Presented at 30th CDC, Dec. 11-13 1991, Brighton, UK

UNCERTAINTY WEIGHT SELECTION FOR H-INFINITY AND MU-CONTROL METHODS

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Abstract

Design and analysis of robust control systems by use of H_∞-methods (*i.e.*, using the structured singular value, μ) requires frequency dependent weights to define performance and uncertainty. The purpose of this paper is to give some insight into H_∞-weight selection from a process control perspective. In an other paper [8] we studied the performance specifications, but in this paper we concentrate on uncertainty modelling. We study how to transform parametric gain-delay uncertainties into frequency dependent weights specifying norm bounded uncertainties.

1 Introduction

The control problem studied in this paper is based on the ill-conditioned plant presented by Skogestad, Morari and Doyle [10]. We use the same simplified distillation column (the LV-configuration) as our example:

$$G(s) = \frac{1}{75s+1}\begin{bmatrix} 0.878 & -0.864 \\ 1.082 & -1.096 \end{bmatrix} \qquad (1)$$

This is a very crude model of a distillation column and does not describe an actual column very well. However, it is an excellent example for demonstrating the problems with ill-conditioned plants.

Skogestad *et al.* [10] show that this type of plant is very sensitive to input uncertainty. This demonstrates that any controller design method has to take model uncertainty into account. The structured singular value, μ, introduced by Doyle [2], allows us to include structured norm bounded perturbations (uncertainties) in the H_∞-framework. Skogestad *et al.* [10] use μ and a synthesis method called "D-K" iteration [4] to synthesize a "μ-optimal" controller which yield a feedback system not sensitive to the uncertainty.

Freudenberg [6] and Yaniv and Barlev [12] also use this distillation model to demonstrate design methods for robust control of ill-conditioned plants. Freudenberg uses a controller based on the singular value decomposition of the plant, while Yaniv and Barlev use the quantitative feedback theory (QFT), by Horowitz [11], to design a decentralized two degree of freedom controller.

The uncertainty and performance specifications in [10], [6] and [12] are slightly different. [10] and [6] use norm bounded uncertainties and a performance requirement for the maximum singular value of the sensitivity function for the worst case uncertainty. [12] use parametric gain and delay uncertainty, and specify the performance by of use magnitude bounds on the *elements* of the transfer function from set points to controlled outputs.

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2 A general framework for uncertainty modelling

Linear Fractional Transformations (LFT) provide a general framework for modelling norm bounded perturbations [3]. An LFT may be written on the following form (see Fig. 1)

$$z = F_u(P,\Delta)w = [P_{22} + P_{21}\Delta(I - P_{11}\Delta)^{-1}P_{12}]w \qquad (2)$$

P_{22} is the nominal mapping from w to z and Δ is the H_∞-norm bounded perturbation; $||\Delta||_\infty = \sup_\omega \bar{\sigma}(\Delta(j\omega)) \le 1$.

This uncertainty description has mainly been used for "unstructured" [1] uncertainty where the block-diagonal elements of Δ are complex. An example of such an uncertainty description is additive uncertainty on the whole plant matrix or additive element-by element uncertainty.

However, also "structured" parametric uncertainty may be written within the LFT framework. In this case let the real variable δ_i denote a physical parameter variation, and let $|\delta_i| \le 1$. For example, a state space model with uncertain (and possibly correlated) coefficients

$$G_p(s) = (C + \Sigma\delta_i C_i)(sI - A - \Sigma\delta_i A_i)^{-1}(B + \Sigma\delta_i B_i) + (D + \Sigma\delta_i D_i) \qquad (3)$$

(the subscript p denotes perturbed, *i.e.* with uncertainty) may be rewritten on the general LFT-form with Δ a diagonal matrix with δ_i's (possibly repeated) along its diagonal ([9] pp.8–11). Here the matrices A_i, B_i, C_i and D_i reflect how the i'th uncertainty δ_i affects the state space model.

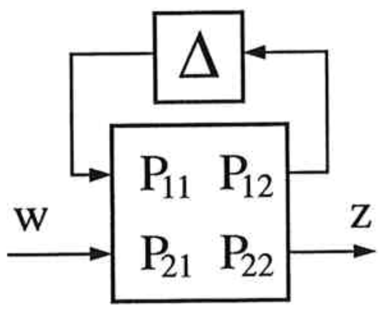

Figure 1. Linear Fractional Transformation.

[1] By "unstructured" uncertainty we mean that several separate uncertainty blocks have been combined into one "full" complex Δ-block.

A different example is from a study of a reactor, where we found that one of the elements in the B-matrix depended on the operating point in the following manner

$$b(\delta) = \frac{1.5 + 0.1\delta}{0.5 + 0.1\delta} \; ; \; -1 \leq \delta \leq 1. \quad (4)$$

This nonlinear dependency on the parameter δ may be written on the LFT-form , $b(\delta) = F_u(P, \delta)$ (Fig. 1), with $P = \begin{pmatrix} -0.2 & 1 \\ -0.4 & 3 \end{pmatrix}$.

In that same reactor example the parameter δ also appeared in the A-matrix

$$A = \begin{pmatrix} -1.5 + 0.1\delta & 0 \\ 1 & -1.5 + 0.1\delta \end{pmatrix} \quad (5)$$

To write this uncertainty as an LFT we may use

$$A = \begin{pmatrix} -1.5 & 0 \\ 1 & -1.5 \end{pmatrix} + 0.1 \begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix} \quad (6)$$

In this case we need two repeated scalar δ's (the rank of the matrix A_i is two), or three repeated δ's if also the variation in the B-matrix is included.

One example of an uncertainty which cannot be written on LFT-form is time delay uncertainty, and we we shall consider later some useful approximations.

We prefer LFT models where the P_{ij}'s are proper rational transfer functions and Δ is complex. This preference is of computational reasons: 1) the H_∞-synthesis in the "D-K" iteration uses a state space model of the problem, and 2) At present the algorithms to compute μ can not deal with combined real and complex perturbations, which would be needed to analyze robust performance with real parametric uncertainty. Therefore, at least at present, most of the general representations of parametric uncertainty, cannot be used in practice.

3 Performance specifications

If w in Fig. 1 represents normalized external inputs, z represents normalized errors, then a general H_∞-performance specification becomes for the nominal case (NP)

$$||P_{22}||_\infty = \sup_\omega \bar{\sigma}(P_{22}(j\omega)) < 1 \quad (7)$$

and for robust performance (RP)

$$||F_u(P, \Delta)||_\infty < 1 \; , \; \forall \Delta \quad (8)$$

The performance and uncertainty weights are included in P. As shown by Doyle [2] a computationally useful condition for robust performance may be written in terms of the structured singular value

$$\text{RP} \quad \text{iff} \quad \mu_{\tilde{\Delta}}(N_{RP}) < 1 \; , \; \forall \omega \quad (9)$$

where $N_{RP} = P$ and $\tilde{\Delta} = \text{diag}(\Delta, \Delta_P)$. The peak μ-value as a function of frequency is denoted μ_{RP}. Δ_P is a "full" complex perturbation matrix which stems from the performance requirement of wanting the singular value from w to z less than 1.

This is of course a very general framework. In practice, performance specifications are based on engineering judgement. In another paper [8] we discuss two different approaches: 1) Based on specifying bounds on important transfer functions such as the sensitivity, S, or the complementary sensitivity, T; 2) Based on considering signals as a function of frequency, and specify bounds on their amplification through the system. The latter approach is more general, but one often looses the direct handle on specifications such as bandwidth and maximum peak.

4 The original problem from [10]

Problem definition

Skogestad *et al.* [10] consider as a performance specification a simple bound on $S_p = (I + G(I + \Delta_I W_I)C)^{-1}$; the sensitivity function for the worst case input uncertainty.

$$RP \Leftrightarrow ||W_e S_p||_\infty < 1 \; , \; \forall \Delta_I \quad (10)$$

where the nominal model $G(s)$ is given in Eq.1, and

$$W_I(s) = \begin{bmatrix} w_I(s) & 0 \\ 0 & w_I(s) \end{bmatrix} \; ; \; w_I(s) = 0.2\frac{(5s+1)}{(0.5s+1)} \quad (11)$$

$$\Delta_I(s) = \begin{bmatrix} \delta_1(s) & 0 \\ 0 & \delta_2(s) \end{bmatrix} \; ; \; |\delta_i(j\omega)| \leq 1, \; \forall \omega \quad (12)$$

and

$$W_e(s) = \begin{bmatrix} w_e(s) & 0 \\ 0 & w_e(s) \end{bmatrix} \; ; \; w_e(s) = 0.5\frac{(10s+1)}{10s}. \quad (13)$$

w_I is a bound on multiplicative (relative) input uncertainty. Eq.11 shows that the uncertainty is 20% at low frequencies and reaches 1 at a frequency of approximately 1 rad/min. Note that the corresponding uncertainty matrix, Δ_I, is a diagonal matrix since it is assumed that uncertainty does not "spread" from one channel to another (for example, a large input signal in channel 1 does not affect the signal in channel 2). The performance weight in Eq.13 requires integral action, a bandwidth of approximately 0.05 rad/min and a maximum peak for $\bar{\sigma}(S_p)$ of 2.

