrees-of-freedom controller'. Except for W_f it is only the magnitude of these weights that matters; they should therefore be stable and minimum phase.

3.1 Performance Approach A. Weights on transfer functions (loop shaping)

Many performance specifications may be translated into an upper bound $1/|w_P|$ on the frequency plot of the magnitude of the sensitivity function $S=(I+GC)^{-1}$.

$$\sigma(S(j\omega)) < 1/|w_P(j\omega)|, \quad \forall\omega \qquad \ldots(12)$$

This is equivalent to Eqn (11) with $E = w_P S$ (weighted sensitivity). The concept of bandwidth, which is here defined as the frequency ω_B where the asymptote of $\sigma(S)$ first crosses one, is closely related to this kind of performance specification, and most classical frequency-domain specifications may be captured by this approach.

Classical frequency-domain specifications. For example, assume that the following specifications are given in the frequency domain:

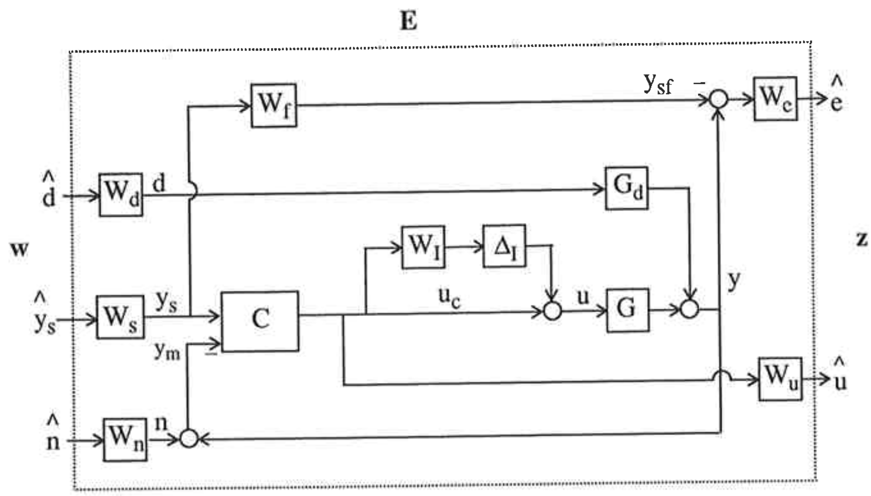

Fig 3 General feedback system with weights, a two-degree-of-freedom controller and input uncertainty. It is assumed that the outputs are measured directly. The transfer function E is used for H-infinity performance with Approach B

1. Steady-state offset less than A;
2. Closed-loop bandwidth higher than ω_B;
3. Amplification of high-frequency noise less than a factor M.

These specifications may be reformulated in terms of Eqn (12) using

$$w_P(s) = \frac{1}{M}\frac{\tau_{cl}s + M}{\tau_{cl}s + A}, \quad \text{with } \tau_{cl} = 1/\omega_B \qquad \ldots(13)$$

and the resulting bound $1/|w_P(j\omega)|$ is shown graphically in Fig 4.

In many cases a steeper slope on S is desired at frequencies below the bandwidth to improve performance. For example, this may be the case if the disturbances are relatively slow as discussed below (Section 3.3).

Several transfer functions. As mentioned in the introduction, one may define similar performance objectives in terms of other transfer functions, and consider the combined effect by stacking them together when computing the H-infinity norm.

Matrix-valued weights. In the multivariable case, the generalised weighted sensitivity is $W_P S W'_P$. For example, one may use different bounds on the sensitivity function for various outputs. Assume that we want the response in channel 1 to be about 10-times faster than that in channel 2. Then we might use the performance specification

$$\|W_P S\|_\infty < 1; \quad W_P = \begin{pmatrix} w_{P11} & 0 \\ 0 & w_{P22} \end{pmatrix} \qquad \ldots(14)$$

with $\omega_{B11} = 10\omega_{B22}$. Later we shall study the use of different weights in each channel for the distillation example.

Introducing matrix-valued weights is necessary if the disturbances have strong directionality. However, the direct implications for the shape of the sensivitity function then become less clear, and it is then probably better to shift to the more general signal-oriented approach discussed next.

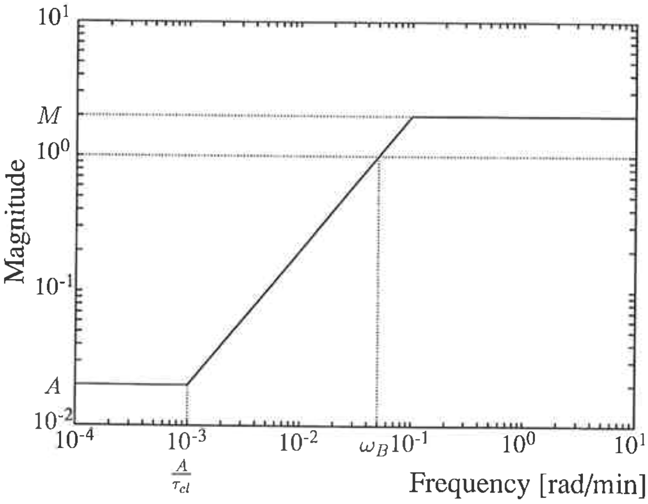

Fig 4 Asymptotic plot of $1/w_P = M(\tau_{cl}s + A)/(\tau_{cl}s + M)$ where $\tau_{cl} = 1/\omega_B$. $|S(j\omega)|$ should lie below $1/|w_P|$ to satisfy classical frequency-domain specifications in terms of A, M and ω_B

3.2 Performance Approach B. Frequency by frequency sinusoidal signals

In many cases it is not possible, or at least it is difficult, to specify directly appropriate weights on selected transfer functions. A sinusoidal signal-oriented approach is then more suitable. For example, this approach is used in the space-shuttle application study by Doyle *et al* (1987).

