Greek Symbols

α = dimensionless parameter = $u_i hC/r_i A\Delta P$
π = osmotic pressure of solute in the solution of concentration C_{2j}, atm
γ = dimensionless osmotic pressure = $bCX_{21f}/\Delta P$
Δ = fractional solvent recovery = $1 - V$
θ = dimensionless solute permeability = $(D_{2m}/K\delta)/(A\Delta P/C)$
η = dimensionless mass-transfer coefficient = $kC/A\Delta P$
δ = effective membrane thickness of the hollow fiber, cm

Subscripts

f = high-pressure-side feed at $r = r_i$
j = 1 = feed stream, 2 = high-pressure side of the hollow fiber, 3 = permeate stream
0 = values for high rejection membranes, i.e., $X_{23} = 0$
1, 2 = conditions at $r = r_i$ and at $r = r_b$, respectively

Superscripts

– = average value over the module

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Implications of Large RGA Elements on Control Performance

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Large elements in the RGA imply a plant which is fundamentally difficult to control. (1) The plant is very sensitive to *uncorrelated* uncertainty in the transfer matrix elements. (2) The closed-loop system with an inverse-based controller is very sensitive to diagonal input uncertainty. With a diagonal controller, the system is *not* sensitive to diagonal input uncertainty, but the controller does not correct for the strong directionality of the plant and may therefore give poor performance even without uncertainty.

1. Introduction

Each element in the Relative Gain Array (RGA) is defined as the open-loop gain divided by the gain between the same two variables when all *other* loops are under "perfect" control (Bristol, 1966)

$$\lambda_{ij} = \frac{(\partial y_i/\partial u_j)_{u_{k\neq j}}}{(\partial y_i/\partial u_j)_{y_{k\neq i}}} = \frac{\text{gain all other loops open}}{\text{gain all other loops closed}} \tag{1}$$

The elements λ_{ij} form the RGA Λ. Definition 1 is compactly written in terms of the transfer matrix $\mathbf{G}(s) = \{g_{ij}\} = \{(\partial y_i/\partial u_j)_{u_k}\}$ and its inverse $\mathbf{G}^{-1}(s) = \{\hat{g}_{ij}\} = \{(\partial u_i/\partial y_j)_{y_k}\}$ as

$$\Lambda(\mathbf{G}) = \{\lambda_{ij}\} = \{g_{ij}\hat{g}_{ij}\} = \mathbf{G}(s) \times \mathbf{G}^{-1}(s)^T \tag{2a}$$

where $\times$ denotes element-by-element multiplication. For 2×2 plants,

$$\Lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} = \begin{pmatrix} \lambda_{11} & 1 - \lambda_{11} \\ 1 - \lambda_{11} & \lambda_{11} \end{pmatrix}$$

$$\lambda_{11} = 1\bigg/\left(1 - \frac{g_{12}g_{21}}{g_{11}g_{22}}\right) \tag{2b}$$

Although definition 1 is limited to steady state ($s = 0$), formulas 2 may be used to compute the RGA as a function of frequency ($s = jw$).

Since its introduction more than 20 years ago (Bristol, 1966), the RGA has found widespread use in industry. In his original paper, Bristol makes a number of intersting claims about the RGA, but few of these were actually proved. Therefore, for a number of years the RGA remained an empirical tool with little rigorous theoretical basis. The RGA was originally defined as a measure of interactions (eq 1) when using single-loop controllers on a multivariable plant (decentralized control), and Grosdidier et al. (1985) have proved a number of results which demonstrate the usefulness of the RGA in this respect. In particular, pairings corresponding to negative RGA elements should be avoided whenever possible. Shinskey (1967, 1984) uses the RGA extensively as a tool for selecting control configurations for distillation columns. Also he interprets the RGA primarily as an interaction measure. However, if this were the case, then an RGA analysis would be of no interest if multivariable rather than decentralized controllers were chosen. Experience indicates that this is not the case, and that the RGA is a measure of achievable control quality in a much wider sense than just as a tool for choosing pairings for decentralized control. In fact, Bristol indicates in his original paper that large RGA elements imply a plant which is fundamentally difficult to control. This claim has also been made more recently (McAvoy, 1983; Grosdidier et al., 1985). The following identity, which is easily derived from the results of Grosdidier et al. (1985), gives a good intuitive feeling for why large RGA elements may cause problems:

$$\mathrm{d}\hat{g}_{ji}/\hat{g}_{ji} = -\lambda_{ij}\,\frac{\mathrm{d}g_{ij}}{g_{ij}} \tag{3}$$

This identity shows that the elements $\hat{g}_{ji}$ of the inverse ($\mathbf{G}^{-1}$) are extremely sensitive to small changes in the elements g_{ij} of $\mathbf{G}$ if the RGA elements are large. This seems to indicate that plants with large RGA elements are very sensitive to modeling errors, and this is indeed true as we show in this paper.

More specifically, the objective of this paper is to answer the following two questions. (A) Is a plant with large

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elements in the RGA always difficult to control? (B) Is a plant with small elements in the RGA always easy to control (in the absence of other limitations on control performance, such as constraints and RHP zeros)? We will look at the questions in the context of model *uncertainty*. The results are illustrated with a simplified distillation column as an example. All results below involving the RGA, singular values, and condition numbers apply to any frequency ($s = jw$) unless otherwise stated.