N_{RP} in the Robust Performance μ-condition (Eq.9) for this problem becomes:

$$N_{RP} = \begin{bmatrix} -W_I CSG & W_I CS \\ W_e SG & -W_e S \end{bmatrix}; \; \tilde{\Delta} = \begin{bmatrix} \begin{bmatrix} \delta_1 & \\ & \delta_2 \end{bmatrix} & \\ & \Delta_P \end{bmatrix} \quad (14)$$

Controller designs

Skogestad *et al.* [10] use "D-K" iteration [4] based on "the 1984-approach" [3] H_∞-minimization to design a "μ-optimal" controller. Their controller has six states and gives μ_{RP}=1.067 for both structured and unstructured Δ_I. Lundström *et al.* [8] use the new MATLAB μ-toolbox [1], based on the state-space H_∞-solution by Doyle *et al.* [5], to synthesize a better "μ-optimal"[2] controller (denoted $C_{\mu new}$) giving $\mu_{RP} = 0.978$. This controller has 22 states. Freudenberg [6] use another design method and achieve a controller with five states giving μ_{RP}=1.054 for unstructured Δ_I. Yaniv and Barlev [12] do not present a μ value for their design. We obtained $\mu_{RP} = 2.28$ for their controller[3] (without their prefilter) applied to the original formulation in [10].

Comments on the uncertainty weight

The uncertainty formulation by Skogestad *et al.* [10] is a complex *norm bounded* uncertainty. In each channel they allow the following input model

$$g_{Ip}(j\omega) = \left(1 + \delta(j\omega)\left|0.2\frac{5j\omega+1}{0.5j\omega+1}\right|\right) \; , \; |\delta(j\omega)| \leq 1 \; \forall\omega \quad (15)$$

(nominally $g_I = 1$). Yaniv and Barlev [12] use *parametric* uncertainty in their design specifications

$$g_{Ip}(j\omega) = ke^{-\theta j\omega} \; ; \; k \in [0.8 \; 1.2] \; ; \; \theta \in [0 \; 1] \quad (16)$$

These two sets are not identical for two reasons: 1) The uncertainty defined by Eq.15 is more general as it allows all kind of transfer functions $g_{Ip}(s)$ as long as their norm is bounded by the

[2] We will use "μ-optimal" to denote the best *obtained* solution using the "D-K" iteration in MATLAB μ-toolbox.

[3] There is a misprint in [12]; the second order pole in controller "g1" of 0.22 should be with damping 0.5.

weight. For example, the following input model is allowed by Eq.15

$$g_{Ip}(s) = \begin{pmatrix} \frac{-0.5s+1.2}{0.5s+1} & 0 \\ 0 & \frac{-0.5s+0.8}{0.5s+1} \end{pmatrix} \tag{17}$$

2) On the other hand, the uncertainty allowed by Eq.16 is *not* quite covered by the uncertainty weight in Eq.15, although one might get this impression from the paper of Skogestad *et al.* [10]. Eq.15 does cover gain uncertainty of $\pm 20\%$. It also covers *approximately* a 1 min. delay (a first order Padé approximation is covered). However, it does not cover all *combinations* of these gain and delay uncertainties, especially not at high frequencies.

The two sets defined in Eq.15 and 16 are compared in Fig. 2. At $\omega = 1$ rad/min most of the parametric uncertainty is covered by the norm bounded set, but the corner corresponding to +20% and 1 min delay is not covered. This turns out to be critical, since this often is the "worst case" uncertainty. As $\omega \to \infty$ there is a region corresponding to positive gain error and phase error of about -180^o which is not covered.

Comments on two-degree of freedom controller

The problem specifications as given in [10] is on the sensitivity function $S = (I + GC)^{-1}$. Here C denotes the feedback part of the controller. Thus, a two-degree of freedom controller will not improve the design. However, if one interprets the performance specification from a *signal* point of view, that is, considers the response from y_s (reference signals) to $e = y - y_s$ (errors), then one may get improvement by a two-degree of freedom controller. Specifically, if C_f denotes the part of the controller which filters the reference signals, then $e = -SC_f y_s$. Yaniv and Barlev considered the problem from this signal point of view and designed a two-degree of freedom controller with a prefilter

$$C_f = \frac{1}{25s^2 + 5s + 1} I_{2\times 2}. \tag{18}$$

The time response is improved by adding this prefilter, but the value of μ_{RP} increases from 2.28 to 2.33 because the response gets slower (we show time domain simulations towards the end of this paper). We also considered this signal performance specification [8], but the improvement in performance by using a two degree of freedom controller was rather small; we were able to reduce μ_{RP} from 0.978 to 0.926. However, we then have no direct handle on the sensitivity function S (its H_∞-norm increases from 2 to about 3.5), and the robustness in terms of robust stability is worse.

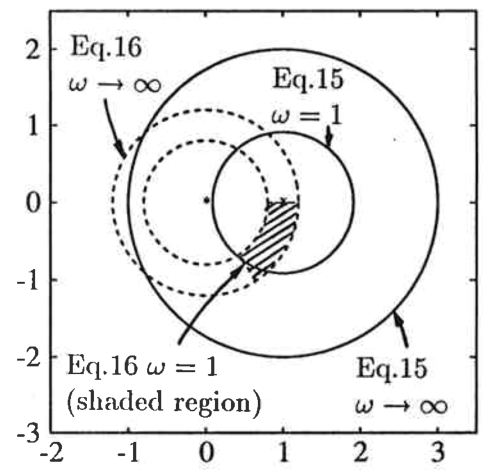

Figure 2. Representation in the complex plane of the uncertainty set in Eq.15 (solid discs) and Eq.16 (region between dashed circles).

Comments on the performance weight

The original problem formulation [10] was intentionally made very simple. For instance, the only performance requirement at high frequencies is to keep the magnitude of the sensitivity function below 2. This leads to an optimal controller with a rather high gain at high frequencies. A better problem formulation should include some penalty which forces the controller gain to roll off at high frequencies. We can achieve this, for example, by including a weight on the manipulated inputs, which will limit the transfer function CS, and/or including measurement noise, which will limit the complementary sensitivity function T. The uncertainty weight could also be used with similar effect, since large uncertainty at high frequencies will force the *input* complementary sensitivity function ($T_I = CG(I + CG)^{-1}$) to roll off.

5 New uncertainty weights

In the following we will assume that the true input model in each channel is given by the parametric uncertainty of Yaniv and Barlev (Eq.16). In Fig. 2 we showed that the norm bounded set of Skogestad *et al.* (Eq.15) does not quite cover this uncertainty. Here we will derive norm bounded sets which do cover the gain and delay uncertainty.

5.1 Real perturbations

Exact description: We may describe the process exactly by

$$g_{Ip}(s) = (\bar{k} + \delta_k \epsilon_k) e^{-(\bar{\theta} + \delta_\theta \epsilon_\theta)s} \tag{19}$$

where $\bar{k}$ and $\bar{\theta}$ are the average parameter values and δ_k and δ_θ are *real* scalars, $-1 \leq \delta \leq 1$. For the uncertainty in Eq.16 we get: $\bar{k} = 1$, $\bar{\theta} = 0.5$, $\epsilon_k = 0.2$ and $\epsilon_\theta = 0.5$. This uncertainty description has two problems: 1) The time delay cannot be modelled by an LFT, and 2) The perturbations are real.

Real perturbations and first order Padé: In order to achieve a rational transfer functions we use a first order Padé approximation of the time delay uncertainty.

$$e^{-\theta s} = e^{-\bar{\theta} s} e^{-\delta_\theta \epsilon_\theta s} \approx e^{-\bar{\theta} s} \left(\frac{1 - \delta_\theta \frac{\epsilon_\theta}{2} s}{1 + \delta_\theta \frac{\epsilon_\theta}{2} s} \right) \tag{20}$$

The time delay uncertainty may now be rearranged as an LFT.

$$\left(\frac{1 - \delta_\theta \frac{\epsilon_\theta}{2} s}{1 + \delta_\theta \frac{\epsilon_\theta}{2} s} \right) = \left(1 - \delta_\theta \epsilon_\theta s \left(1 + \delta_\theta \frac{\epsilon_\theta}{2} s \right)^{-1} \right) \tag{21}$$

Which gives $P = \begin{bmatrix} -\epsilon_\theta s/2 & 1 \\ -\epsilon_\theta s & 1 \end{bmatrix}$ in Eq.2. Problems with this description are: 1) The perturbations are real, and 2) The time delay is approximated.

For this approximation we cannot simply relax the requirement that δ_θ has to be real and allow it to be complex; a simple analysis shows that with $\delta_\theta = j$ this would imply that the norm of $e^{-\theta s}$ would be infinity at the frequency $\omega = 2/\epsilon_\theta$.

Real perturbations and n^{th} order approximation: We may achieve an arbitrary good approximation of the time delay by the following n'th order approximation.

$$e^{-\theta s} \approx e^{-\bar{\theta} s} \left(\frac{1 - \delta_\theta \frac{\epsilon_\theta}{2n} s}{1 + \delta_\theta \frac{\epsilon_\theta}{2n} s} \right)^n \tag{22}$$

This may also be written on the general LFT-form, but since all δ_θ's are equal, we need *repeated* perturbations. Thus, the problems with this description are: 1) The perturbations are real, 2) Repeated real δ_θ's are needed, and 3) High order model.