In this approach we consider the effect of persistent sinusoidal input signals of a given frequency. Consider again Fig 3, the signal weights W_d, W_s and W_n will be diagonal matrices which give the expected magnitude of each input signal at each frequency. Typically, the disturbance weight, W_d, and the setpoint weight, W_s do not vary very much with frequency[1], while the noise weight, W_n, usually has its peak value at high frequency. W_e is a diagonal matrix which at each frequency specifies the *inverse* of the allowed magnitude of a specific output error. If we want no steady-state offset, the weight should include an integrator such that its magnitude is infinite at steady-state (we require offset-free response to slow-varying sinusoids)[2]. Typically, we let the weight (W_e) level off at high frequencies at a value $(1/M)\,W_s^{-1}$. The factor of M limits the maximum peak of S and is a way of including some loop-shaping ideas from Approach A. Often the value of M is about 2. The corner frequency for the weight W_e (where it levels off) should be approximately M/τ_{cle} where τ_{cle} is the maximum allowed closed-loop time constant for that output. The actuator penalty weight, W_u, is usually small or zero at steady-state[3]. W_u may be close to a pure differentiator (s) if we want to penalise fast changes in the inputs.

It is important to check that the various performance requirements are consistent. This may be done by evaluating their influence of the required loop shapes (Approach A), in particular, at low and high frequencies. Alternatively, one may test if it is possible to get $\mu < 1$ for NP by performing an H-infinity synthesis with no uncertainty.

The approach described above tends to give a large number of weights, and this is a disadvantage. First, the dimension of the problem grows and the solution takes more time. Second, with too many independent sources of noise and disturbances it may become very unlikely for the worst case to occur in practice (note that the H-infinity norm represents a worst-case approach as the singular value picks out the worst direction). For example, if we have a large number of measurements (it may be possible to have 100 measurements in a

[1]That is, in this Approach B we should *not* add a integrator ($1/s$) to the weight even if step changes in disturbances or setpoints are expected (however, in Approach C this is correct). The reason is that in Approach B we consider the response frequency-by-frequency, and a step change cannot really be modelled very well, and certainly not as a slow-varying sinusoid of infinite magnitude. A more-reasonable approach is to consider a range of sinusoids and use a nearly constant weight with the same magnitude as of the step.

[2]Note that we may not require offset-free response for $y - y_s$ if the measurement noise is non-zero at steady-state ($\omega = 0$). Therefore, to get a controller with integral action we should select W_n to be zero at $\omega = 0$. Alternatively, we may require no offset for $y_m - y_s$, where $y_m = y + n$ is the measurement.

[3]The use of actuators inputs of a certain magnitude is often unavoidable (independent of the controller) in order to reject slow-varying disturbances, and penalising the inputs at low frequencies makes little sense in such cases.

distillation column), it will be very unlikely that the worst combination of measurement noise will ever occur in practice. Mejdell (1990) encountered this problem and found it necessary to reduce the number of independent noise directions.

3.3 Combining performance weights (going from Approach B to Approach A)

There are, of course, cases in which Approaches A and B are identical, but in general there is a significant difference between considering in A the shape of the transfer functions (eg in terms of its slope and frequency where it crosses one), and in B considering the magnitude of a specific output signal to sinusoidal disturbances.

To illustrate how specifications on setpoints and disturbance rejection (Approach B) may be reformulated as bounds on the weighted sensitivity (Approach A) consider Fig 3 and evaluate the transfer function from normalised disturbances and setpoints to normalised errors. We have

$$z = \hat{e} = E\begin{pmatrix}\hat{d}\\ \hat{y}_s\end{pmatrix} = Ew \qquad \ldots(15)$$

With conventional feedback control (one-degree of freedom), no setpoint filtering ($W_f = I$) and no uncertainty ($\Delta_I = 0$) we have (Fig 3)

$$E = (W_e S G_d W_d \;\; W_e S W_s). \qquad \ldots(16)$$

The performance specification is $\|E\|_\infty < 1$, but we want to obtain a bound on $\sigma(S)$. To this end we find a weight $w_P(j\omega)$ such that at each frequency $\sigma(w_P S) = \sigma(E)$. For the SISO (scalar) case we get $\sigma(E) = |W_e S| \sqrt{|G_d W_d|^2 + |W_s|^2}$ and $\sigma(w_P S) = |w_P S|$, and we have at each frequency

$$|w_P| = |W_e| \sqrt{|G_d W_d|^2 + |W_s|^2} \qquad \ldots(17)$$

Consider the following special SISO case where we assume: (1) outputs have been scaled such that for setpoints $W_s = 1$; (ii) G_d has been scaled such that disturbances d are less than 1 in magnitude, thus $W_d = 1$; (iii) disturbance model $G_d = k_d/(1+\tau_d s)$; and (iv) the errors, e, should be less than M in magnitude at high frequencies, and we want integral action and require a response time better than about τ_{cle}, ie, $W_e = (\tau_{cle} s + M)/M\tau_{cle} s$.

With the exception of at most a factor $\sqrt{2}$ (at frequencies where $|G_d| \approx 1$) we may then use the following approximation for Eqn (17):

$$w_P(s) \approx W_e(s)\left(\frac{|k_d|}{1+\tau_d s} + 1\right) = \frac{s + M/\tau_{cle}}{Ms} \; \frac{s + \dfrac{|k_d|+1}{\tau_d}}{s + 1/\tau_d} \qquad \ldots(18)$$

Obviously, if the disturbance gain, $|k_d|$, is small compared to 1 (the magnitude of the setpoints), then $w_P(s) \approx W_e(s)$, and the disturbance does not affect the bound on $S(j\omega)$. However, in general the requirement of disturbance rejection may require a faster response than the response time, τ_{cle}, required by the weight W_e. The most important feature of the performance weight, $w_P(s)$, is its bandwidth requirement, ω_B^*,