2. Relationships between the RGA and the Condition Number

Let $\mathbf{G}(s)$ denote the linear transfer function model of the plant. The plant outputs $y(s)$ are then related to the plant inputs by $y(s) = \mathbf{G}(s)u(s)$. Let the magnitude of the vectors y and u be defined in terms of the 2-norm, which is obtained by taking the root mean square of the components at any frequency ($s = jw$)

$$\|y\|_2 = (\sum y_i^2)^{1/2} \qquad \|u\|_2 = (\sum u_i^2)^{1/2}$$

A multivariable plant has the property that the magnitude of the output vector (y) depends on the *direction* of the input vector (u). We define the maximum gain of the plant at any frequency as

$$\text{max gain:}\quad \sigma_{\max}(\mathbf{G}) = \max_{u\neq 0} \frac{\|y\|_2}{\|u\|_2} = \max_{u\neq 0} \frac{\|\mathbf{G}u\|_2}{\|u\|_2}$$

That is, $\sigma_{\max}(\mathbf{G})$ is obtained by choosing the direction of u such that $\|y\|_2/\|u\|_2$ is maximized. Similarly

$$\text{min gain:}\qquad \sigma_{\min}(\mathbf{G}) = \min_{u\neq 0} \frac{\|\mathbf{G}u\|_2}{\|u\|_2}$$

$\sigma_{\max}(\mathbf{G})$ and $\sigma_{\min}(\mathbf{G})$ are the maximum and minimum singular values of $\mathbf{G}$ (Klema and Laub, 1980). The condition number of the plant is the ratio between the maximum and minimum gain at any frequency

$$\gamma(\mathbf{G}) = \sigma_{\max}(\mathbf{G})/\sigma_{\min}(\mathbf{G})$$

We say that a plant has a strong "directionality" and is "ill-conditioned" if $\gamma(\mathbf{G})$ is large.

Note the property $\sigma_{\min}(\mathbf{G}) = 1/\sigma_{\max}(\mathbf{G}^{-1})$ which yields $\gamma(\mathbf{G}) = \sigma_{\max}(\mathbf{G})\sigma_{\max}(\mathbf{G}^{-1})$. Bristol (1966) himself pointed out the resemblance between the definition of the condition number and expression 2 for the RGA. However, Grosdidier et al. (1985) were the first to establish a rigorous relationship between the condition number and the RGA, and these results have later been extended by Skogestad and Morari (1987a) and Nett and Manousiouthakis (1987). The usefulness of these results in the control context is somewhat limited, since the results which link the condition number and control quality are somewhat weak. These results include (1) the bound on the matrix additive uncertainty by Grosdidier et al. (1985) (theorem 14 in their paper) and the slightly more powerful extension to element uncertainty by Skogestad and Morari (1987a) (also presented as result 1 in section 3 below) (these results demonstrate that ill-conditioned plants are sensitive to additive kinds of uncertainty) and (2) the result presented by Morari and Doyle (1986) (eq 63 in their paper) which indicates that the control performance of ill-conditioned plants may be very sensitive to input uncertainty.

Next we will briefly review some of the relationships between the RGA and the condition number. First note that the RGA is scaling independent:

$$\Lambda(\mathbf{G}) = \Lambda(\mathbf{S}_1\mathbf{G}\mathbf{S}_2)$$

Here $\mathbf{S}_1$ and $\mathbf{S}_2$ are diagonal "scaling" matrices with real positive entries and $\mathbf{S}_1\mathbf{G}\mathbf{S}_2$ corresponds physically to another choice of units for the inputs and outputs of the plant $\mathbf{G}$. On the other hand, the condition number $\gamma(\mathbf{G})$ is scaling dependent. One way of making it scaling independent is to minimize $\gamma(\mathbf{S}_1\mathbf{G}\mathbf{S}_2)$ over all possible scalings

$$\gamma^*(\mathbf{G}) = \min_{\mathbf{S}_1,\mathbf{S}_2} \gamma(\mathbf{S}_1\mathbf{G}\mathbf{S}_2) \tag{4}$$

Not surprisingly, the tightest relationships between the RGA and the condition number are in terms of $\gamma^*(\mathbf{G})$. The following inequalities show that plants with large elements in the RGA are *always* ill-conditioned (Nett and Manousiouthakis, 1987)

$$\gamma(\mathbf{G}) \geq \gamma^*(\mathbf{G}) \geq \|\Lambda\|_m - 1/\gamma^*(\mathbf{G}) \geq \|\Lambda\|_m - 1 \tag{5}$$

Here the "m-norm" is defined as (also see Nomenclature section)

$$\|\Lambda\|_m = 2 \max \{\|\Lambda\|_{i1}, \|\Lambda\|_{i\infty}\} \tag{6}$$

From (5) we see that large elements in the RGA always imply a large value of $\gamma^*(\mathbf{G})$ and $\gamma(\mathbf{G})$. Since ill-conditioned plants as indicated above are generally believed to cause control problems, (5) gives some justification to the claim that plants with large elements in the RGA are fundamentally difficult to control. Note that $\gamma(\mathbf{G})$ can be significantly larger than $\gamma^*(\mathbf{G})$, and the plant may therefore be ill-conditioned ($\gamma(\mathbf{G})$ large) even if all the elements in the RGA are small. In particular, 2×2 plants with an odd number of negative elements in $\mathbf{G}$ always have $\gamma^*(\mathbf{G}) = 1$ and $\|\Lambda\|_m = 2$, but $\gamma(\mathbf{G})$ may be arbitrary large. For example,

$$\mathbf{G} = \begin{pmatrix} 1 & -0.01 \\ 1 & 0.01 \end{pmatrix}, \qquad \gamma(\mathbf{G}) = 100, \qquad \gamma^*(\mathbf{G}) = 1, \qquad \|\Lambda\|_m = 2$$

From (5) we know that $\gamma^*(\mathbf{G})$ is always large when there are large elements in the RGA. And, similarly, a large value of $\gamma^*(\mathbf{G})$ always implies large elements in the RGA. This is seen from the following bound in terms of $\|\Lambda\|_1$ (sum of element magnitudes) which applies to 2×2 plants (Grosdidier et al., 1985)

$$2 \times 2: \qquad \gamma^*(\mathbf{G}) \leq \|\Lambda\|_1 \tag{7}$$

and from the following *conjecture* for $n \times n$ plants (Skogestad and Morari, 1987a; Nett and Manousiouthakis, 1987)

$$\text{conjecture:}\qquad \gamma^*(\mathbf{G}) \leq \|\Lambda\|_1 + k(n) \tag{8}$$

with $k(2) = 0$, $k(3) = 1$, and $k(4) = 2$. Note that for 2×2 plants, the 1- and the m-norm of the RGA are identical

$$2 \times 2: \qquad \|\Lambda\|_1 = \|\Lambda\|_m = 2\|\Lambda\|_{i\infty} = 2\|\Lambda\|_{i1}$$

Combining (5) and (7), one shows that $\|\Lambda\|_1$ and $\gamma^*(\mathbf{G})$ are always close in magnitude (in particular when they are large):

$$2 \times 2: \qquad \|\Lambda\|_1 - \frac{1}{\gamma^*(\mathbf{G})} \leq \gamma^*(\mathbf{G}) \leq \|\Lambda\|_1 \tag{9}$$

Consequently, for 2×2 plants, the difference between $\|\Lambda\|_1$ and $\gamma^*(\mathbf{G})$ is at most equal to $1/\gamma^*(\mathbf{G})$ and $\gamma^*(\mathbf{G}) \to \|\Lambda\|_1$ as $\|\Lambda\|_1 \to \infty$. Numerical evidence suggests that this also holds for $n \times n$ plants.