5.2 Complex perturbations

Here we will consider modelling the parametric uncertainty in each input channel (the perturbations in k and θ) by a *single* complex scalar perturbation δ. That is, we use a structured approach with respect to the channels, but an "unstructured" (lumped) approach with respect to the parametric uncertainties in each channel.

We use a slightly generalized uncertainty description compared to the input uncertainty considered in Section 4:

$$G_p(s) = G_c(s)(G_I(s) + \Delta_I(s)W_I(s)) \tag{23}$$

(subscript c for column model and I for input model; above we used $G_c = G, G_I = I$). This is simply an additive uncertainty description for the input model. We assume the same nominal model and same uncertainty in both inputs, *i.e.*, $W_I = w_I I$ and $G_I = g_I I$. Δ_I is given in Eq.12. At each frequency the true plant is allowed to be within a disc in the complex plane which has the nominal input model $g_I(j\omega)$ as a center and a radius equal to $|w_I(j\omega)|$. With this uncertainty description it is less conservative to approximate the real uncertainties in k and θ in each channel by one complex perturbation than by two separate complex perturbations.

To generate the uncertainty description we use the following procedure: i) Define the nominal "center" model, ii) Determine the radius w_I at each frequency such that the complex region generated by the real parameter variations is covered, iii) Approximate this radius by a transfer function (the approximation should always be "conservative", *i.e.*, such that all allowed plants are included). We will consider three different choices of the nominal input model.

1) Select $g_{I1} = 1$, *i.e.* the center point is fixed at $(1+j0)$ for all frequencies. This is the choice made by Skogestad *et al.* [10], but as mentioned above their weight $w_I(s)$ does not include the whole set generated by the uncertainty in Eq.16. To derive a weight which covers the set we did as follows: a) At low frequencies ($\omega < \pi/\theta_{max}$) the point furthest away from g_{I1} corresponds to a gain of $(1+\epsilon_k)$ and a time delay of θ_{max}, b) Using a second order Padé for the time delay then gives

$$w'_{I1}(s) = 1 - (1+\epsilon_k)\frac{\theta_{max}^2 s^2 - 6\theta_{max}s + 12}{\theta_{max}^2 s^2 + 6\theta_{max}s + 12} \tag{24}$$

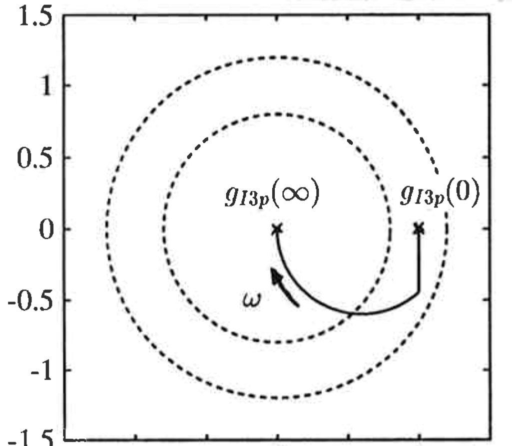

Figure 3. Trajectory of the "tight" model $g_{I3}(j\omega)$ (center in the smallest set) in the complex plane. The region between the dashed circles shows the parametric set Eq.16 as $\omega \to \infty$.

Since we are interested in only the magnitude of w_I this may be replaced by

$$w''_{I1}(s) = \frac{\epsilon_k\theta_{max}^2 s^2 + 6(2+\epsilon_k)\theta_{max}s + 12\epsilon_k}{\theta_{max}^2 s^2 + 6\theta_{max}s + 12} \tag{25}$$

c) The magnitude of this weight is too small at high frequencies. To compensate for this we multiply the weight by the factor $\frac{\tau s+1}{(\tau\epsilon_k/(2+\epsilon_k))s+1}$ where τ is found for each specific case (in our case $\tau = 0.167$). The final weight for our case is shown in Table 1.

2) Select $g_{I2} = \bar{k}e^{-\bar{\theta}j\omega}$. This is the average of the parametric set and was chosen by Laughlin *et al.* [7] who studied transformations from parametric uncertainty to a norm bounded complex set. We used a procedure similar to the one outlined above to approximate the radius.

3) Select the nominal model g_{I3} such that the radius of the uncertainty set at each frequency is minimized. At a given frequency g_{I3} is the "average" in the complex plane and will be denoted the "tight" model in the following. The trajectory of the "tight" $g_{I3}(j\omega)$ is shown in Fig. 3. Unfortunately, this $g_{I3}(j\omega)$ is not a rational transfer function, so we cannot use it and the corresponding uncertainty weight in the "D-K" algorithm. However, for a given controller we may compute μ for this uncertainty set, since the μ computations are performed on a frequency-by-frequency basis.

The three different approximations are illustrated in Fig. 4. Set g_{I1p} includes prediction, since it covers (at low frequencies) a region with positive imaginary part. g_{I2p} and g_{I3p} are similar at low frequencies, but differs at high. g_{I3p} is always the smallest set, and g_{I2p} is at most frequencies a smaller set than g_{I1p}. However, the smallest set may not necessarily yield the best design. The reason is that the smallest set, g_{I3p}, is not always contained in g_{I1p} (or g_{I2p}), *i.e.*, it may include plants which are outside the original set, and which are not covered by g_{I1p} (or g_{I2p}). If these plants then are the "worst case" plants, then approach 3 may in fact be more conservative than approach 1 (at least at some frequencies).

Table 1. New uncertainty models ($G_{Ip} = G_I + \Delta_I W_I$).

Set	G_I	W_I
1	$I_{2\times2}$	$\frac{0.2s^2+13.2s+2.4}{s^2+6s+12}\frac{0.167s+1}{(0.167/11)s+1}I_{2\times2}$
2	$\frac{0.5^2s^2-6*0.5s+12}{0.5^2s^2+6*0.5s+12}I_{2\times2}$	$\frac{0.05s^2+6.6s+2.4}{0.25s^2+3s+12}\frac{(0.167/2)s+1}{(0.167/22)s+1}I_{2\times2}$

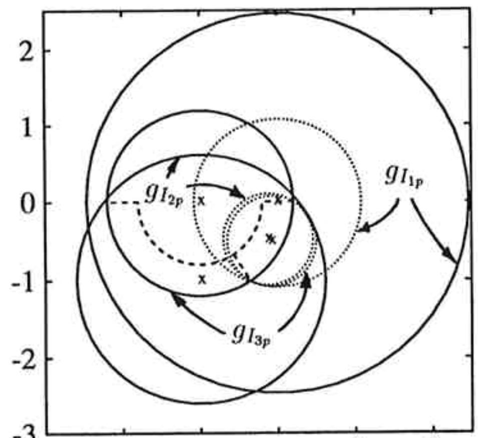

Figure 4. The three sets g_{I1p}, g_{I2p} and g_{I3p} (Table 1) at $\theta_{max}\omega = 1$ rad (dotted discs) and at $\theta_{max}\omega = \pi$ rad (solid discs).

6 μ-analysis

To study in more detail the uncertainty descriptions mentioned above, we designed μ-optimal controllers for the example problem in [10], but used the different G_I and W_I from Table 1 in the uncertainty description. It turns out that controller C_1 (22 states) designed for the larger set, g_{Ip1}, gives a lower μ_{RP} than controller C_2 (22 states) tuned for the smaller set, g_{Ip2}. This demonstrates that a smaller set not necessarily gives the best design.

We also compare controllers C_1 and C_2 with the optimal controller from [8], $C_{\mu new}$, and the controller from [12], which we denote C_{QFT}. The results are summarized in Table 2. Two μ-values are shown for each design: The first column shows the μ_{RP}-value for the uncertainty set used in the design of the controller. The second column shows the μ_{RP}-value for the "tight" complex uncertainty description, g_{I3p} (computed frequency-by-frequency without any need to approximate g_{I3}). Fig. 5 shows $\mu(N_{RP})$ as a function of frequency for the "tight" uncertainty for all four controllers (solid curves), and for controller $C_{\mu new}$ and the original uncertainty (dashed curve).

Controller $C_{\mu new}$ is designed for the original uncertainty set, and μ_{RP} for this controller increases when the "tight" set is used. This demonstrates that the original set does not include all worst case plants in the "tight" set. For example, +20% gain error and 1 min delay is not included. However, in the frequency range $0.1 < \omega < 1$, the "tight" set gives lower $\mu(N_{RP})$ than the original uncertainty , showing that the original uncertainty allows "worst case" plants outside the "tight" set (and thereby also outside the parametric set) at these frequencies.

C_{QFT} is designed for the parametric uncertainty (Eq.16). We cannot compute μ_{RP} exactly for this uncertainty (it would require real δ's), however, this controller gives a high μ_{RP}-value for the "tight" set. Controllers C_1 and C_2 are designed for uncertainty sets which cover the parametric uncertainty and μ_{RP} for these two controllers does not change very much when the uncertainty is changed to "tight" uncertainty.

The differencies between the original uncertainty set and the parametric set is also illustrated in Fig. 6. Here the worst case plants for controller $C_{\mu new}$ applied to the original uncertainty set are plotted in the complex plane. The worst case plants are computed for $0.001 < \omega < 100$ rad/min. Each "*" mark the worst plant in channel 1 at one specific frequency. The worst plants in channel 2 are marked with "o". At very low frequencies the worst plant is 0.8[4] in both channels. This plant is also allowed by the parametric uncertainty. As the frequency increases, we see that some of the worst case plants for the original set lie outside the parametric set, *i.e.*, at these frequencies the original set is conservative.