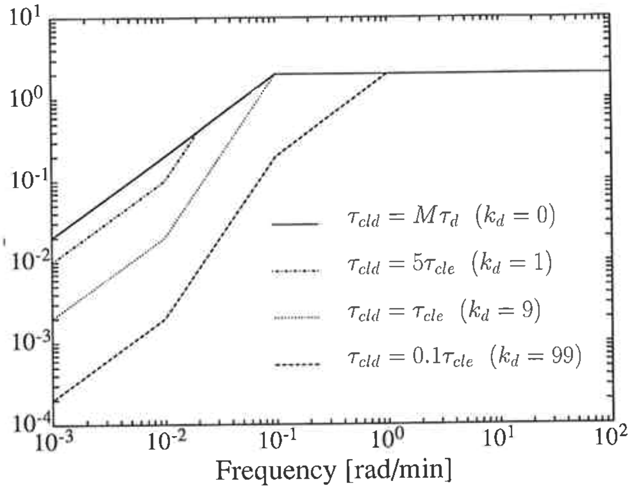

Fig 5 Asymptotic plot of $1/w_P$ (Eqn (18)) for cases where $\tau_{cle} < \tau_d$ ($M = 2$, $\tau_{cle} = 20$ min and $\tau_d = 100$ min is used in the plot)

which we define as the frequency at which the asymptote of $w_P(s)$ crosses 1. Introduce the response-time constant imposed by disturbances

$$\tau_{cld} = M\frac{\tau_d}{|k_d|+1} \qquad \ldots(19)$$

A closer analysis of Eqn (18) shows that $\omega_B^* = \max\{1/\tau_{cle}, 1/\tau_{cld}\}$. For $\tau_{cld} < \tau_{cle}$, the bandwidth requirement is determined by disturbance rejection. This is illustrated in Fig 5 where we show the bound $1/|w_P|$ on $|S|$ as a function of frequency. The solid line shows the requirement for setpoint tracking only, whereas the various dotted lines show the requirement for increasing magnitude of the disturbance. In this case with 'slow disturbances' ($\tau_{cle}/M < \tau_d$) the weight in Eqn (18) has a region at low frequencies where $|w_P(j\omega)|$ has a slope of -2 on a $\log|w_P| - \log\omega$ plot.

In the multivariable case we must use matrix-valued weights, and it is not possible to transform Approach B into a scalar bound on S. Specifically, $\sigma(SG_d(j\omega))$ may be significantly smaller than $\sigma(S(j\omega))\,\sigma(G_d(j\omega))$ when G_d is in the 'good' direction corresponding to the large plant gains (see Skogestad *et al*, 1988).

3.4 Performance Approach C. Power signals – power spectrum weights

This is not a frequency-by-frequency approach. Rather, one must consider the entire frequency spectrum. One may think of the weights W_d, W_n and W_s as upper bounds on the power spectral density of the input signals, whereas W_e and W_u are equal to the inverse of the upper bounds on the power spectral density of the output signals. For example, if we allow for step changes of the setpoints, we may choose a weight $W_s = 1/s$ (but we will also allow a lot of other signals bounded by this spectral density). We will not discuss this approach any further, but just note that, compared to Approach B, it corresponds in many cases to shifting integrators from the output weights to the input weights.

4. Fixed or adjustable weights

One advantage with *H*-infinity or μ-optimal control is that it is relatively well-defined as to what an objective function with a value close to 1 means: the worst-case response will satisfy our performance objective. If μ at a given frequency is different from 1, then the interpretation is that at this frequency we can tolerate $1/\mu$-times more uncertainty and still satisfy our performance objective with a margin of $1/\mu$.

Controller synthesis almost always consists of a series of steps in which the designer iterates between mathematical formulation of the control problem, synthesis and analysis. In μ-synthesis the designer will usually redefine the control problem by adjusting some performance or uncertainty weight until the final optimal μ-value is reasonably close to 1. In most cases this is done in a more or less *ad hoc* fashion, but it may also be done systematically. One attractive option is to keep the uncertainty weight fixed (of course, it must be possible to satisfy RS) and evaluate the achievable performance with this level of uncertainty, that is, adjust some performance weight to make $\sup_\omega \mu(N_{RP}) = 1$. There are two obvious options to adjust the performance weight

(1) Scale the performance frequency-by-frequency such that $\mu(N_{RP}) = 1$ at all frequencies, that is, at each frequency find a $k(\omega)$ which solves

$$\mu\begin{pmatrix} N_{RP_{11}} & N_{RP_{12}} \\ kN_{RP_{21}} & kN_{RP_{22}} \end{pmatrix} = 1 \qquad \ldots(20)$$

This option is most attractive for analysis with a given controller. The numerical search for k is straightforward, since μ increases monotonically with k, and since a solution always exists provided that we have RS.

(2) Adjust some parameter in the performance weight such that the peak value of $\mu(N_{RP})$ is 1. This option is most reasonable for μ-synthesis, that is, if the controller is not given. For example, with the performance weight of Eqn (13) we may adjust the time constant τ_{cl} such that the optimisation problem becomes

$$\min_{\tau_{cl}, C} |\tau_{cl}|; \quad \text{s.t.} \quad \mu(N_{RP}(C, \tau_{cl})) \leq 1, \quad \forall\omega \qquad \ldots(21)$$

Different plants may then be compared based on their maximum achievable bandwidth. A disadvantage with this approach is that it may be impossible to achieve $\mu(N_{RP}) = 1$ by adjusting τ_{cl} in the performance weight if, for example, the high-frequency specification (value of M) is limiting. Skogestad and Lundström (1990) have used this approach to compare alternative control structures for a distillation-column example. An other approach is to keep τ_{cl} and M in the weight Eqn (13) fixed, and rather adjust the weight of *all* frequencies with the same constant. However, sometimes this does not make sense from a physical point of view since we cannot adjust the weight very much at high frequencies (since $S \approx I$ at high frequencies).

In this paper we do not employ these approaches, but use fixed weights only.

5. Skogestad *et al* (1988) example revisited

We shall use the same plant as studied previously by Skogestad *et al* (1988). The plant model is

$$y(s) = \begin{pmatrix} y_D \\ x_B \end{pmatrix} = G(s)\,u(s); \quad G(s) = \frac{1}{75s+1}\begin{pmatrix} 0.878 & -0.864 \\ 1.082 & -1.096 \end{pmatrix} \qquad \ldots(22)$$

which has a condition number of 141.7 and a RGA-value of 35.5 at all frequencies. The unit for time is minutes. This is a very crude model of a distillation column, but it is an excellent example for demonstrating the problems with ill-conditioned plants. Freudenberg (1989) and Yaniv and Barlev (1990) also used this model to demonstrate design methods for robust control of ill-conditioned plants.