3. The RGA and Model Uncertainty

In this section we present two results which directly relate the RGA to control stability and performance. Result 1 has been derived previously (Skogestad and Morari, 1987a) and is presented here to more clearly show the relationship to the RGA. Result 1 (for example, eq 12) shows that plants with large RGA elements will easily become singular if small relative errors (r) on each transfer matrix element (g_{ij}) occur. Tight control of a plant which may become singular is not possible. The uncertainties ("errors") on each element have to be *independent* for result 1 (eq 12) to be nonconservative. However, the element uncertainties are usually correlated, and result 1 is not very useful in this case.

The most important result in this paper is therefore result 2, which directly shows that inverse-based controller should *never* be used for plants with large RGA elements because of the presence of input uncertainty. This result is subsequently used to argue that plants with large RGA elements are fundamentally difficult to control.

3.1. Independent Relative Element Uncertainty. This result introduces $||\Lambda(\mathbf{G})||_1$ as a sensitivity measure with respect to independent uncertainty on the plant elements.

Result 1 (2 × 2) (Skogestad and Morari, 1987a). *Assume each transfer matrix element has a relative uncertainty of magnitude r; that is, the actual ("perturbed") plant is*

$$\mathbf{G}_p = \begin{pmatrix} g_{11}(1+\Delta_{11}) & g_{12}(1+\Delta_{12}) \\ g_{21}(1+\Delta_{21}) & g_{22}(1+\Delta_{22}) \end{pmatrix}, \quad |\Delta_{ij}| < r \tag{10}$$

The uncertainties on each element are assumed to be ***independent****; that is, there is no correlation between the Δ_{ij}'s. Then the plant $\mathbf{G}_p$ remains nonsingular at steady state ($w = 0$) for any real perturbations, $-r \le \Delta_{ij} \le r$ if and only if*

$$r < \frac{1}{\gamma^*(\mathbf{G})} \tag{11}$$

which is satisfied if

$$r < \frac{1}{||\Lambda||_1} \qquad ||\Lambda||_1 = 2|\lambda_{11}| + 2|1 - \lambda_{11}| \tag{12}$$

Comment. Condition 12 also holds for complex perturbations $|\Delta_{ij}| \le r$ and $w > 0$ (Skogestad and Morari, 1987a). Condition 11 does *not* hold in these cases.

Condition 11 is necessary and sufficient. Condition 12 is only sufficient, but it is also "tight" because of the close relationship between $\gamma^*(\mathbf{G})$ and $||\Lambda||_1$ shown in (9). Conditions similar to (12) are derived for $n \times n$ *plants* using conjecture 8 above and theorem 6 in Skogestad and Morari (1987a).

Conjecture 1 ($n \times n$). *The plant remains nonsingular at any frequency w for complex relative errors of magnitude r(w) on each element if*

$$r(w) < 1/(||(jw)||_1 + k(n)) \tag{13}$$

with k(2) = 0 (in this case the conjecture is proven to be correct), k(3) = 1, and k(4) = 2.

The control implications of conditions 11–13 follow from the fact that if a plant is singular at a certain frequency (w), then the plant has a zero on the jw axis. The presence of this RHP zero limits the achievable control quality (Morari, 1983). In particular, it is impossbile to have integral control for a plant which may become singular at steady state ($w = 0$) (Skogestad and Morari, 1987a). Consequently, if there are large elements in the RGA and $||\Lambda||_1$ is large, we can allow only very small uncertainties in the elements without having control problems.

A very similar result has recently been published by Yu and Luyben (1987). It applies to $n \times n$ plants at steady state when only *one* element varies at the time: Let the ijth element be $g_{ij}(1 + \Delta_{ij})$. Then the plant becomes singular with a relative pertubation $\Delta_{ij} = -1/\lambda_{ij}$. Consequently, a large value of λ_{ij} means that only small relative errors on the corresponding plant element g_{ij} are tolerated.

The main restriction inherent in these results is the assumption of *independent* element uncertainty. Conditions 12 and 13 and the result of Yu and Luyben (1987) may be *very* conservative if the element uncertainties are correlated. For example, for distillation columns, even though $||\Lambda||_1$ is large and the elements in $\mathbf{G}$ may vary widely with operating conditions (r may be close to 1), it can be shown that the elements are correlated such that the plant never becomes singular (Skogestad and Morari, 1986). Therefore, for distillation column control, these results do not "explain" why plants with large values of the RGA in general are difficult to control.

3.2. Uncertainty on Each Manipulated Input. Result 2 below introduces the RGA as a measure of how performance is affected by uncertainty on each manipulated input. This result is of more general interest than result 1, because uncertainty on the manipulated inputs is *always* present: We never know the exact value of the inputs $\boldsymbol{u}$ which are applied to the plant.