Table 2. Optimal μ_{RP}-values for the design uncertainty and for the "tight" complex uncertainty for four different controllers.

Controller	design	$\mu_{RP\ design}$	$\mu_{RP\ tight}$
$C_{\mu new}$	Eq.15	0.978	1.30
C_{QFT}	Eq.16	–	2.36
C_1	g_{I1p} (Table 1)	1.04	1.05
C_2	g_{I2p} (Table 1)	1.12	1.12

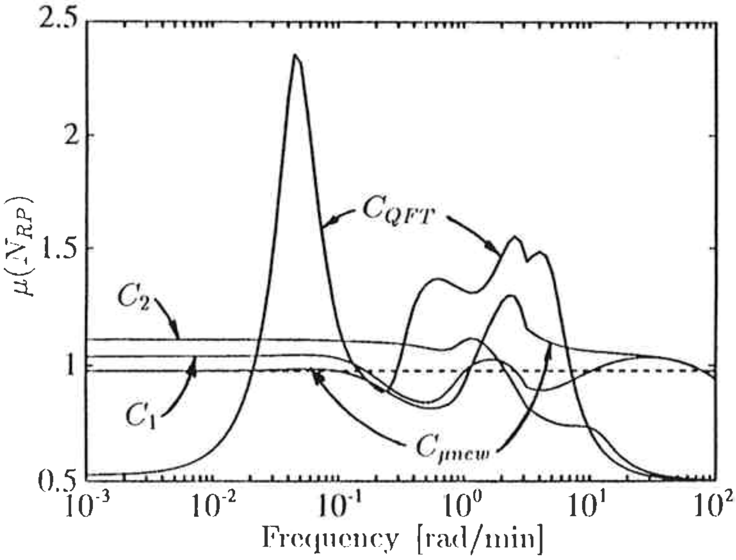

Figure 5. Solid curves: $\mu(N_{RP})$ for four different controllers and "tight" uncertainty (g_{I3p}). Dotted curve: $\mu(N_{RP})$ for controller $C_{\mu new}$ and original uncertainty (Eq.15).

7 Simulations

Simulations of set point changes for controller C_1 are shown in Fig. 7, and are compared to responses for controller C_{QFT} in Fig. 8. (Controller $C_{\mu new}$ gives essentially the same responses as for controller C_1). Responses are shown for the four corner points as defined by the parametric gain uncertainty description (Eq.16), and with a time delay of 1 min approximated by a second order Padé approximation. In the simulations with controller C_1 we have used a prefilter

$$C_f = 1/(5s+1)I_{2\times2} \tag{26}$$

for the set points. In the other simulation we used the filter by Yaniv and Barlev (Eq.18).

The responses shows that C_1 reaches the desired set point much faster than C_{QFT}. Controller C_{QFT} is initially somewhat less sensitive to the uncertainty, but shows a rather slow and oscillatory settling towards the new steady-state. Indeed, this behavior may be expected from the μ-plot in Fig. 5 where we see that the peak value occurs at a frequency $\omega \approx 0.04$ rad/min. In the time domain this corresponds to a resonant sinusoid with period $T = 2\pi/\omega \approx 150$ min, and indeed, this is the period found in the simulations. The better initial response may also be expected from the μ-plot, since $\mu(N_{RP})$ for C_{QFT} is lower than for C_1 at high frequencies.

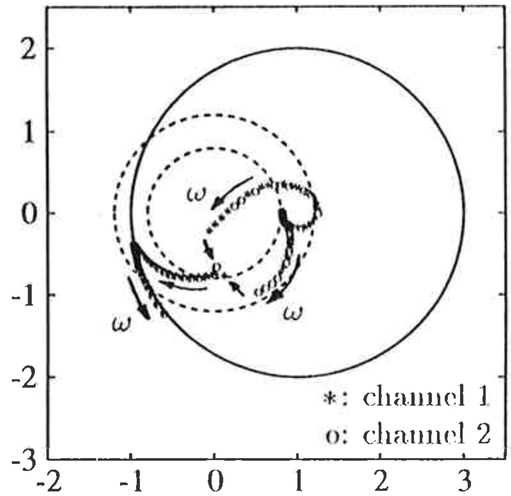

Figure 6. Worst case plants for controller $C_{\mu new}$ plotted in the complex plane, frequency-by-frequency for $0.001 < \omega < 100$ rad/min. Solid disc: g_{I1p} as $\omega \to \infty$. Region between the dashed circles: parametric uncertainty (Eq.16) as $\omega \to \infty$.

[4]The worst case for an inverse-based controller (a decoupler) is 0.8 and 1.2

8 Discussion and Conclusions

In Section 5 we postulated a parametric gain-delay uncertainty set as the "true" uncertainty. We then tried to minimize an uncertainty set generated by a complex additive perturbation which covers the "true" set. (If we had pole uncertainty we would probably have used inverse additive rather than additive perturbations.) The common practice of specifying uncertainty as gain and delay uncertainties, has probably to do with the popularity of first order-dead-time models in process control. However, by restricting the uncertainty description into a specific parametric form we may enlarge the set, since we have to choose parameter uncertainties such that the actual worst case is covered. If we then approximate the parametric set with a complex norm bounded set (for computational convenience), then the set is even further enlarged.

A much better approach would be to consider the nominal plant model *and* the uncertainty model as *one* unit. Both parts of this unit should be defined together such that a minimal set is achieved. It is very possible that discs in the complex plane will approximate the actual uncertainty better than , for example, parametric gain and delay uncertainty does. In those cases where this is true controllers designed by use of μ for *complex* perturbations would be less conservative than controllers designed for the gain delay uncertainty.

Acknowledgements. Support from NTNF is gratefully acknowledged.

NOMENCLATURE

$C(s)$ - controller
$C_f(s)$ - set point filter
$G(s)$ - MIMO linear model of process
$g(s)$ - SISO linear model of process
k - gain
$S(s) = (I + G(s)C(s))^{-1}$ - sensitivity function
$T(s) = G(s)C(s)(I + G(s)C(s))^{-1}$ - complementary sensitivity
$W(s)$ - weight matrix
$w(s)$ - weight scalar
$\|N\|_\infty = \sup_\omega \bar{\sigma}(N(j\omega))$ - H_∞-norm of N
Δ - perturbation matrix
δ - perturbation scalar (real or complex)
ϵ_k - magnitude of gain error
ϵ_θ - magnitude of delay error
θ - time delay (min)
μ - structured singular value
$\mu_{RP} = \sup_\omega \mu_\Delta(N_{RP}(j\omega))$
$\bar{\sigma}$ - maximum singular value
ω - frequency (rad min^{-1})

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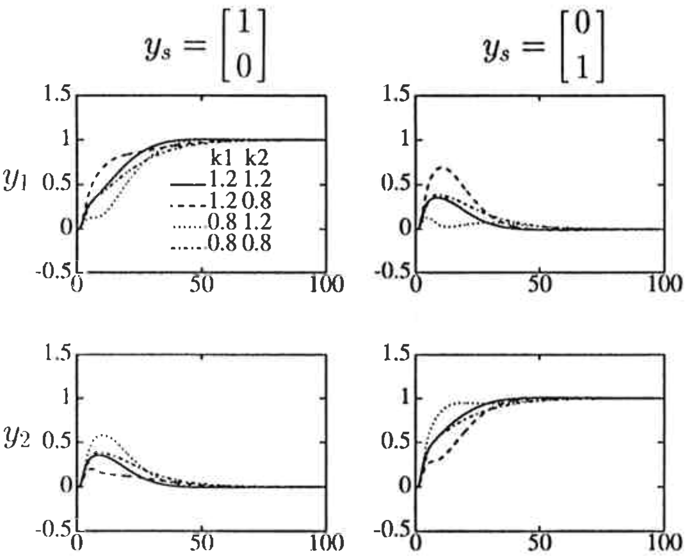

Figure 7. Simulation of set point changes with various input gains (k_1, k_2) using controller C_1 and prefilter Eq.26. All responses are with a 1 min delay (2nd order Padé).

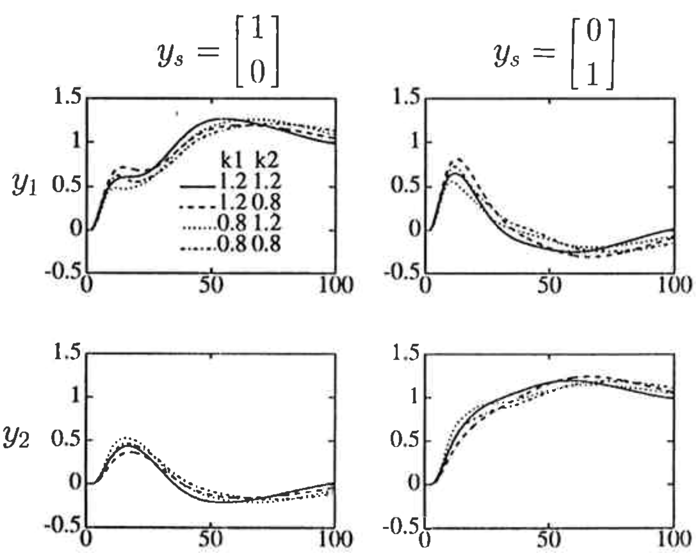

Figure 8. Simulation of set point changes with various input gains (k_1, k_2) using controller C_{QFT} and prefilter Eq.18. All responses are with a 1 min delay (2nd order Padé).