In Skogestad *et al* (1988) the following specifications were used

(1) The relative magnitude of the uncertainty in each of the two input channels is given by

$$w_I(s) = 0.2(5s+1)/(0.5s+1). \qquad \ldots(23)$$

Thus the uncertainty is 20% at low frequencies and reaches 1 at a frequency of approximately 1 rad/min. Note that the corresponding uncertainty matrix, Δ_I is a diagonal matrix since we assume that uncertainty does not 'spread' from one channel to another (for example, we assume that a large input signal in channel 1 does not affect the signal in channel 2).

(2) RP-specification (using performance Approach A): the worst case (in terms of uncertainty) *H*-infinity norm of $w_P S$ should be less than 1.

$$w_P(s) = 0.5(10s+1)/10s \qquad \ldots(24)$$

This requires integral action, a bandwidth of approximately 0.05 rad/min and a maximum peak for $\sigma(S)$ of 2.

The resulting μ-condition for Robust Performance becomes (see Fig 1):

$$\text{RP} \quad \text{iff} \quad \mu_{\tilde\Delta}(N_{RP}) < 1, \quad \forall\omega \qquad \ldots(25)$$

where

$$N_{RP} = \begin{bmatrix} -w_I CSG & w_I CS \\ w_P SG & -w_P S \end{bmatrix}; \quad \tilde\Delta = \begin{bmatrix} \delta_1 & & 0 \\ & \delta_2 & \\ 0 & & \Delta_P \end{bmatrix} \qquad \ldots(26)$$

In the following we shall keep the uncertainty description fixed, but consider alternative performance specifications.

5.1 Original problem formulation (performance Approach A)

Skogestad *et al* (1988) used a software package based on the *H*-infinity minimisation in Doyle (1985) (denoted 'the 1984 approach' in Doyle *et al*, 1989) to design a 'μ-optimal controller'. Their controller has six states and gives $\sup_\omega \mu(N_{RP}) = \mu_{RP} = 1.067$ for both structured and unstructured Δ_I. We will denote this controller $C_{\mu old}$. Freudenberg (1989) used another design method and achieved a controller with five states giving $\mu_{RP} = 1.054$ for unstructured Δ_I. Yaniv and Barlev (1990) do not present a μ value for their

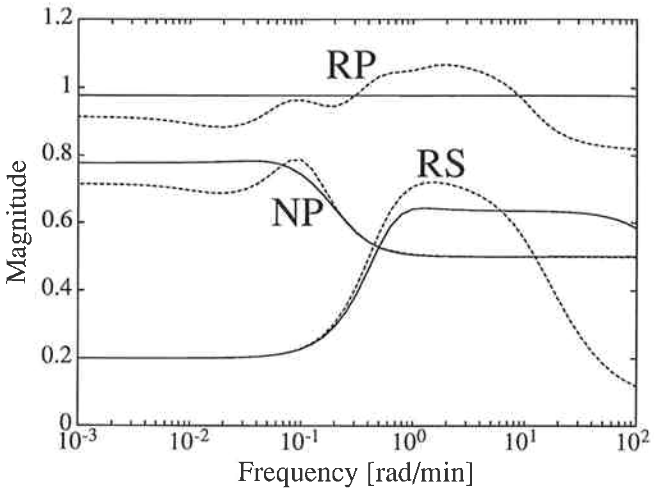

Fig 6 μ-plots for $C_{\mu new}$ (solid curves) and $C_{\mu old}$ (dashed curves)

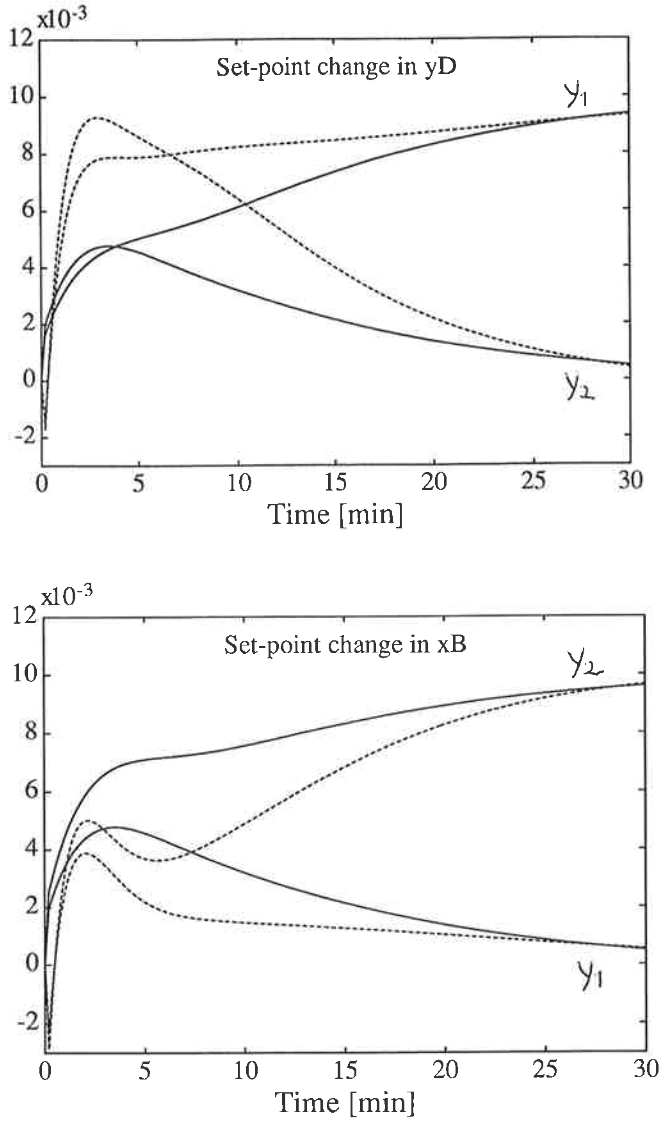

Fig 7 Simulation of setpoint changes using controller $C_{\mu new}$. Responses are shown both for nominal case (solid curves) and with input uncertainty given in Eqn (27) (dashed curves)

design. We obtained $\mu_{RP} = 2.28$ for their controller[1] without pre-filter.

New optimal design. With the new *H*-infinity software (Balas *et al*, 1990) based on the state–space solution of Doyle *et al* (1989), the μ-synthesis ('D–K iteration') performs better than with the 1984 approach. We were able to design a controller which, compared to $C_{\mu old}$, lowered μ_{RP} from 1.067 to 0.978. The new controller will be denoted $C_{\mu new}$. It has 22 states and a state–space representation is given in Appendix 1.