Let u_{ci} denote the desired value of the ith manipulated input as computed by the controller, and let Δ_i represent the relative uncertainty on this input. Then the actual plant input is $u_i = u_{ci}(1 + \Delta_i)$ or in vector form $\boldsymbol{u} = \boldsymbol{u}_c(\mathbf{I} + \Delta_\mathbf{I})$ where $\Delta_\mathbf{I} = \mathrm{diag}\,\{\Delta_i\}$ represents the diagonal input uncertainty. Alternatively, we may define the perturbed plant as

$$\mathbf{G}_p = \mathbf{G}(\mathbf{I} + \Delta_\mathbf{I}) \qquad \Delta_\mathbf{I} = \mathrm{diag}\,\{\Delta_i\} \tag{14}$$

Any real plant has diagonal input uncertainty, and for process control applications, it seems unlikely that one should be able to get the uncertainty on each input lower than about 5–10%. This is probably the case even when cascaded loops based on flow measurements are used to correct for the nonlinear valve characteristics. Note that these errors (uncertainty) are *not* on the absolute value of the flows, but rather on their *change*. For example, assume we want to increase a flow rate from 100 to 110 kmol/min. However, due to errors in the flow measurement, the actual change is from 100 to 111 kmol/min. Although the absolute measurement error in this case is only 1%, the corresponding error in the *change* is 10%, i.e., $|\Delta_i| = 0.1$, for this input. Even though it seems unlikely in a real plant to reduce the magnitude of $|\Delta_i|$ below, say, 0.05, it is nevertheless clear that the effect of input uncertainty on the closed-loop system can be strongly reduced in some cases by using cascaded loops based on flow measurements instead of manipulating the valves directly.

In this paper we consider each manipulated input (valve) as the source of the input uncertainty. Since there is no reason to assume that these manipulated inputs influence each other, this results in a *diagonal* input uncertainty matrix, $\Delta_\mathbf{I}$. The case of *unstructured* input uncertainty ($\Delta_\mathbf{I}$ is a full matrix) is often considered in the literature (e.g., Morari and Doyle, 1986). This may be convenient for mathematical purposes, but for the reasons mentioned above, this is often not a proper description of the actual uncertainty, and the resulting conditions for robustness and performance may be unnnecessary conservative. For example, Morari and Doyle (1986) show that robust performance may be poor for plants with a large value of $\gamma(\mathbf{G})$ when there is *unstructured* input uncertainty. However, below we show that if we consider *diagonal* input uncertainty, then large RGA elements (or equivalently a large value of $\gamma^*(\mathbf{G})$ (eq 9)), rather than a large value of $\gamma(\mathbf{G})$, indicate control problems. For example, we present an ill-conditioned plant ($\gamma(\mathbf{G}) = 142$) with small RGA elements ($||\Lambda(\mathbf{G})||_1 = 2$) and which therefore is easy to control

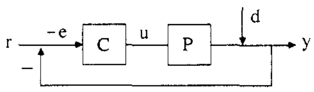

Figure 1. Classical feedback structure. ***d*** represents the effect of the disturbance on the output.

also in the presence of diagonal input uncertainty (Figure 5).

The loop transfer matrix, $\mathbf{G}_p\mathbf{C}$, is closely related to performance because of the identity (Figure 1), $y = (\mathbf{I} + \mathbf{G}_p\mathbf{C})^{-1}\boldsymbol{d}$. $\mathbf{G}_p\mathbf{C}$ may be written in terms of the nominal $\mathbf{GC}$ and an "error term", $\mathbf{C}^{-1}\Delta_\mathbf{I}\mathbf{C}$ or $\mathbf{G}\Delta_\mathbf{I}\mathbf{G}^{-1}$, as

$$\mathbf{G}_p\mathbf{C} = \mathbf{GC}(\mathbf{I} + \mathbf{C}^{-1}\Delta_\mathbf{I}\mathbf{C}) \tag{15a}$$

$$\mathbf{G}_p\mathbf{C} = (\mathbf{I} + \mathbf{G}\Delta_\mathbf{I}\mathbf{G}^{-1})\mathbf{GC} \tag{15b}$$

For SISO plants, a relative input error of magnitude Δ_1 on g results in the *same* relative change in $g_pc = gc(1 + \Delta_1)$, but for multivariable plants the effect of the input uncertainty on $\mathbf{G}_p\mathbf{C}$ may be amplified significanty as shown below.

Result 2. *For 2 × 2 plants the error term* $\mathbf{C}^{-1}\Delta_\mathbf{I}\mathbf{C}$ *in (15a) may be expressed in terms of the RGA of the controller* $\mathbf{C}$ *as*

ab/ac

$$\mathbf{C}^{-1}\Delta_\mathbf{I}\mathbf{C} = \begin{pmatrix} \lambda_{11}(\mathbf{C})\Delta_1 + \lambda_{21}(\mathbf{C})\Delta_2 & \lambda_{11}(\mathbf{C})\dfrac{c_{12}}{c_{11}}(\Delta_1 - \Delta_2) \\ -\lambda_{11}(\mathbf{C})\dfrac{c_{21}}{c_{22}}(\Delta_1 - \Delta_2) & \lambda_{12}(\mathbf{C})\Delta_1 + \lambda_{22}(\mathbf{C})\Delta_2 \end{pmatrix} \tag{16}$$

For n × n plants it is easily shown that the diagonal elements of the error term $\mathbf{C}^{-1}\Delta_\mathbf{I}\mathbf{C}$ *may be written as a straightforward generalization of the 2 × 2 case*

$$(\mathbf{C}^{-1}\Delta_\mathbf{I}\mathbf{C})_{ii} = \sum_{j=1}^{n}\lambda_{ji}(\mathbf{C})\Delta_j \tag{17}$$

Similarly, for 2 × 2 plants, the error term $\mathbf{G}\Delta_\mathbf{I}\mathbf{G}^{-1}$ *in (15b) may be expressed in terms of the RGA of the plant*

ad/ae

$$\mathbf{G}\Delta_\mathbf{I}\mathbf{G}^{-1} = \begin{pmatrix} \lambda_{11}\Delta_1 + \lambda_{12}\Delta_2 & -\lambda_{11}\dfrac{g_{12}}{g_{22}}(\Delta_1 - \Delta_2) \\ \lambda_{11}\dfrac{g_{21}}{g_{11}}(\Delta_1 - \Delta_2) & \lambda_{21}\Delta_1 + \lambda_{22}\Delta_2 \end{pmatrix} \tag{18}$$

(Here $\lambda_{ij} = \lambda_{ij}(\mathbf{G})$ *denotes the RGA elements of the plant.) For n × n plants, the diagonal elements of the error term* $\mathbf{G}\Delta_\mathbf{I}\mathbf{G}^{-1}$ *are*

$$(\mathbf{G}\Delta_\mathbf{I}\mathbf{G}^{-1})_{ii} = \sum_{j=1}^{n}\lambda_{ij}(\mathbf{G})\Delta_j \tag{19}$$

Comment. Similar results, but with, for example, g_{11} replaced by $-g_{22}$ and g_{22} replaced by $-g_{11}$ in the off-diagonal elements in (18), may be derived for the case of output uncertainty and performance measured at the input of the plant. This case is generally of less interest.