Proceedings of the 30th Conference
on Decision and Control
Brighton, England • December 1991

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UNCERTAINTY WEIGHT SELECTION FOR H-INFINITY AND MU-CONTROL METHODS

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Abstract

Design and analysis of robust control systems by use of H_∞-methods (*i.e.*, using the structured singular value, μ) requires frequency dependent weights to define performance and uncertainty. The purpose of this paper is to give some insight into H_∞-weight selection from a process control perspective. In an other paper [8] we studied the performance specifications, but in this paper we concentrate on uncertainty modelling. We study how to transform parametric gain-delay uncertainties into frequency dependent weights specifying norm bounded uncertainties.

1 Introduction

The control problem studied in this paper is based on the ill-conditioned plant presented by Skogestad, Morari and Doyle [10]. We use the same simplified distillation column (the LV-configuration) as our example:

$$G(s) = \frac{1}{75s+1}\begin{bmatrix} 0.878 & -0.864 \\ 1.082 & -1.096 \end{bmatrix} \tag{1}$$

This is a very crude model of a distillation column and does not describe an actual column very well. However, it is an excellent example for demonstrating the problems with ill-conditioned plants.

Skogestad *et al.* [10] show that this type of plant is very sensitive to input uncertainty. This demonstrates that any controller design method has to take model uncertainty into account. The structured singular value, μ, introduced by Doyle [2], allows us to include structured norm bounded perturbations (uncertainties) in the H_∞-framework. Skogestad *et al.* [10] use μ and a synthesis method called "D-K" iteration [4] to synthesize a "μ-optimal" controller which yield a feedback system not sensitive to the uncertainty.

Freudenberg [6] and Yaniv and Barlev [12] also use this distillation model to demonstrate design methods for robust control of ill-conditioned plants. Freudenberg uses a controller based on the singular value decomposition of the plant, while Yaniv and Barlev use the quantitative feedback theory (QFT), by Horowitz [11], to design a decentralized two degree of freedom controller.

The uncertainty and performance specifications in [10], [6] and [12] are slightly different. [10] and [6] use norm bounded uncertainties and a performance requirement for the maximum singular value of the sensitivity function for the worst case uncertainty. [12] use parametric gain and delay uncertainty, and specify the performance by of use magnitude bounds on the *elements* of the transfer function from set points to controlled outputs.

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2 A general framework for uncertainty modelling

Linear Fractional Transformations (LFT) provide a general framework for modelling norm bounded perturbations [3]. An LFT may be written on the following form (see Fig. 1)

$$z = F_u(P,\Delta)w = [P_{22} + P_{21}\Delta(I - P_{11}\Delta)^{-1}P_{12}]w \tag{2}$$

P_{22} is the nominal mapping from w to z and Δ is the H_∞-norm bounded perturbation; $\|\Delta\|_\infty = \sup_\omega \bar{\sigma}(\Delta(j\omega)) \leq 1$.

This uncertainty description has mainly been used for "unstructured" [1] uncertainty where the block-diagonal elements of Δ are complex. An example of such an uncertainty description is additive uncertainty on the whole plant matrix or additive element-by element uncertainty.

However, also "structured" parametric uncertainty may be written within the LFT framework. In this case let the real variable δ_i denote a physical parameter variation, and let $|\delta_i| \leq 1$. For example, a state space model with uncertain (and possibly correlated) coefficients

$$G_p(s) = (C+\Sigma\delta_iC_i)(sI - A - \Sigma\delta_iA_i)^{-1}(B+\Sigma\delta_iB_i)+(D+\Sigma\delta_iD_i) \tag{3}$$

(the subscript p denotes perturbed, *i.e.* with uncertainty) may be rewritten on the general LFT-form with Δ a diagonal matrix with δ_i's (possibly repeated) along its diagonal ([9] pp.8–11). Here the matrices A_i, B_i, C_i and D_i reflect how the i'th uncertainty δ_i affects the state space model.

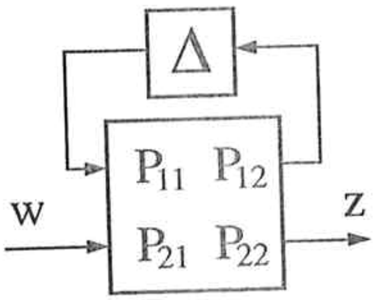

Figure 1. Linear Fractional Transformation.

[1]By "unstructured" uncertainty we mean that several separate uncertainty blocks have been combined into one "full" complex Δ-block.

A different example is from a study of a reactor, where we found that one of the elements in the B-matrix depended on the operating point in the following manner

$$b(\delta) = \frac{1.5 + 0.1\delta}{0.5 + 0.1\delta} \; ; \; -1 \leq \delta \leq 1. \tag{4}$$

This nonlinear dependency on the parameter δ may be written on the LFT-form , $b(\delta) = F_u(P, \delta)$ (Fig. 1), with $P = \begin{pmatrix} -0.2 & 1 \\ -0.4 & 3 \end{pmatrix}$.

In that same reactor example the parameter δ also appeared in the A-matrix

$$A = \begin{pmatrix} -1.5 + 0.1\delta & 0 \\ 1 & -1.5 + 0.1\delta \end{pmatrix} \tag{5}$$

To write this uncertainty as an LFT we may use

$$A = \begin{pmatrix} -1.5 & 0 \\ 1 & -1.5 \end{pmatrix} + 0.1 \begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix} \tag{6}$$

In this case we need two repeated scalar δ's (the rank of the matrix A_i is two), or three repeated δ's if also the variation in the B-matrix is included.

One example of an uncertainty which cannot be written on LFT-form is time delay uncertainty, and we we shall consider later some useful approximations.

We prefer LFT models where the P_{ij}'s are proper rational transfer functions and Δ is complex. This preference is of computational reasons: 1) the H_∞-synthesis in the "D-K" iteration uses a state space model of the problem, and 2) At present the algorithms to compute μ can not deal with combined real and complex perturbations, which would be needed to analyze robust performance with real parametric uncertainty. Therefore, at least at present, most of the general representations of parametric uncertainty, cannot be used in practice.

3 Performance specifications

If w in Fig. 1 represents normalized external inputs, z represents normalized errors, then a general H_∞-performance specification becomes for the nominal case (NP)

$$||P_{22}||_\infty = \sup_\omega \bar{\sigma}(P_{22}(j\omega)) < 1 \tag{7}$$

and for robust performance (RP)

$$||F_u(P, \Delta)||_\infty < 1 \; , \; \forall \Delta \tag{8}$$

The performance and uncertainty weights are included in P. As shown by Doyle [2] a computationally useful condition for robust performance may be written in terms of the structured singular value

$$\text{RP} \quad \text{iff} \quad \mu_{\tilde{\Delta}}(N_{RP}) < 1 \; , \; \forall \omega \tag{9}$$

where $N_{RP} = P$ and $\tilde{\Delta} = \text{diag}(\Delta, \Delta_P)$. The peak μ-value as a function of frequency is denoted μ_{RP}. Δ_P is a "full" complex perturbation matrix which stems from the performance requirement of wanting the singular value from w to z less than 1.

This is of course a very general framework. In practice, performance specifications are based on engineering judgement. In another paper [8] we discuss two different approaches: 1) Based on specifying bounds on important transfer functions such as the sensitivity, S, or the complementary sensitivity, T; 2) Based on considering signals as a function of frequency, and specify bounds on their amplification through the system. The latter approach is more general, but one often looses the direct handle on specifications such as bandwidth and maximum peak.

4 The original problem from [10]

Problem definition

Skogestad *et al.* [10] consider as a performance specification a simple bound on $S_p = (I + G(I + \Delta_I W_I)C)^{-1}$; the sensitivity function for the worst case input uncertainty.

$$RP \Leftrightarrow ||W_e S_p||_\infty < 1 \; , \; \forall \Delta_I \tag{10}$$

where the nominal model $G(s)$ is given in Eq.1, and

$$W_I(s) = \begin{bmatrix} w_I(s) & 0 \\ 0 & w_I(s) \end{bmatrix} \; ; \; w_I(s) = 0.2\frac{(5s+1)}{(0.5s+1)} \tag{11}$$

$$\Delta_I(s) = \begin{bmatrix} \delta_1(s) & 0 \\ 0 & \delta_2(s) \end{bmatrix} \; ; \; |\delta_i(j\omega)| \leq 1, \; \forall \omega \tag{12}$$

and

$$W_e(s) = \begin{bmatrix} w_e(s) & 0 \\ 0 & w_e(s) \end{bmatrix} \; ; \; w_e(s) = 0.5\frac{(10s+1)}{10s}. \tag{13}$$

w_I is a bound on multiplicative (relative) input uncertainty. Eq.11 shows that the uncertainty is 20% at low frequencies and reaches 1 at a frequency of approximately 1 rad/min. Note that the corresponding uncertainty matrix, Δ_I, is a diagonal matrix since it is assumed that uncertainty does not "spread" from one channel to another (for example, a large input signal in channel 1 does not affect the signal in channel 2). The performance weight in Eq.13 requires integral action, a bandwidth of approximately 0.05 rad/min and a maximum peak for $\bar{\sigma}(S_p)$ of 2.