Fig 6 shows μ for RP, NP and RS as a function of frequency for $C_{\mu new}$ (solid curves) and $C_{\mu old}$ (dashed curves). $\mu(N_{RP})$ for the new controller is extremely flat and the peak value, μ_{RP}, is substantially lower than for the old controller. The *nominal* performance is generally worse for the new controller, while robust stabilty is improved for some frequencies.

Fig 7 shows the time response to setpoint changes for controller $C_{\mu new}$. The solid curves show the nominal response, and the dotted curves the response with model error. The specific model error we use is

$$u(s) = \begin{pmatrix} \dfrac{-0.5s+1.2}{0.5s+1} & 0 \\ 0 & \dfrac{-0.5s+0.8}{0.5s+1} \end{pmatrix} u_c(s) \qquad \ldots(27)$$

where u is the true input signal and u_c the input signal computed by the controller.

This uncertainty is covered by the uncertainty description, $w_I(s)$. By comparing the time responses to those presented for the controller $C_{\mu old}$ in Skogestad *et al* (1988) we see that difference is relatively small.

5.2 Alternative controller designs (Approach A)

Table 1 shows minimised μ-values obtained for different controller structures. The second row shows that with a two-degree-of-freedom controller we may reduce the μ-value for RP from 0.978 to 0.926 (to avoid numerical problems when obtaining this controller we had to introduce some measurement noise). In general, a two-degree-of-freedom controller yields improved performance when we have simultaneous disturbance rejection and command following. In our case the model uncertainty in effect introduces disturbances (generated internally) and makes it advantageous to use a controller which shapes the setpoints differently.

In Table 1 we also show the results using two PI- or PID-controllers of the form below.

$$C(s) = \begin{pmatrix} c_{PID_1}(s) & 0 \\ 0 & c_{PID_2}(s) \end{pmatrix};$$

$$c_{PID_i}(s) = k_i \frac{1+\tau_{I_i}s}{\tau_{I_i}s} \frac{1+\tau_{D_i}s}{1+0.1\tau_{D_i}s} \qquad \ldots(28)$$

Optimal PI/PID tunings were obtained using a general-purpose optimisation algorithm to minimise μ_{RP} with respect to the six parameters[2]. We obtained optimal

[1]There is a misprint in Yaniv and Barlev (1990); the second order pole in controller 'g l' of 0.22 should be, with damping, 0.5.

[2]We set up the problem as a min–max problem $\min_c \max_\omega \mu(N_{RP})$, and used the routine 'minimax' in the Optimization toolbox for MATLAB (Grace, 1990).

TABLE 1: Optimal μ_{RP}-values for the original problem formulation, Eqns (25)–(26) (Approach A)

Controller type	μ_{RP}	k_1	k_2	τ_{I1} **min**	τ_{I2} **min**	τ_{D1} **min**	τ_{D2} **min**
$C_{\mu new}$ One – degree	0.978						
$C_{\mu 2}$, Two – degree	0.926						
PI	1.50	142	−25.6	64.4	1.35		
PID	1.32	163	−39.1	41.2	0.896	0.342	0.290

μ_{RP}-values of 1.50 for PI-control and 1.32 for PID-control. Note that the optimal tunings are very different for the two channels in spite of the fact that the problem formulation is nearly symmetric. This is probably caused by the fixed structure and limited number of states of PI/PID controllers. This issue is discussed in more detail by Lundström *et al* (1991).

5.3 Other performance weights (Approach A)

Here we use the same problem formulation as in the previous section, except for using different performance weights in each output channel.

$$W_P(s) = \begin{pmatrix} w_{P_1} & 0 \\ 0 & w_{P_2} \end{pmatrix}; \quad w_{Pi}(s) = \frac{1}{M}\frac{\tau_{cl_i}s + M}{\tau_{cl_i}s} \quad \ldots(29)$$

Intuitively, we may reduce the 'interactions' (this is a term which is relevant for single-loop control) in the system by having one channel with a fast response, and one channel with a slow response. Optimal μ-values for different choices of τ_{cl_1} and τ_{cl_2} are shown in Table 2. We keep the 'average' response time constant by holding $\tau_{cl_1}\tau_{cl_2}$ constant. We see that the μ-values are somewhat lower when we allow different response times in the two channels (of course, this is only true to a limited extent, since the response time of the fast channel is limited by the uncertainty weight which crosses one at a frequency of 1 rad/min). The fourth entry in Table 2 does not have the same 'average' response time, but is included to illustrate that the μ_{RP}-value increases markedly if we require that one loop is made faster without relaxing the requirement of the other loop.

As expected, the reduced interaction becomes even more clear if we study single-loop (decentralised) control using PID controllers. The tuning parameters and μ_{RP} for different choices of performance weights are also given in Table 2. The last entry in the table shows that for these controllers we can increase the speed of one channel at almost no cost in terms of μ_{RP}.

5.4 Performance Approach B

Consider a revised problem that includes disturbance rejection as shown in the block diagram in Fig 3. We shall design a two-degree-of-freedom controller using Approach B. The plant G is given in Eqn (22). G_d describes the effect of disturbances (feed flow, F, and feed composition, z_F) on the two controlled variables (top and bottom composition, y_D and x_B).

$$G_d(s) = \frac{1}{75s+1}\begin{pmatrix} 0.394 & 0.881 \\ 0.586 & 1.119 \end{pmatrix} \quad \ldots(30)$$

We use the following weights to define the problem.

$$W_d(s) = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.2 \end{pmatrix}; \quad W_s(s) = 0.01 I_{2x2};$$

$$W_n(s) = 0.01\frac{s}{s+1} I_{2x2} \quad \ldots(31)$$

$$W_f(s) = \frac{1}{5s+1} I_{2x2}; \quad W_I(s) = 0.2\frac{5s+1}{0.5s+1} I_{2x2} \quad \ldots(32)$$

$$W_e(s) = \frac{100}{2}\frac{20s+2}{20s} I_{2x2}; \quad W_u(s) = 0.01\frac{50s+1}{0.0005s+1} I_{2x2} \quad \ldots(33)$$

These weights are plotted in Fig 8.

