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μ ANALYSIS AND SYNTHESIS OF TIME DELAY SYSTEMS USING SMITH PREDICTOR

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1. Introduction

The dynamic behavior of many industrial processes contain inherent time-delays. Time-delays not only make it difficult to achieve satisfactory performance but also greatly complicate the analytical and computational aspects of system design. A few authors have worked on robust control of time delay systems, e.g., Laughlin et al. (1987) who studied robust performance of the SISO Smith predictor within the IMC structure. In this paper, the robust performance of Smith predictor is addressed using the structured singular value (μ). By employing the properties of the Smith predictor, this time-delay design problem is converted to a corresponding delay-free problem. This conversion not only makes the analysis and synthesis easier, but also avoids problems due to rational time delay approximations.

2. Robust Performance of Time-Delay Systems Using Smith Predictor

In Fig. 1, $G_p(s) = \{\hat{g}_{ij}e^{-\hat{\theta}_{ij}s}\}$ is the actual plant, $G(s) = \{g_{ij}e^{-\theta_{ij}s}\}$ is the nominal model, and $G_0(s) = \{g_{ij}\}$ is the delay-free part of G. All these transfer function are assumed to be stable. The overall controller $K(s)$ includes the "primary" controller $K_0(s)$ for the time-delay free system and the Smith predictor, $G-G_0$. Besides the sensitivity matrix

$$S = (I + GK)^{-1} \quad (1)$$

we also introduce the delay-free sensitivity matrix

$$S_0 = (I + G_0K_0)^{-1} \quad (2)$$

Nominal Stability (NS)

The time delay control system with Smith predictor shown in Fig. 1 should be internally stable. Because the plant is assumed stable this is equivalent to the nominal stability of the corresponding delay-free control system, S_0.

Robust Stability (RS)

If nominal stability holds, then

1. Robust stability will be guaranteed for *additive* uncertainty bounded by $w_A(s)$ if and only if

$$\mu_{\Delta_A}(w_AK_0S_0) < 1, \quad \forall\omega \quad (3)$$

2. Robust stability will be guaranteed for *input* multiplicative uncertainty bounded by $w_I(s)$ if and only if

$$\mu_{\Delta_I}(w_IK_0S_0G) < 1, \quad \forall\omega \quad (4)$$

If we assume $G = G_0D, D = diag\{e^{-\theta_is}\}$, i.e. the time delays are on the *inputs* only, then (4) is equivalent to

$$\mu_{\Delta_I}(w_IK_0S_0G_0) < 1, \quad \forall\omega \quad (5)$$

3. Robust stability will be guaranteed for *output* multiplicative uncertainty bounded by $w_O(s)$ if and only if

$$\mu_{\Delta_O}(w_OGK_0S_0) < 1, \quad \forall\omega \quad (6)$$

If similarly we assume $G = DG_0, D = diag\{e^{-\theta_is}\}$, i.e. the time delays are on the *output* only, then (6) is equivalent to

$$\mu_{\Delta_O}(w_OG_0K_0S_0) < 1, \quad \forall\omega \quad (7)$$

Note that the RS-conditions (3), (5) and (7) for the system *with* time delays appear as conditions on the delay-free systems only. (5) and (7) follow from a property of μ (Doyle 1982) since D is a unitary matrix.

Nominal Performance (NP)

Nominal performance is here defined in terms of the weighted sensitivity matrix

$$NP \quad \Leftrightarrow \quad \bar{\sigma}(w_PS) < 1, \quad \forall\omega \quad (8)$$

However, we want to obtain a delay-free design procedure. We then have to define performance in terms of the corresponding delay-free system

$$NP_0 \quad \Leftrightarrow \quad \bar{\sigma}(w_{P0}S_0) < 1, \quad \forall\omega \quad (9)$$

Here the subscript 0 in w_{P0} is used to explicitly show this weight is on delay free sensitivity function.

Of course, NP is our real objective, and we will try to achieve this by satisfying NP_0. We need to obtain a reasonable weight w_{P0}. Ideally, we want to select w_{P0} such that NP and NP_0 are equivalent, that is, such that $\bar{\sigma}(w_{P0}S_0) = \bar{\sigma}(w_PS)$. However, this is not possible before we start the design, because S_0 and S are unknown at this point. Intuitively, we expect that we must use tighter performance specifications on S_0 than on S, that is, w_{P0} must be larger in magnitude than w_P. This is confirmed by considering the following equality

$$S = (I - GG_0^{-1}) + GG_0^{-1}S_0 \quad (10)$$

We want S small. The first term in the right side of Eq. (10) is an "unavoidable error", and the second term is a "delayed" error of the delay free system.