N_{RP} in the Robust Performance μ-condition (Eq.9) for this problem becomes:

$$N_{RP} = \begin{bmatrix} -W_I CSG & W_I CS \\ W_e SG & -W_e S \end{bmatrix} ; \; \tilde{\Delta} = \begin{bmatrix} \begin{bmatrix} \delta_1 & \\ & \delta_2 \end{bmatrix} & \\ & \Delta_P \end{bmatrix} \tag{14}$$

Controller designs

Skogestad *et al.* [10] use "D-K" iteration [4] based on "the 1984-approach" [3] H_∞-minimization to design a "μ-optimal" controller. Their controller has six states and gives μ_{RP}=1.067 for both structured and unstructured Δ_I. Lundström *et al.* [8] use the new MATLAB μ-toolbox [1], based on the state-space H_∞-solution by Doyle *et al.* [5], to synthesize a better "μ-optimal"[2] controller (denoted $C_{\mu new}$) giving $\mu_{RP} = 0.978$. This controller has 22 states. Freudenberg [6] use another design method and achieve a controller with five states giving μ_{RP}=1.054 for unstructured Δ_I. Yaniv and Barlev [12] do not present a μ value for their design. We obtained $\mu_{RP} = 2.28$ for their controller[3] (without their prefilter) applied to the original formulation in [10].

Comments on the uncertainty weight

The uncertainty formulation by Skogestad *et al.* [10] is a complex *norm bounded* uncertainty. In each channel they allow the following input model

$$g_{Ip}(j\omega) = \left(1 + \delta(j\omega)\left|0.2\frac{5j\omega+1}{0.5j\omega+1}\right|\right) \; , \; |\delta(j\omega)| \leq 1 \; \forall \omega \tag{15}$$

(nominally $g_I = 1$). Yaniv and Barlev [12] use *parametric* uncertainty in their design specifications

$$g_{Ip}(j\omega) = ke^{-\theta j\omega} \; ; \; k \in [0.8 \; 1.2] \; ; \; \theta \in [0 \; 1] \tag{16}$$

These two sets are not identical for two reasons: 1) The uncertainty defined by Eq.15 is more general as it allows all kind of transfer functions $g_{Ip}(s)$ as long as their norm is bounded by the

[2]We will use "μ-optimal" to denote the best *obtained* solution using the "D-K" iteration in MATLAB μ-toolbox.

[3]There is a misprint in [12]; the second order pole in controller "g1" of 0.22 should be with damping 0.5.

weight. For example, the following input model is allowed by Eq.15

$$g_{Ip}(s) = \begin{pmatrix} \frac{-0.5s+1.2}{0.5s+1} & 0 \\ 0 & \frac{-0.5s+0.8}{0.5s+1} \end{pmatrix} \tag{17}$$

2) On the other hand, the uncertainty allowed by Eq.16 is *not* quite covered by the uncertainty weight in Eq.15, although one might get this impression from the paper of Skogestad *et al.* [10]. Eq.15 does cover gain uncertainty of ±20%. It also covers *approximately* a 1 min. delay (a first order Padé approximation is covered). However, it does not cover all *combinations* of these gain and delay uncertainties, especially not at high frequencies.

The two sets defined in Eq.15 and 16 are compared in Fig. 2. At $\omega = 1$ rad/min most of the parametric uncertainty is covered by the norm bounded set, but the corner corresponding to +20% and 1 min delay is not covered. This turns out to be critical, since this often is the "worst case" uncertainty. As $\omega \to \infty$ there is region corresponding to positive gain error and phase error of about −180° which is not covered.

Comments on two-degree of freedom controller

The problem specifications as given in [10] is on the sensitivity function $S = (I + GC)^{-1}$. Here C denotes the feedback part of the controller. Thus, a two-degree of freedom controller will not improve the design. However, if one interprets the performance specification from a *signal* point of view, that is, considers the response from y_s (reference signals) to $e = y - y_s$ (errors), then one may get improvement by a two-degree of freedom controller. Specifically, if C_f denotes the part of the controller which filters the reference signals, then $e = -SC_f y_s$. Yaniv and Barlev considered the problem from this signal point of view and designed a two-degree of freedom controller with a prefilter

$$C_f = \frac{1}{25s^2 + 5s + 1} I_{2\times 2}. \tag{18}$$

The time response is improved by adding this prefilter, but the value of μ_{RP} increases from 2.28 to 2.33 because the response gets slower (we show time domain simulations towards the end of this paper). We also considered this signal performance specification for but the improvement in performance by using a two degree of freedom controller was rather small; we were able to reduce μ_{RP} from 0.978 to 0.926. However, we then have no direct handle on the sensitivity function S (its H_∞-norm increases from 2 to about 3.5), and the robustness in terms of robust stability is worse.

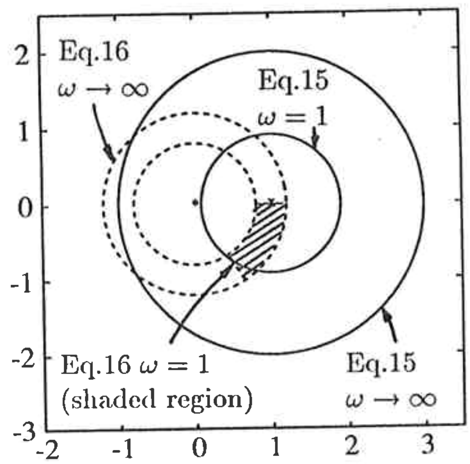

Figure 2. Representation in the complex plane of the uncertainty set in Eq.15 (solid discs) and Eq.16 (region between dashed circles).

Comments on the performance weight

The original problem formulation [10] was intentionally made very simple. For instance, the only performance requirement at high frequencies is to keep the magnitude of the sensitivity function below 2. This leads to an optimal controller with a rather high gain at high frequencies. A better problem formulation should include some penalty which forces the controller gain to roll off at high frequencies. We can achieve this, for example, by including a weight on the manipulated inputs, which will limit the transfer function CS, and/or including measurement noise, which will limit the complementary sensitivity function T. The uncertainty weight could also be used with similar effect, since large uncertainty at high frequencies will force the *input* complementary sensitivity function ($T_I = CG(I + CG)^{-1}$) to roll off.

5 New uncertainty weights

In the following we will assume that the true input model in each channel is given by the parametric uncertainty of Yaniv and Barlev (Eq.16). In Fig. 2 we showed that the norm bounded set of Skogestad *et al.* (Eq.15) does not quite cover this uncertainty. Here we will derive norm bounded sets which do cover the gain and delay uncertainty.

5.1 Real perturbations

Exact description: We may describe the process exactly by

$$g_{Ip}(s) = (\bar{k} + \delta_k \epsilon_k) e^{-(\bar{\theta} + \delta_\theta \epsilon_\theta)s} \tag{19}$$

where $\bar{k}$ and $\bar{\theta}$ are the average parameter values and δ_k and δ_θ are *real* scalars, $-1 \leq \delta \leq 1$. For the uncertainty in Eq.16 we get: $\bar{k} = 1$, $\bar{\theta} = 0.5$, $\epsilon_k = 0.2$ and $\epsilon_\theta = 0.5$. This uncertainty description has two problems: 1) The time delay cannot be modelled by an LFT, and 2) The perturbations are real.

Real perturbations and first order Padé: In order to achieve a rational transfer functions we use a first order Padé approximation of the time delay uncertainty.

$$e^{-\theta s} = e^{-\bar{\theta} s} e^{-\delta_\theta \epsilon_\theta s} \approx e^{-\bar{\theta} s} \left(\frac{1 - \delta_\theta \frac{\epsilon_\theta}{2} s}{1 + \delta_\theta \frac{\epsilon_\theta}{2} s} \right) \tag{20}$$

The time delay uncertainty may now be rearranged as an LFT.

$$\left(\frac{1 - \delta_\theta \frac{\epsilon_\theta}{2} s}{1 + \delta_\theta \frac{\epsilon_\theta}{2} s} \right) = \left(1 - \delta_\theta \epsilon_\theta s \left(1 + \delta_\theta \frac{\epsilon_\theta}{2} s \right)^{-1} \right) \tag{21}$$

Which gives $P = \begin{bmatrix} -\epsilon_\theta s/2 & 1 \\ -\epsilon_\theta s & 1 \end{bmatrix}$ in Eq.2. Problems with this description are: 1) The perturbations are real, and 2) The time delay is approximated.

For this approximation we cannot simply relax the requirement that δ_θ has to be real and allow it to be complex; a simple analysis shows that with $\delta_\theta = j$ this would imply that the norm of $e^{-\theta s}$ would be infinity at the frequency $\omega = 2/\epsilon_\theta$.

Real perturbations and n^{th} order approximation: We may achieve an arbitrary good approximation of the time delay by the following n'th order approximation.

$$e^{-\theta s} \approx e^{-\bar{\theta} s} \left(\frac{1 - \delta_\theta \frac{\epsilon_\theta}{2n} s}{1 + \delta_\theta \frac{\epsilon_\theta}{2n} s} \right)^n \tag{22}$$

This may also be written on the general LFT-form, but since all δ_θ's are equal, we need *repeated* perturbations. Thus, the problems with this description are: 1) The perturbations are real, 2) Repeated real δ_θ's are needed, and 3) High order model.