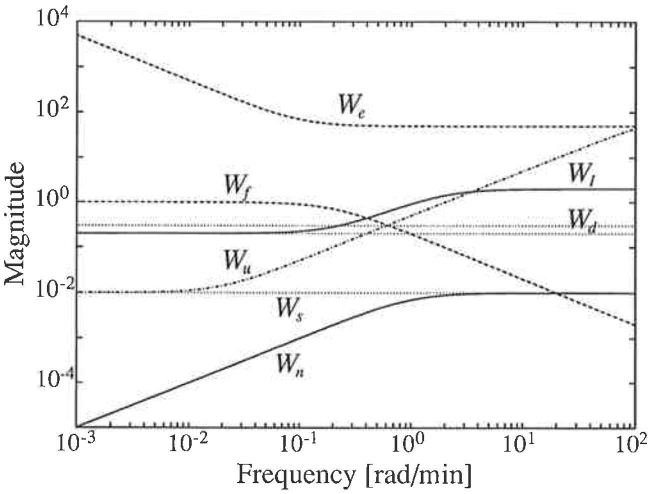

Fig 8 Frequency plot of the weights used in Approach B in section 5.4

TABLE 2: Optimal μ_{RP}-values for different performance weights, Eqn (13), (Approach A)

Controller type	μ_{RP}	τ_{cl1} **min**	τ_{cl2} **min**	k_1	k_2	τ_{I1} **min**	τ_{I2} **min**	τ_{D1} **min**	τ_{D2} **min**
$C_{\mu new}$, One – degree	0.978	20	20						
One – degree	0.970	40	10						
One – degree	0.937	80	5						
One – degree	1.098	20	10						
PID	1.32	20	20	163	−39.1	41.2	0.896	0.342	0.290
PID	1.15	40	10	98.4	−17.7	67.5	0.769	0.385	0.529
PID	1.09	80	5	56.2	−39.3	68.1	1.48	0.332	0.582
PID	1.33	20	10	164	−37.2	39.2	0.674	0.398	0.327

G and G_d were found by linearising a non-linear model at an operating point where $y_D = 0.99$, $x_B = 0.01$, $F = 1.0$ and $z_F = 0.5$ (Skogestad and Morari, 1987). W_d shows that we are expecting up to $0.3/1 = 30\%$ variation in F and $0.2/0.5 = 40\%$ in z_F. Similarly, W_s specifies the setpoint variations, $0.98 \leq y_{D_{sp}} \leq 1.00$ and $0.00 \leq x_{B_{sp}} \leq 0.02$. These weights reflect the relative importance of the external inputs, ie, we consider 30% variation in F to be comparable to a setpoint variation of 0.01 kmol/kmol. The noise at high frequency is allowed to be of magnitude 0.01. We see from the weight W_e that the allowed output error $y - y_s$ is $2/100 = 0.02$ at high frequencies and the required response time $\tau_{cle} = 20$ min. The factors 0.01 and 100 in the weights for W_s, W_n and W_e correspond to an output scaling and could, alternatively, have been accomplished by multiplying the elements in G and G_d by 100.

The optimal controller, $C_{\mu B}$, gives μ_{RP} value for this problem definition of 1.04, whereas the controller $C_{\mu new}$ (with input $y_s - y_m$), which is essentially tuned for setpoints only, gives a value of 1863 at high frequencies and 1.75 at low frequencies. The reason for the extremely high μ-value is that $C_{\mu new}$ is tuned without any direct penalty on manipulated inputs, while in the new formulation (Approach B) such a penalty (W_u) is included. Conversely, when applied to the original problem definition, $C_{\mu B}$, gives $\mu_{RP} = 1.18$, whereas $C_{\mu new}$ gives 0.978.

Recall the analysis of Eqn (17) where we analysed the relative importance of disturbance and setpoint tracking on performance. If in this example we look at the disturbance rejection from a scalar point of view, the performance time constants, τ_{cld} in Eqn (19), for the effect of the two disturbances in F and z_F on output 2 are about $2{\cdot}75/0.3{\cdot}58.6+1) = 8.0$ min and $2{\cdot}75/(0.2{\cdot}111.9+1) = 6.4$ min, respectively, whereas τ_{cle} for setpoints is 20 min. However, this does not take into account the direction of the disturbances. In our cases the disturbance condition number (Skogestad *et al*, 1988) for the two disturbances are 11.5 and 1.8, respectively, whereas the 'disturbance' condition number for the two setpoints are 111 and 89 (Skogestad and Morari, 1987). Thus, the disturbances are in the 'good' directions of the plant, and the bandwidth requirements imposed by the disturbances are not as hard as computed above. However, the disturbances do put tighter restrictions at lower frequencies (the 'slope two' requirement) than the setpoint requirement. This is also clear from the simulations discussed next.

In Fig 9 controller $C_{\mu new}$ and $C_{\mu B}$ are compared with respect to disturbance rejection. The disturbances are in F (+30%) at time $t = 0$ and in z_F (+40%) at $t = 50$ min. Solid curves show the response for controller $C_{\mu new}$ and dashed curves are for controller $C_{\mu B}$. We note that controller $C_{\mu new}$ gives a rather sluggish return to the setpoint. This dominant (low-frequency) part of the response is significantly improved with the controller $C_{\mu B}$. The controller, $C_{\mu new}$ for Approach A, could have been improved by using a performance weight, ω_P, with slope two at intermediate frequencies. Also, note that the disturbance in z_F is simpler to reject because it is almost exclusively in the 'good' direction.