The RGA is independent of scaling, but the off-diagonal elements in (16) and (18) will depend on the scaling of the plant outputs. For the correct interpretation of these elements, the plant outputs should be scaled such that an output deviation of magnitude 1 has equal significance for all outputs.

Controllers with large RGA elements will lead to large elements in the matrix $\mathbf{C}^{-1}\Delta_\mathbf{I}\mathbf{C}$, and plants with large RGA elements will lead to large elements in the matrix $\mathbf{G}\Delta_\mathbf{I}\mathbf{G}^{-1}$. Equations 15a and 15b *seem* to imply that either of these cases will lead to large elements in $\mathbf{G}_p\mathbf{C}$ and therefore poor performance when there is input uncertainty ($\Delta_\mathbf{I} \neq 0$). However, this interpretation is generally *not* correct since the directionality of $\mathbf{GC}$ may be such that the elements in $\mathbf{G}_p\mathbf{C}$ remain small even though $\mathbf{C}^{-1}\Delta_\mathbf{I}\mathbf{C}$ *or* $\mathbf{G}\Delta_\mathbf{I}\mathbf{G}^{-1}$ have large elements. This should be clear from the following two extreme cases.

(1) Assume the controller has small RGA elements (small elements in $\Lambda(\mathbf{C})$). In this case, the elements in the error term $\mathbf{C}^{-1}\Delta_\mathbf{I}\mathbf{C}$ are similar to $\Delta_\mathbf{I}$ in magnitude (eq 16 and 17). Consequently, $\mathbf{G}_p\mathbf{C}$ is not particularly influenced by input uncertainty, even though the plant itself may be strongly ill-conditioned with large RGA elements (large elements in $\Lambda(\mathbf{G})$).

(2) Assume the plant has small RGA elements (small elements in $\Lambda(\mathbf{G})$). In this case, the elements in the error term $\mathbf{G}\Delta_\mathbf{I}\mathbf{G}^{-1}$ are similar to $\Delta_\mathbf{I}$ in magnitude (eq 18 and 19). Consequently, $\mathbf{G}_p\mathbf{C}$ is not particularly influenced by input uncertainty, even though the controller itself may have large RGA elements. (Comment: From a practical point of view, one might argue that it is unlikely that anyone would design a controller with large RGA elements for a plant with small RGA elements.)

From (1) and (2), we conclude that for a system to be sensitive to input uncertainty, *both* the controller and the plant must have large RGA elements. These results agree with Doyle's conditions for robust performance (RP) as presented in a paper by Skogestad and Morari (1986) (eq 34 in their paper): RP in the presence of *unstructured* input uncertainty ($\Delta_\mathbf{I}$ is a full matrix) is automatically implied by nominal performance (NP) and robust stability (RS) provided the condition number of *either* the controller, $\gamma(\mathbf{C})$, *or* the plant, $\gamma(\mathbf{G})$, is close to 1. Note that our results (eq 15–19 above) are in terms of *diagonal* input uncertainty ($\Delta_\mathbf{I}$ diagonal) and involve the RGA rather than the condition number.

3.3. Inverse-Based Controller. For "tight" control, it is desirable to use an inverse-based controller, $\mathbf{C}(s) = \mathbf{G}^{-1}(s)\mathbf{K}(s)$, where $\mathbf{K}(s)$ is a *diagonal* matrix. A special case of such an inverse-based controller is a decoupler. With $\mathbf{C}(s) = \mathbf{G}^{-1}(s)\mathbf{K}(s)$, we find $\Lambda(\mathbf{C}) = \Lambda(\mathbf{G}^{-1}\mathbf{K}) = \Lambda(\mathbf{G}^{-1}) = \Lambda^T(\mathbf{G})$. Thus, if the elements of $\Lambda(\mathbf{G})$ are large, so will be the elements of $\Lambda(\mathbf{C})$, and from the discussion above, we expect high sensitivity to input uncertainty. We also see directly from

$$\mathbf{G}_p\mathbf{C} = \mathbf{K}(s)(\mathbf{I} + \mathbf{G}\Delta_\mathbf{I}\mathbf{G}^{-1}) = \mathbf{K}(s)(\mathbf{I} + \mathbf{C}^{-1}\Delta_\mathbf{I}\mathbf{C}) \tag{20}$$

that large elements in $\mathbf{G}\Delta_\mathbf{I}\mathbf{G}^{-1}$ (or equivalently large elements in $\mathbf{C}^{-1}\Delta_\mathbf{I}\mathbf{C}$) imply that the loop transfer matrix $\mathbf{G}_p\mathbf{C}$ is very different from the nominal one $\mathbf{GC} = \mathbf{K}(s)$, and poor response or even instability is expected with $\Delta_\mathbf{I} \neq 0$ (in this case $\mathbf{GC} = \mathbf{K}$ has no "directionality" that may make $\mathbf{G}_p\mathbf{C}$ remain small).

Decouplers have been discussed extensively in the chemical engineering literature, in particular in the context of distillation columns (e.g., Luyben (1970) and Arkun et al. (1984)). The idea of using a decoupler ($\mathbf{D}$) is that the multivariable aspects are taken care of by the decoupler and tuning of the control system is reduced to a series of single-loop problems. Let the diagonal matrix $\mathbf{K}(s)$ denote these "single-loop" controllers. The overall controller, $\mathbf{C}$, including the decoupler is

$$\mathbf{C}(s) = \mathbf{DK}(s) \tag{21}$$

A steady-state decoupler is obtained with $\mathbf{D} = \mathbf{G}(0)^{-1}$. The sensitivity of decouplers to *decoupler errors* has been discussed in the literature (e.g., Toijala (Waller) and Fagervik (1972)), and the observed sensitivity for such errors is in fact easily explained from result 1 (eq 12). However, the most important reason for the robustness problems encountered with decouplers is probably not decoupler errors but rather *input uncertainty*. Recall from (20) that *any* controller of the form $\mathbf{C}(s) = \mathbf{G}^{-1}(s)\mathbf{K}(s)$ is sensitive to input uncertainty if the plant has large RGA elements. Decouplers are generally of this form and should therefore *not* be used for plants with large RGA elements. Let $\mathbf{G}_{\text{diag}}$ denote the matrix consisting of the diagonal elements in $\mathbf{G}$. Then for the decouplers most commonly studied in the literature, we find

$$\text{"ideal decoupling":}\quad \mathbf{D} = \mathbf{G}^{-1}\mathbf{G}_{\text{diag}} \tag{22a}$$

$$\text{"simplified decoupling":}\quad \mathbf{D} = \mathbf{G}^{-1}((\mathbf{G}^{-1})_{\text{diag}})^{-1} \tag{22b}$$