5.2 Complex perturbations

Here we will consider modelling the parametric uncertainty in each input channel (the perturbations in k and θ) by a *single* complex scalar perturbation δ. That is, we use a structured approach with respect to the channels, but an "unstructured" (lumped) approach with respect to the parametric uncertainties in each channel.

We use a slightly generalized uncertainty description compared to the input uncertainty considered in Section 4:

$$G_p(s) = G_c(s)(G_I(s) + \Delta_I(s)W_I(s)) \tag{23}$$

(subscript c for column model and I for input model; above we used $G_c = G, G_I = I$). This is simply an additive uncertainty description for the input model. We assume the same nominal model and same uncertainty in both inputs, *i.e.*, $W_I = w_I I$ and $G_I = g_I I$. Δ_I is given in Eq.12. At each frequency the true plant is allowed to be within a disc in the complex plane which has the nominal input model $g_I(j\omega)$ as a center and a radius equal to $|w_I(j\omega)|$. With this uncertainty description it is less conservative to approximate the real uncertainties in k and θ in each channel by one complex perturbation than by two separate complex perturbations.

To generate the uncertainty description we use the following procedure: i) Define the nominal "center" model, ii) Determine the radius w_I at each frequency such that the complex region generated by the real parameter variations is covered, iii) Approximate this radius by a transfer function (the approximation should always be "conservative", *i.e.*, such that all allowed plants are included). We will consider three different choices of the nominal input model.

1) Select $g_{I1} = 1$, *i.e.* the center point is fixed at $(1+j0)$ for all frequencies. This is the choice made by Skogestad *et al.* [10], but as mentioned above their weight $w_I(s)$ does not include the whole set generated by the uncertainty in Eq.16. To derive a weight which covers the set we did as follows: a) At low frequencies ($\omega < \pi/\theta_{max}$) the point furthest away from g_{I1} corresponds to a gain of $(1+\epsilon_k)$ and a time delay of θ_{max}, b) Using a second order Padé for the time delay then gives

$$w'_{I1}(s) = 1 - (1+\epsilon_k)\frac{\theta^2_{max}s^2 - 6\theta_{max}s + 12}{\theta^2_{max}s^2 + 6\theta_{max}s + 12} \tag{24}$$

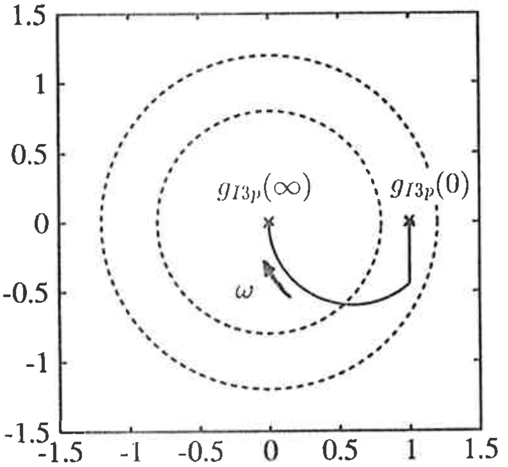

Figure 3. Trajectory of the "tight" model $g_{I3}(j\omega)$ (center in the smallest set) in the complex plane. The region between the dashed circles shows the parametric set Eq.16 as $\omega \to \infty$.

Since we are interested in only the magnitude of w_I this may be replaced by

$$w''_{I1}(s) = \frac{\epsilon_k\theta^2_{max}s^2 + 6(2+\epsilon_k)\theta_{max}s + 12\epsilon_k}{\theta^2_{max}s^2 + 6\theta_{max}s + 12} \tag{25}$$

c) The magnitude of this weight is too small at high frequencies. To compensate for this we multiply the weight by the factor $\frac{\tau s+1}{(\tau\epsilon_k/(2+\epsilon_k))s+1}$ where τ is found for each specific case (in our case $\tau = 0.167$). The final weight for our case is shown in Table 1.

2) Select $g_{I2} = \bar{k}e^{-\bar{\theta}j\omega}$. This is the average of the parametric set and was chosen by Laughlin *et al.* [7] who studied transformations from parametric uncertainty to a norm bounded complex set. We used a procedure similar to the one outlined above to approximate the radius.

3) Select the nominal model g_{I3} such that the radius of the uncertainty set at each frequency is minimized. At a given frequency g_{I3} is the "average" in the complex plane and will be denoted the "tight" model in the following. The trajectory of the "tight" $g_{I3}(j\omega)$ is shown in Fig. 3. Unfortunately, this $g_{I3}(j\omega)$ is not a rational transfer function, so we cannot use it and the corresponding uncertainty weight in the "D-K" algorithm. However, for a given controller we may compute μ for this uncertainty set, since the μ computations are performed on a frequency-by-frequency basis.

The three different approximations are illustrated in Fig. 4. Set g_{I1p} includes prediction, since it covers (at low frequencies) a region with positive imaginary part. g_{I2p} and g_{I3p} are similar at low frequencies, but differs at high. g_{I3p} is always the smallest set, and g_{I2p} is at most frequencies a smaller set than g_{I1p}. However, the smallest set may not necessarily yield the best design. The reason is that the smallest set, g_{I3p}, is not always contained in g_{I1p} (or g_{I2p}), *i.e.*, it may include plants which are outside the original set, and which are not covered by g_{I1p} (or g_{I2p}). If these plants then are the "worst case" plants, then approach 3 may in fact be more conservative than approach 1 (at least at some frequencies).

Table 1. New uncertainty models ($G_{Ip} = G_I + \Delta_I W_I$).

Set	G_I	W_I
1	$I_{2\times2}$	$\frac{0.2s^2+13.2s+2.4}{s^2+6s+12}\frac{0.167s+1}{(0.167/11)s+1}I_{2\times2}$
2	$\frac{0.5^2s^2-6*0.5s+12}{0.5^2s^2+6*0.5s+12}I_{2\times2}$	$\frac{0.05s^2+6.6s+2.4}{0.25s^2+3s+12}\frac{(0.167/2)s+1}{(0.167/22)s+1}I_{2\times2}$

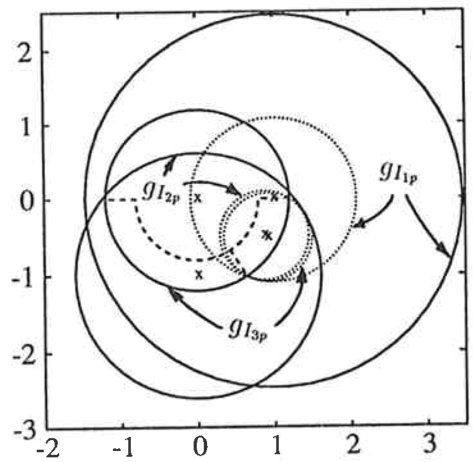

Figure 4. The three sets g_{I_1p}, g_{I_2p} and g_{I_3p} (Table 1) at $\theta_{max}\omega = 1$ rad (dotted discs) and at $\theta_{max}\omega = \pi$ rad (solid discs).

6 μ-analysis

To study in more detail the uncertainty descriptions mentioned above, we designed μ-optimal controllers for the example problem in [10], but used the different G_I and W_I from Table 1 in the uncertainty description. It turns out that controller C_1 (22 states) designed for the larger set, g_{Ip1}, gives a lower μ_{RP} than controller C_2 (22 states) tuned for the smaller set, g_{Ip2}. This demonstrates that a smaller set not necessarily gives the best design.

We also compare controllers C_1 and C_2 with the optimal controller from [8], $C_{\mu new}$, and the controller from [12], which we denote C_{QFT}. The results are summarized in Table 2. Two μ-values are shown for each design: The first column shows the μ_{RP}-value for the uncertainty set used in the design of the controller. The second column shows the μ_{RP}-value for the "tight" complex uncertainty description, g_{I3p} (computed frequency-by-frequency without any need to approximate g_{I3}). Fig. 5 shows μ_{RP}) as a function of frequency for the "tight" uncertainty for all four controllers (solid curves), and for controller $C_{\mu new}$ and the original uncertainty (dashed curve).

Controller $C_{\mu new}$ is designed for the original uncertainty set, and μ_{RP} for this controller increases when the "tight" set is used. This demonstrates that the original set does not include all worst case plants in the "tight" set. For example, +20% gain error and 1 min delay is not included. However, in the frequency range $0.1 < \omega < 1$, the "tight" set gives lower $\mu(N_{RP})$ than the original uncertainty , showing that the original uncertainty allows "worst case" plants outside the "tight" set (and thereby also outside the parametric set) at these frequencies.

C_{QFT} is designed for the parametric uncertainty (Eq.16). We cannot compute μ_{RP} exactly for this uncertainty (it would require real δ's), however, this controller gives a high μ_{RP}-value for the "tight" set. Controllers C_1 and C_2 are designed for uncertainty sets which cover the parametric uncertainty and μ_{RP} for these two controllers does not change very much when the uncertainty is changed to "tight" uncertainty.

Table 2. Optimal μ_{RP}-values for the design uncertainty and for the "tight" complex uncertainty for four different controllers.