Fig 10 shows the setpoint response with and without model error (Eqn (27)) for controller $C_{\mu B}$. We note

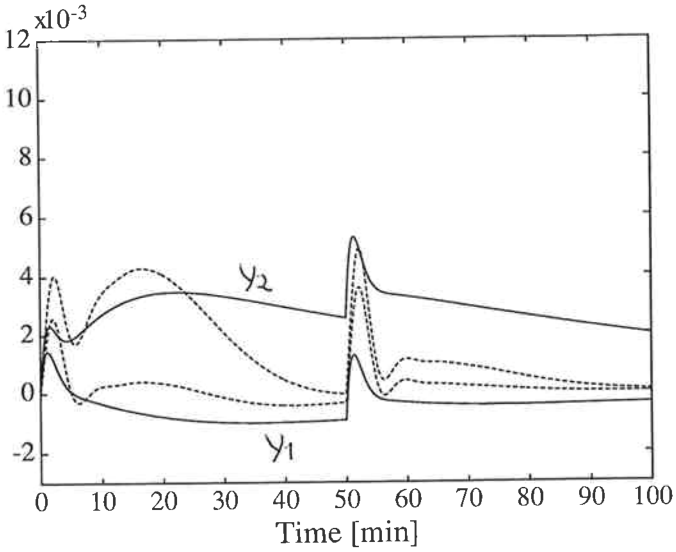

Fig 9 Simulation of a disturbance in F (+30%) at time $t = 0$ and in z_F (+40%) at $t = 50$ min using controller $C_{\mu new}$ (solid curves) and $C_{\mu B}$ (dashed curves). The input error in Eqn (27) is used in the simulations

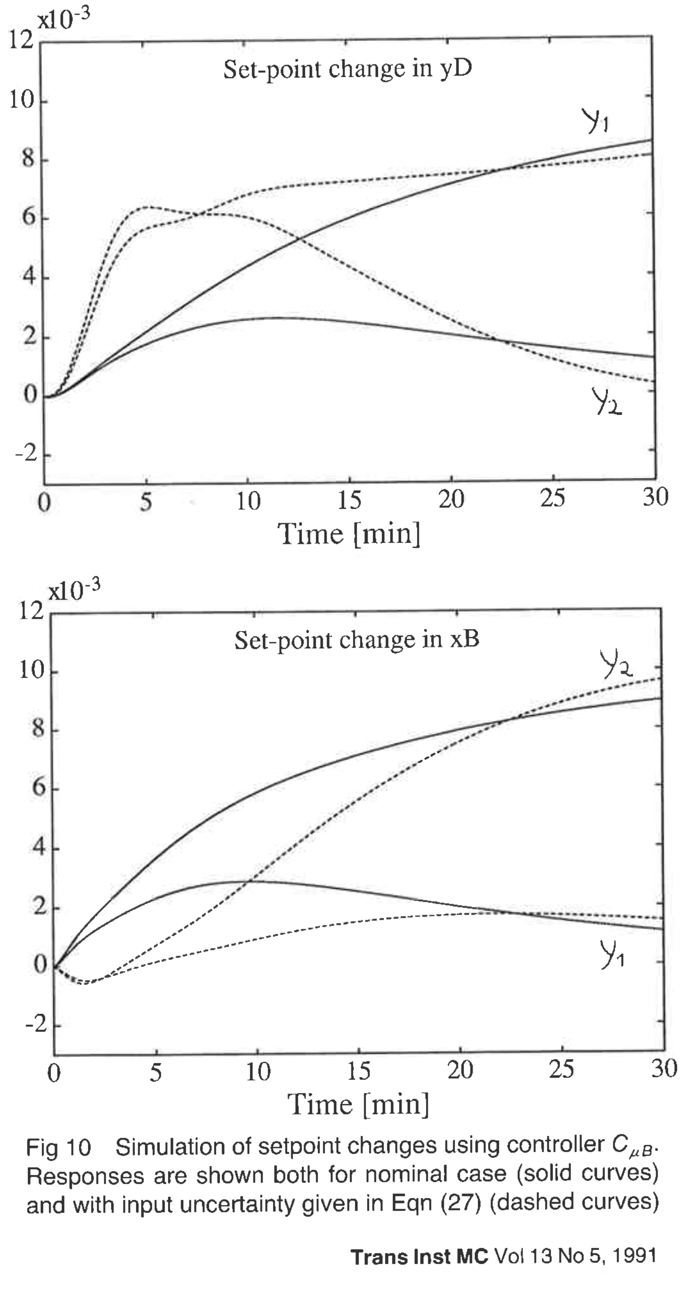

Fig 10 Simulation of setpoint changes using controller $C_{\mu B}$. Responses are shown both for nominal case (solid curves) and with input uncertainty given in Eqn (27) (dashed curves)

that in terms of setpoints the response is not better than with controller $C_{\mu new}$ (Fig 7).

6. Some comments on μ-synthesis

The μ-synthesis procedure employed today makes use of the upper bound of μ, trying to 'solve'

$$\min_{C,D} \| DN_{RP}(C)D^{-1} \|_\infty \quad \ldots(34)$$

The algorithm, often called 'D–K' iteration', is as follows:

1 Scale the original problem with a stable and minimum-phase transfer matrix D with appropriate structure;
2 Find a controller C by minimising the H-infinity norm of $DN_{RP}(C)D^{-1}$;
3 Compute $\mu(N_{RP}(C))$ and obtain at each frequency the optimal 'D-scales' from $\min_D \sigma(DN_{RP}D^{-1})$;
4 Fit the magnitude of each element of $D(\omega)$ to a stable and minimum-phase transfer function; and
5 Test a stop criterion. Stop or go to 1.

The major problem with μ-synthesis is that the D–K iteration is not guaranteed to find the global optimum of Eqn (34). Both step 2 and step 3 in this algorithm are convex optimisation problems, but this does not imply joint convexity for the whole algorithm (Doyle and Chu, 1985). A second problem is the difficulty in defining a stop criterion for the optimisation.

Good initial D-scales in step 1 of the algorithm, reduce the number of iterations, and may even, because of local minimas, affect the final minimum μ-value. For our example problem with the original problem definition, we observed that a natural physical scaling of the problem (using 'logarithmic compositions' as discussed by Skogestad and Morari, (1988)), that corresponds to multiplying all elements in $G(s)$ by a factor of 100, gave very good initial D-scales. With this simple scaling, the μ-value after the first iteration was 1.2, as compared to 14.9 without scaling. Another way to obtain good scalings is to start the algorithm at step 3 using a 'good' controller obtained by any design method.

The D–K iteration depends heavily on optimal solutions in step 2 and 3, and also on good fits in step 4. The H-infinity design (step 2) generally works fine using the μ-toolbox. The μ software does sometimes not compute a sufficiently tight upper bound of μ. Thereby the D-scales are not optimal, and the D–K iteration suffers. We have experienced cases in which, for some frequencies, the computed μ-value has been larger than the maximum singular value. When this occurs, the D–K iteration often starts diverging.