In both of these cases, the controller is of the form $\mathbf{C}(s) = \mathbf{G}^{-1}\mathbf{K}(s)$ and will lead to serious robustness problems if the plant has large RGA elements. On the other hand, if "one-way" decoupling is used, then $\mathbf{D}$ is triangular and $\Lambda(\mathbf{C}) = \Lambda(\mathbf{DK}) = \mathbf{I}$. A "one-way" decoupler is therefore much less sensitive to input uncertainty (recall (16) and (17)).

Control Implications of (17), (19), and (20). (i) An inverse-based controller (and in particular a decoupler) should *never* be used for a plant with large elements in the RGA. (ii) One-way decouplers are much less sensitive to input uncertainty. (iii) Inverse-based controllers may give poor response even if the elements in the RGA are small. This may happen if g_{12}/g_{22} or g_{21}/g_{11} is large (eq 18 and 20). One example is a triangular plant which always has $\lambda_{11} = 1$, but where the response obtained with an inverse-based controller may display large "interactions" in the presence of uncertainty.

It should be added that it is the behavior of $\mathbf{G}_p\mathbf{C}$ around crossover ($\|\mathbf{G}_p\mathbf{C}\| \approx 1$) which is primary importance for the stability and performance of the closed-loop system. Therefore, control problems are expected if the RGA has large elements in this frequency range.

3.4. Diagonal Controller. A diagonal controller always has $\lambda_{11}(\mathbf{C}) = 1$, and the error term in (15a) becomes

$$\mathbf{C}^{-1}\Delta_\mathbf{I}\mathbf{C} = \Delta_\mathbf{I} \tag{23}$$

Therefore, the response is only weakly influenced by the presence of input uncertainty. However, it may be difficult to achieve a good *nominal* response when the controller is restricted to being diagonal (this may be the case even if λ_{11} is close to one as for a nearly triangular plant): The diagonal controller gives limited correction for the "directionality" of the plant and $\gamma(\mathbf{GC})$ may be large. In this case, the response depends strongly on the "disturbance direction": Let $\boldsymbol{d}$ represent the effect of the disturbance on the output. The response is poor for a disturbance ($\boldsymbol{d}$) with a large disturbance condition number (Skogestad and Morari, 1987b):

$$\gamma_d(\mathbf{GC}) = \frac{\|(\mathbf{GC})^{-1}\boldsymbol{d}\|_2}{\|\boldsymbol{d}\|_2}\sigma_{\max}(\mathbf{GC}) \tag{24}$$

$\gamma_d(\mathbf{GC})$ ranges in value between 1 and $\gamma(\mathbf{GC})$. A value close to 1 indicates that the disturbance is in the "good" direction, corresponding to the high loop gain, $\sigma_{\max}(\mathbf{GC})$. A value close to $\gamma(\mathbf{GC})$ indicates that the disturbance is in the "bad" direction, corresponding to the low loop gain, $\sigma_{\min}(\mathbf{GC})$. $\boldsymbol{d}$ may also represent the effect of a *set-point change*. If arbitrary set-point changes are allowed, then there exists a set-point change $\boldsymbol{y}_s$ such that $\gamma_{ys}(\mathbf{GC}) = \gamma(\mathbf{GC})$.

Diagonal controllers do *not* generally correct for the directionality of the plant and $\gamma(\mathbf{GC})$ is large whenever $\Lambda(\mathbf{G})$ has large elements (see (26) below). The inequality

$$\mathbf{C}\text{ diagonal:}\quad \gamma(\mathbf{GC}) \geq \gamma^*(\mathbf{G}) \tag{25}$$

follows since a diagonal controller merely corresponds to a scaling of the input to the plant. Applying (5) yields

$$\mathbf{C}\text{ diagonal:}\quad \gamma(\mathbf{GC}) \geq \|\Lambda(\mathbf{G})\|_m - 1 \tag{26}$$

and we see that a plant with large RGA values always will have $\gamma(\mathbf{GC})$ large and will yield poor performance (at least if arbitrary set-point changes are considered).

One *special case* when a diagonal controller may yield acceptable performance for an ill-conditioned plant ($\gamma(\mathbf{G})$ large) is when the plant is naturally "decoupled" at the input ($\mathbf{V} = \mathbf{I}$). This plant has all RGA elements less than 1 as shown below. Write the singular value decomposition (SVD) of $\mathbf{G}$ as

$$\mathbf{G} = \mathbf{U}\Sigma\mathbf{V}^H,\quad \Sigma = \begin{pmatrix}\sigma_{\max}(\mathbf{G}) & 0\\ 0 & \sigma_{\min}(\mathbf{G})\end{pmatrix} \tag{27}$$

For the case $\mathbf{V} = \mathbf{I}$ (or, more generally, when $\mathbf{V}$ has only one nonzero element in each row and column, which give $\mathbf{V} = \mathbf{I}$ by rearranging the inputs), a diagonal controller can be found which removes *most* of the directionality in the plant: Choose $\mathbf{C}(s) = c(s)\Sigma^{-1}$ to get $\mathbf{GC} = c(s)\mathbf{U}$ which has $\gamma_d(\mathbf{GC}) = 1$ for all disturbances. Note, however, that the response is not decoupled (unless $\mathbf{U}$ is diagonal). Also note that $\gamma^*(\mathbf{G}) = \gamma^*(\mathbf{U}\Sigma) = 1$ in this case (Σ is diagonal and $\gamma(\mathbf{U}) = 1$), and it follows from (5) that the elements in $\Lambda(\mathbf{G})$ are less than 1 in magnitude.