For the case with time-delays on the plant *outputs* only we have $G = DG_0$, where $D = diag\{e^{-\theta_is}\}$ and $GG_0^{-1} = D$. It then follows that

$$S = (I - D) + DS_0 \quad (11)$$

and

$$|\bar{\sigma}(S) - \bar{\sigma}(S_0)| \le \bar{\sigma}(I - D) \quad (12)$$

So for the case of *output* time-delays (and for SISO systems in general), the difference between $\bar{\sigma}(S)$ and $\bar{\sigma}(S_0)$ is relatively small, and by selecting

$$|w_{P0}^{-1}| \le |w_P^{-1}| - \bar{\sigma}(I - D) \quad (13)$$

NP is satisfied if NP$_0$ is satisfied (we may modify w_{P0}^{-1} at high frequencies to make it physically more reasonable).

For the general MIMO case Eq. (10) yields

$$\bar{\sigma}(S) \le \bar{\sigma}(I - GG_0^{-1}) + \bar{\sigma}(GG_0^{-1})\bar{\sigma}(S_0) \quad (14)$$

By choosing

$$|w_{P0}^{-1}| \le (|w_P^{-1}| - \bar{\sigma}(I - GG_0^{-1}))/\bar{\sigma}(GG_0^{-1}) \quad (15)$$

NP will be satisfied if NP_0 is satisfied. With this weight NP_0 is only a sufficient condition for NP, and it may result in a too tight performance weight if $\bar{\sigma}(GG_0^{-1})$ is large ($\gg 1$).

Robust Performance (RP)

Robust performance is here defined as

$$RP \quad \Leftrightarrow \quad \bar{\sigma}(w_P S_p) < 1, \quad \forall \omega, \quad \forall \Delta_A, \Delta_I \text{ or } \Delta_O \tag{16}$$

where S_p is the sensitivity function for the case with model error

$$S_p = (I + G_p K)^{-1} \tag{17}$$

The corresponding μ-test in terms of the system with time-delay (K, G and uncertainty) is

$$RP \quad \Leftrightarrow \quad \mu_{RP} = \sup_{\omega} \mu_{\Delta}(N) < 1 \tag{18}$$

where Δ is the block structure including both uncertainty and performance block, N can be derived from the standard '$N - \Delta$' structure and is different for different kind of uncertainty.

For the same reason as mentioned for nominal performance, we would like to have μ-tests on the delay-free system which guarantee RP. One approach is to first define RP_0 of the delay-free system

$$RP_0 \quad \Leftrightarrow \quad \bar{\sigma}(w_{P0} S_{p0}) < 1, \quad \forall \omega, \quad \forall \Delta_A, \Delta_I \text{ or } \Delta_O \tag{19}$$

where

$$\begin{aligned} S_{p0} &= S_0(I + w_A \Delta_A K_0 (I + G_0 K_0)^{-1})^{-1} \\ &= S_0(I + w_I G_0 \Delta_I K_0 (I + G_0 K_0)^{-1})^{-1} \\ &= S_0(I + w_O \Delta_O G_0 K_0 (I + G_0 K_0)^{-1})^{-1} \end{aligned} \tag{20}$$

We then get the following equivalent μ-condition

$$RP_0 \quad \Leftrightarrow \quad \mu_{RP_0} = \sup_{\omega} \mu_{\Delta}(N_0) < 1, \tag{21}$$

where N_0 is in terms of K_0, G_0 and uncertainty.

Unfortunately, RP_0 is not in general equivalent to RP except for SISO systems even for cases when NP and NP_0 are equivalent. However, we can still specify RP_0 to satisfy RP. We have for additive uncertainty

$$\begin{aligned} S_p &= S(I + (G_p - G)K(I + GK)^{-1})^{-1} \\ &= S(I + w_A \Delta_A K_0 (I + G_0 K_0)^{-1})^{-1} \\ &= (I - GG_0^{-1})(I + w_A \Delta_A K_0 (I + G_0 K_0)^{-1})^{-1} + GG_0^{-1} S_{p0} \end{aligned} \tag{22}$$

The last equation is the same as Eq. (10) except that the first term on the right hand side is changed by a factor $(I + w_A \Delta_A K_0 (I + G_0 K_0)^{-1})^{-1}$. By choosing

$$|w_{P0}^{-1}| \leq (|w_P^{-1}| - \bar{\sigma}(I - GG_0^{-1}) \bar{\sigma}((I + w_A \Delta_A K_0 (I + G_0 K_0)^{-1})^{-1})) / \bar{\sigma}(GG_0^{-1}) \tag{23}$$

RP will be satisfied if RP_0 is satisfied. Comparing (23) with (15), we see that it requires a tighter performance weight w_{P0} to satisfy RP than to satisfy NP, since $\bar{\sigma}((I + w_A \Delta_A K_0 (I + G_0 K_0)^{-1})^{-1})$ is always larger than 1. Although $\bar{\sigma}((I + w_A \Delta_A K_0 (I + G_0 K_0)^{-1})^{-1})$ involves the controller K_0, it is generally independent of K_0 at low frequencies (for example, a controller with integrator), so we may be able to select a reasonable w_{P0} before we start the design.