The last critical factor is the fitting of the D-scales. It is important to get a food fit, preferably by a transfer function of low order. The software for D-scale fitting in the μ-toolbox requires that the user specifies the order of the transfer function and decides if the fit is good enough. The optimal order of the transfer function D varies as the D–K iteration progress. It is sometimes better to increase the order, and sometimes the order should be decreased. If it is difficult to obtain a good fit, it often helps to use a different frequency range for the fit. It may also help to use a finer frequency grid.

The final problem is to determine when to stop the iteration. Two intuitive candidates for criterion for terminating the iteration are:

(1) An iteration criterion

$$\mu_{k-1} - \mu_k < \epsilon_1 \quad \ldots(35)$$

and

(2) A 'flatness' criterion

$$\max_\omega (\mu_{\text{peak}} - \mu(\omega)) < \epsilon_2 \quad \ldots(36)$$

(1) In Eqn (35) the subscript denotes the $k-1^{\text{th}}$ and the k^{th} iteration, respectively. This is a standard criterion, the iteration terminates if the objective function (μ) does not improve. There are two problems with this criterion. First, we may have found a local minimum, which means it is possible to improve μ by using a different D. Second, this criterion would terminate the iteration if μ increases. That may sound reasonable, but we have experienced situations in which μ increases for a number of iterations and then starts to decrease again. (2) Eqn (36) relies on the optimal controller giving a flat μ-versus-frequency plot. However, this is not always possible to obtain. For instance, the optimal solution to the problem in Skogestad *et al* (1988) does not give a flat μ-plot; instead, μ always goes to 0.5 at high frequencies (since S goes to I, and w_P goes to 0.5 I). As the number of iterations are increased one is able to extend the frequency where μ starts dropping down to 0.5, but the curve never becomes flat at all frequencies.

The results presented in this paper are obtained by terminating the D–K iteration when the H-infinity norm equals μ and μ is totally flat for frequencies less than 10 rad/min.

7. Conclusions

In this paper we have addressed performance weight selection when using the H-infinity norm. We have stressed the difference between an approach by which we try to shape directly a few important transfer functions such as S and T, which gives us the opportunity, for example, to specify directly minimum and maximum bandwidth requirements (Approach A), and an approach by which we consider the magnitude of signals (Approach B). The difference between the two performance approaches is probably most clear when one considers disturbance rejection; in this case the required bandwidth to achieve acceptable disturbance rejection is not clear (at least not for multivariable systems), and Approach B is preferable. An important advantage with Approach A, is that one may, to some extent, combine performance and robustness issues. For example, there may be bandwidth limitations related to robustness or the sampling time. With Approach B, robustness is generally handled by modelling the uncertainty explicitly and evaluating Robust Performance using the structure singular value. In practice, most engineers will probably use a combination of Approaches A and B when selecting weights, but when doing this it is important to realise the different ways of thinking that are involved.

Acknowledgements

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Appendix 1

State–space description of $C_{\mu new}(s) = C(sI-A)^{-1}B+D$. The controller has 22 states, two inputs and two outputs. The A matrix is given in tridiagonal form with the complex conjugate roots in real two-by-two form, ie, A is a bandmatrix with all non-zero elements on the main diagonal and the two adjacent diagonals. The D matrix is a zero 2-by-2 matrix.

A:

Row number	Diagonal below main	Main diagonal	Diagonal above main
1		−1.0000e−07	0
2	0	−1.0000e−07	0
3	0	−5.3681e−04	0
4	0	−6.8364e−04	0
5	0	−3.4883e−03	0
6	0	−5.5976e−02	0
7	0	−5.7017e−02	0
8	0	−2.0050e−01	0
9	0	−2.6267e−01	−1.1744e−01
10	1.1744e−01	−2.6267e−01	0
11	0	−4.8527e−01	0
12	0	−3.1117e+00	−6.9774e−01
13	6.9774e−01	−3.1117e+00	0
14	0	−1.9255e+01	0
15	0	−4.1007e+01	0
16	0	−1.1341e+02	0
17	0	−1.2966e+02	−8.7070e+01
18	8.7070e+01	−1.2966e+02	0
19	0	−1.3042e+02	−8.6556e+01
20	8.6556e+01	−1.3042e+02	0
21	0	−1.8112e+02	0
22	0	−6.3929e+05	

B and C^T:

B column one	B column two	C row one	C row two
8.5088e−01	1.0625e+00	9.6138e−01	−9.6366e−01
1.6792e+00	−1.3426e+00	1.5210e+00	1.5194e+00
2.6054e−02	−2.0838e−02	−2.3158e−02	−2.3122e−02
1.1099e−01	1.3877e−01	1.1712e−01	−1.1731e−01
−7.4077e−02	−9.2619e−02	7.4475e−02	−7.4592e−02
1.1255e+00	1.4072e+00	1.2294e+00	−1.2313e+00
6.1913e−01	−4.9518e−01	5.1127e−01	5.1046e−01
−1.4905e+00	−1.8635e+00	−1.6054e+00	1.6079e+00
−6.5002e+00	5.1989e+00	−5.6709e+00	−5.6620e+00
7.4959e+00	−5.9952e+00	−2.4034e+00	−2.3997e+00
−1.0500e+00	−1.3128e+00	−7.4675e−01	7.4792e−01
8.4252e−01	1.0534e+00	−6.4611e−01	6.4712e−01
−3.0556e+00	−3.8204e+00	−1.9699e−01	1.9730e−01
−8.0530e+01	6.4408e+01	5.4282e+01	5.4197e+01
4.1453e+01	−3.3154e+01	−3.3348e+01	−3.3296e+01
7.7061e+02	9.6349e+02	−1.7348e+02	1.7375e+02
−2.4073e+01	1.9254e+01	−1.8973e+02	−1.8944e+02
3.3649e+02	−2.6913e+02	2.4277e+01	2.4239e+01
−5.1808e+02	−6.4776e+02	−5.7626e+01	5.7717e+01
−3.8541e+02	−4.8188e+02	3.3056e+02	−3.3108e+02
1.3625e+03	1.7036e+03	2.8059e+02	−2.8103e+02
−1.2773e+04	−1.5970e+04	−1.4449e+04	1.4471e+04