3.5. General Controller Structure. (1) From (20) we concluded that the system is *always* sensitive to input uncertainty if an *inverse-based* controller is used for a plant with large RGA elements (in this case, $\Lambda(\mathbf{C})$ has large elements). (2) On the other hand, we know from (23) that the system is never sensitive to this uncertainty if a *diagonal* controller is used. (In this case $\Lambda(\mathbf{C}) = \mathbf{I}$).

What can be said in other cases? Is the RGA of the controller, $\Lambda(\mathbf{C})$, a useful indicator of a system's sensitivity to input uncertainty? In general, the answer is "no" (this is clear from (18) and (19) above). However, for practical purposes, where $\mathbf{C}(s)$ is designed based on $\mathbf{G}(s)$, the answer is "yes". The reason is that one property of any well-designed multivariable controller is to remove some of the directionality in $\mathbf{G}$ by making $\mathbf{GC}$ "more diagonal" than $\mathbf{G}$. (This excludes, for example, using a controller with large RGA elements for a diagonal plant.) Therefore, large elements in the error term $\mathbf{C}^{-1}\Delta_\mathbf{I}\mathbf{C}$ will lead to some degree to large elements in $\mathbf{G}_p\mathbf{C}$ (eq 15a). Consequently, a plot of the magnitude of the elements of $\Lambda(\mathbf{C})$ as a function of frequency may be useful for evaluating the system's sensitivity to input uncertainty: A controller with small RGA elements at all frequencies is generally insensitive to input uncertainty. On the other hand, a controller with large RGA elements is likely to result in a system which is sensitive to input uncertainty.

3.6. Finding Worst-Case Conditions from the RGA. It is of interest to know the "worst case" combination of Δ_j's (input uncertainty) to use in simulation studies. Consider the error term $\mathbf{G}\Delta_\mathbf{I}\mathbf{G}^{-1}$ which for an inverse-based controller is directly related to the change in $\mathbf{GC}$ (eq 20). If all Δ_j's have the same magnitude ($|\Delta_j| < r_\mathrm{I}$), then from (19) the largest possible magnitude (worst case) of any diagonal element in $\mathbf{G}\Delta_\mathbf{I}\mathbf{G}^{-1}$ is given by $r_\mathrm{I}\|\Lambda(\mathbf{G})\|_{i\infty}$ ("maximum row sum"). To obtain this worst case value, the signs of the Δ_j's should be the same as those in the row of $\Lambda(\mathbf{G})$ with the largest elements.

Table I. Guidelines for Choice of Best Multivariable Controller Structure ("Large" Implies a Comparison with One, Typically >10)

		$\max_d \gamma_d(\mathbf{G})$	
		large	small
$\|\Lambda(\mathbf{G})\|_1$	large	(diagonal)	diagonal
$\|\Lambda(\mathbf{G})\|_1$	small	inverse-based ($\mathbf{V} = \mathbf{I}$: diagonal)	inverse-based (diagonal)

Example. Consider a plant with steady-state gain matrix

$$\mathbf{G}(0) = \begin{pmatrix} 1 & 0.1 & -2 \\ 1 & 2 & -3 \\ -0.1 & -1 & 1 \end{pmatrix} \tag{28}$$

The RGA is

$$\Lambda(\mathbf{G}(0)) = \begin{pmatrix} -1.89 & -0.13 & 3.02 \\ 3.59 & 3.02 & -5.61 \\ -0.70 & -1.89 & 3.59 \end{pmatrix}$$

Assume that the relative uncertainties Δ_1, Δ_2, and Δ_3 on each manipulated input have the same magnitude. Then the second row of $\Lambda(\mathbf{G})$ has the largest row sum ($\|\Lambda(\mathbf{G})\|_{i\infty} = 12.2$), and the worst combination of input uncertainty for an inverse-based controller is

$$\Delta_1 = \Delta_2 = -\Delta_3 = \Delta$$

We find

$$\mathbf{G}\Delta_I\mathbf{G}^{-1} = \begin{pmatrix} -5.0 & 7.5 & 14.3 \\ -9.1 & 12.2 & 21.5 \\ 3.0 & -3.7 & -6.2 \end{pmatrix} \Delta$$

Note that in this specific example, we would arrive at the same worst case diagonal elements by considering row 1 or row 3. Therefore, the worst case will always be obtained with Δ_1 and Δ_2 of the same sign and Δ_3 with a different sign even if their magnitudes are different. In some cases we may arrive at a different conclusion by considering other frequencies. Also note that, unless an inverse-based controller is used, it is not guaranteed that the worse case uncertainties are deduced by using this approach.

4. Choice of Controller Structure

An important decision facing the engineer is the choice of the controller *structure*. Two extremes are considered here: diagonal controller and inverse-based controller. The diagonal controller has advantages: it has fewer tuning parameters, is easier to understand and retune, and can be made failure tolerant more easily. These issues are *not* considered here. We want to decide which of the two choices above may result in the best *multivariable* controller. On the basis of the discussion above, Table I was prepared to assist the engineer in making this choice. The table should be used only as a rough guideline, since diagonal input uncertainty is the only source of uncertainty considered.

5. Large RGA Elements Are Bad News

Let us now answer the two questions presented in section 1.

(A) Is a plant with large elements in the RGA always difficult to control? Yes. This follows from results 1 and 2. However, if the following conditions are satisfied, control may still be acceptable. (1) The transfer matrix elements are correlated, and despite the large values in the RGA, the plant is not likely to become singular. (2) There exists a controller with small RGA elements (e.g., a diagonal controller) which gives an acceptable response for all important disturbances. This is the case if all important disturbances are in the "good" direction (i.e., $\gamma_d(\mathbf{G})$ is small despite the fact that $\gamma(\mathbf{G})$ is large).

Note that condition 2 implies that the plant is actually *not* ill-conditioned for the expected disturbances. We will give an example of such a case below (response to y_{s2} in Figure 4).