If $\bar{\sigma}((I + w_A \Delta_A K_0 (I + G_0 K_0)^{-1})^{-1})$ is not too large (compared to 1), or equivalently if $\underline{\sigma}(I + w_A \Delta_A K_0 (I + G_0 K_0)^{-1})$ is not too small (compared to 1), then the "unavoidable error" will not be much different from the nominal case. $\underline{\sigma}(I + w_A \Delta_A K_0 (I + G_0 K_0)^{-1})$ is indeed not small in SISO systems, but it may be very small in MIMO systems, particularly when Δ is structured. However, in practice our design approach in terms of RP_0 may work well even for general MIMO systems. This follows since large $\underline{\sigma}(I + w_A \Delta_A K_0 (I + G_0 K_0)^{-1})$ is sufficient but not necessary. Similar results exist for *input* multiplicative uncertainty with *input* time-delay and *output* multiplicative uncertainty with *output* delay.

3. Design Example

Here we consider the same SISO example as Laughlin et al. (1987). The plant model is

$$G(s) = \frac{k}{\tau s + 1} e^{-\theta s}, \quad G_0(s) = \frac{k}{\tau s + 1} \tag{24}$$

The nominal values of k, τ and θ are all equal to 1. We use the same multiplicative uncertainty weight $w_I(s)$ as Laughlin et al. which was derived by considering 50% uncertainty in the parameters k, τ and θ.

$$w_I(s) = 1.5 \left(\frac{s + 1}{0.5s + 1}\right) \left(\frac{1 - 0.25s}{1 + 0.25s}\right) - 1 \tag{25}$$

We use the following performance weights

$$w_P(s) = \frac{1}{2} \frac{3.397s + 1}{3.397s}, \quad w_{P0}(s) = \frac{1}{1.5} \frac{3.2s + 1}{3.2s} \tag{26}$$

The weight $w_P(s)$ is also from Laughlin et al. (1987), whereas the performance weight $w_{P0}(s)$ on the delay free sensitivity function S_0 was chosen to match the equality in (23) at low frequencies. Note that the allowed peak on the sensitivity function is 2 for S and 1.5 for S_0.

For the time-delay free process with uncertainty we obtained (using "$D - K$ iteration" and order reduction) a 4-order μ-optimal primary controller $K_{0\mu}$ with peak value $\mu_{RP_0} = 0.9679$. With this controller the corresponding robust performance for the real system with time delay is $\mu_{RP} = 0.9965$. On the other hand, the μ_{RP} values of the "μ-optimal controller" and the Smith predictor controller given by Laughlin et al. (1987) are about 1.08 and 1.1 (observed from their plot), respectively. Clearly, the μ_{RP} value of the μ-optimal controller should be less than ours (0.9965) which was designed based on the delay-free system. The reason for their higher value may be that the software they used is not so good. Indeed, using the newer μ software (Balas et al., 1991), we are able to get a 14'th order μ-optimal controller, K_μ with $\mu_{RP} = 0.9771$.

In order to synthesize the μ-optimal controller, one need to approximate the time-delay by a rational function. Laughlin et al. used a fourth order Pade approximation, and so did we. Generally, one would believe that a fourth-order Pade approximation should be sufficiently accurate, but when we calculate the μ-curves with the μ-optimal controller, K_μ, applied to the plant with the real time-delay, we find that te μ-plot contains several large peaks which locations are identical with the peaks of Pade approximation error. The bandwidth of our system is about 0.2, while the fourth Pade approximation is quite accurate for frequencies less than 5. However, if we consider the μ-plot, we find that μ for robust performance is very flat up to frequency of about 100; this requires the approximation to be good at least up to frequency 100. This means that robust performance problem may put much more severe restriction on approximation of time-delay. To overcome it, one may add a filter to the μ-optimal controller or combine the approximation error explicitly into the model uncertainty. One simple methods is to let the uncertainty weight approach infinity at high frequency (whereas $w_I(s)$ levels off at 4). Since the method proposed in this paper is completely delay-free, it does not suffer from those problems. This is another advantage of our method.

Reference

Laughlin, D. L., D. E. Rivera and M. Morari (1987), "Smith predictor design for robust performance," *Int. J. Control*, 46(2), 477-504

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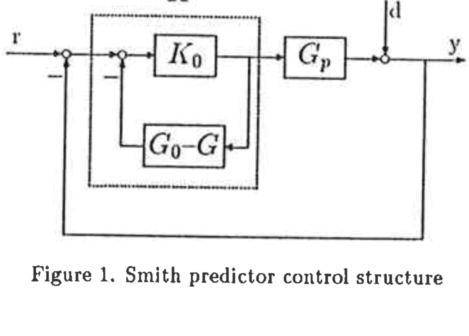

Figure 1. Smith predictor control structure