Controller	design	$\mu_{RP\ design}$	$\mu_{RP\ tight}$
$C_{\mu new}$	Eq.15	0.978	1.30
C_{QFT}	Eq.16	–	2.36
C_1	g_{I1p} (Table 1)	1.04	1.05
C_2	g_{I2p} (Table 1)	1.12	1.12

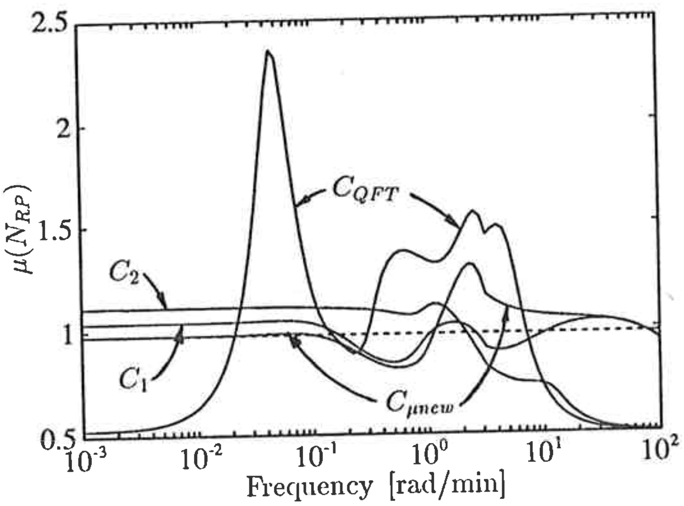

Figure 5. Solid curves: $\mu(N_{RP})$ for four different controllers and "tight" uncertainty (g_{I3p}). Dotted curve: $\mu(N_{RP})$ for controller $C_{\mu new}$ and original uncertainty (Eq.15).

The differencies between the original uncertainty set and the parametric set is also illustrated in Fig. 6. Here the worst case plants for controller $C_{\mu new}$ applied to the original uncertainty set are plotted in the complex plane. The worst case plants are computed for $0.001 < \omega < 100$ rad/min. Each "*" mark the worst plant in channel 1 at one specific frequency. The worst plants in channel 2 are marked with "o". At very low frequencies the worst plant is 0.8^4 in both channels. This plant is also allowed by the parametric uncertainty. As the frequency increases, we see that some of the worst case plants for the original set lie outside the parametric set, *i.e.*, at these frequencies the original set is conservative.

7 Simulations

Simulations of set point changes for controller C_1 are shown in Fig. 7, and are compared to responses for controller C_{QFT} in Fig. 8. (Controller $C_{\mu new}$ gives essentially the same responses as for controller C_1). Responses are shown for the four corner points as defined by the parametric gain uncertainty description (Eq.16), and with a time delay of 1 min approximated by a second order Padé approximation. In the simulations with controller C_1 we have used a prefilter

$$C_f = 1/(5s+1)I_{2\times 2} \tag{26}$$

for the set points. In the other simulation we used the filter by Yaniv and Barlev (Eq.18).

The responses shows that C_1 reaches the desired set point much faster than C_{QFT}. Controller C_{QFT} is initially somewhat less sensitive to the uncertainty, but shows a rather slow and oscillatory settling towards the new steady-state. Indeed, this behavior may be expected from the μ-plot in Fig. 5 where we see that the peak value occurs at a frequency $\omega \approx 0.04$ rad/min. In the time domain this corresponds to a resonant sinusoid with period $T = 2\pi/\omega \approx 150$ min, and indeed, this is the period found in the simulations. The better initial response may also be expected from the μ-plot, since $\mu(N_{RP})$ for C_{QFT} is lower than for C_1 at high frequencies.

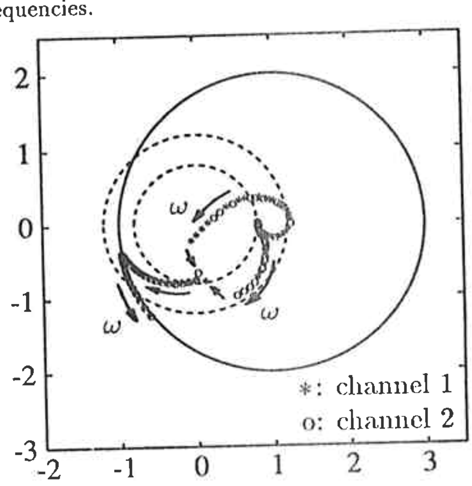

Figure 6. Worst case plants for controller $C_{\mu new}$ plotted in the complex plane, frequency-by-frequency for $0.001 < \omega < 100$ rad/min. Solid disc: g_{I1p} as $\omega \to \infty$. Region between the dashed circles: parametric uncertainty (Eq.16) as $\omega \to \infty$.

[4] The worst case for an inverse-based controller (a decoupler) is 0.8 and 1.2

8 Discussion and Conclusions

In Section 5 we postulated a parametric gain-delay uncertainty set as the "true" uncertainty. We then tried to minimize an uncertainty set generated by a complex additive perturbation which covers the "true" set. (If we had pole uncertainty we would probably have used inverse additive rather than additive perturbations.) The common practice of specifying uncertainty as gain and delay uncertainties, has probably to do with the popularity of first order-dead-time models in process control. However, by restricting the uncertainty description into a specific parametric form we may enlarge the set, since we have to choose parameter uncertainties such that the actual worst case is covered. If we then approximate the parametric set with a complex norm bounded set (for computational convenience), then the set is even further enlarged.

A much better approach would be to consider the nominal plant model *and* the uncertainty model as *one* unit. Both parts of this unit should be defined together such that a minimal set is achieved. It is very possible that discs in the complex plane will approximate the actual uncertainty better than , for example, parametric gain and delay uncertainty does. In those cases where this is true controllers designed by use of μ for *complex* perturbations would be less conservative than controllers designed for the gain delay uncertainty.

Acknowledgements. Support from NTNF is gratefully acknowledged.

NOMENCLATURE

$C(s)$ - controller
$C_f(s)$ - set point filter
$G(s)$ - MIMO linear model of process
$g(s)$ - SISO linear model of process
k - gain
$S(s) = (I + G(s)C(s))^{-1}$ - sensitivity function
$T(s) = G(s)C(s)(I + G(s)C(s))^{-1}$ - complementary sensitivity
$W(s)$ - weight matrix
$w(s)$ - weight scalar
$\|N\|_\infty = \sup_\omega \bar{\sigma}(N(j\omega))$ - H_∞-norm of N
Δ - perturbation matrix
δ - perturbation scalar (real or complex)
ϵ_k - magnitude of gain error
ϵ_θ - magnitude of delay error
θ - time delay (min)
μ - structured singular value
$\mu_{RP} = \sup_\omega \mu_{\hat{\Delta}}(N_{RP}(j\omega))$
$\bar{\sigma}$ - maximum singular value
ω - frequency (rad min^{-1})

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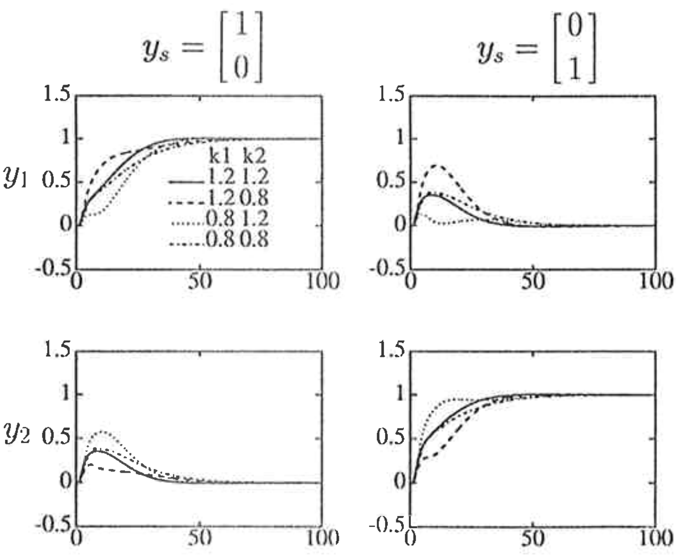

Figure 7. Simulation of set point changes with various input gains (k_1, k_2) using controller C_1 and prefilter Eq.26. All responses are with a 1 min delay (2nd order Padé).

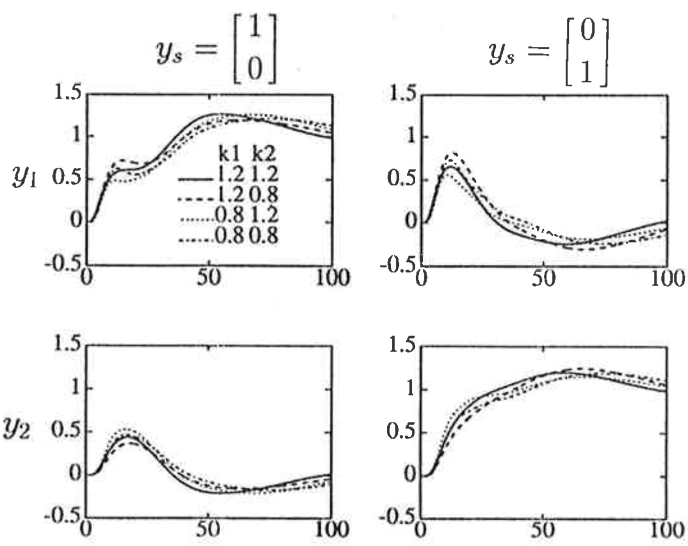

Figure 8. Simulation of set point changes with various input gains (k_1, k_2) using controller C_{QFT} and prefilter Eq.18. All responses are with a 1 min delay (2nd order Padé).