(B) Is a plant with small elements in the RGA always easy to control? No. As seen from (18), an *inverse*-based controller results in serious "interactions" if there is input uncertainty and some of the off-diagonal elements in the plant are large. A *diagonal* controller gives large interactions even in the absence of uncertainty, if the plant is nearly triangular. (Consider, for example, the plant

$$\mathbf{G} = \begin{pmatrix} 1 & 100 \\ 0 & 1 \end{pmatrix}$$

which has $\Lambda = \mathbf{I}$).

Let us also answer the following additional question.

(C) Is a plant with a large condition number always difficult to control? No. On the basis of the uncertainty descriptions investigated in this paper, the RGA rather than $\gamma(\mathbf{G})$ gives a measure of the plants sensitivity to diagonal input uncertainty. We will show in an example below that an inverse-based controller gives very good control for a plant with $\gamma(\mathbf{G}) = 71$ even in the presence of uncertainty (Figure 5).

Table II. Steady-State Data for Distillation Column

Binary Separation, Constant Molar Flows, Feed Liquid

rel volatility	$\alpha = 1.5$
no. of theoretical trays	$N = 40$
feed tray (1 = reboiler)	$N_F = 21$
feed composition	$z_F = 0.5$
product compositions	$y_D^0 = 0.99$, $x_B^0 = 0.01$
product rates	$D/F = B/F = 0.5$
reflux rate	$L/F = 2.706$

Steady-State Gain Matrices

$$\begin{pmatrix} dy_D \\ dx_B \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \mathbf{G}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \mathbf{E}\begin{pmatrix} dF \\ dz_F \end{pmatrix}$$

$$LV \text{ config.: } \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} dL \\ dV \end{pmatrix} \quad \mathbf{G}_{LV} = \begin{pmatrix} 0.878 & -0.864 \\ 1.082 & -1.096 \end{pmatrix}$$

$$DV \text{ config.: } \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} dD \\ dV \end{pmatrix} \quad \mathbf{G}_{DV} = \begin{pmatrix} -0.878 & 0.014 \\ -1.082 & -0.014 \end{pmatrix}$$

$$\text{disturbance matrix (both config.): } \mathbf{E} = \begin{pmatrix} 0.394 & 0.881 \\ 0.586 & 1.119 \end{pmatrix}$$

6. Examples

The distillation column described in Table II is used as an example. The product compositions y_D and x_B are to be controlled by manipulating the reflux (L) and either the boilup (V) or the distillate flow (D). The column is assumed to have no dynamics. (This is, of course, not true. However, we make the crude assumption that the dynamics are given in terms of a single first-order lag, which is exactly cancelled by a zero in the controller.) We stress that the objective of the examples is to demonstrate the usefulness of the RGA as a tool for screening design alternatives and to support the results presented in Table I, rather than to provide a realistic study of distillation column control.

We show simulations for two different configurations of manipulated inputs: LV configuration, $\gamma(\mathbf{G}_{LV}) = 142$, $\lambda_{11}(\mathbf{G}_{LV}) = 35$, $\|\Lambda\|_1 = 142$; DV configuration, $\gamma(\mathbf{G}_{DV}) = 71$, $\lambda_{11}(\mathbf{G}_{DV}) = 0.45$, $\|\Lambda\|_1 = 2$. We also consider two controllers for each of these: inverse-based controller ($\mathbf{GC} = \mathbf{I}(0.7/s)$); diagonal controller.

The controllers are given in the figure texts, and their gains were adjusted to guarantee robust stability for rel-

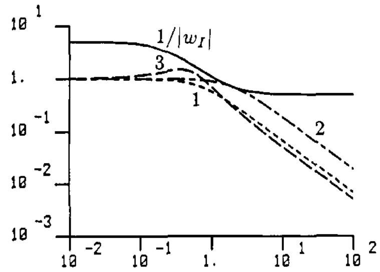

Figure 2. Controllers satisfy the robust stability condition (29). $\mu(\mathbf{CG}(\mathbf{I} + \mathbf{CG})^{-1})$ is shown as a function of frequency for (1) inverse-based controllers for LV and DV configurations, (2) diagonal LV controller, and (3) diagonal DV controller.

Table III. RGA, Condition Numbers, and SVD for Distillation Column

	configuration	
	LV	DV
RGA; λ_{11}	35.1	0.45
$\|\Lambda\|_1$	138.3	2
condition no., $\gamma(\mathbf{G})$	141.7	70.8
dist. condition no., $\gamma_d(\mathbf{G})$		
$d = F$ (feed rate)	11.8	4.3
$d = z_F$ (feed composition)	1.5	1.4
$d = \begin{pmatrix}1\\0\end{pmatrix}$ (set point in y_D)	110.7	54.9
$d = \begin{pmatrix}0\\1\end{pmatrix}$ (set point in x_B)	88.5	44.6
SV decomp., $\mathbf{G} = \mathbf{U\Sigma V}^H$		
$\mathbf{U}$	$\begin{pmatrix}-0.625 & 0.781\\ -0.781 & -0.625\end{pmatrix}$	$\begin{pmatrix}-0.630 & 0.777\\ -0.777 & -0.630\end{pmatrix}$
$\mathbf{\Sigma}$	$\begin{pmatrix}1.972 & 0\\ 0 & 0.0139\end{pmatrix}$	$\begin{pmatrix}1.393 & 0\\ 0 & 0.0197\end{pmatrix}$
$\mathbf{V}$	$\begin{pmatrix}-0.707 & 0.708\\ 0.708 & 0.707\end{pmatrix}$	$\begin{pmatrix}1.000 & -0.001\\ 0.001 & 1.000\end{pmatrix}$

ative uncertainty on each manipulated input with a magnitude bound

$$w_I(s) = 0.2\frac{5s + 1}{0.5s + 1}$$

This implies an input error of up to 20% at low frequencies, as is used in the simulations. The uncertainty increases at high frequency, reaching 100% at about $w = 1$ min^{-1}. This increase at high frequency may take care of neglected flow dynamics. Robust stability is guaranteed for this uncertainty if and only if (Skogestad and Morari, 1987a)

$$\mu(\mathbf{CG}(\mathbf{I} + \mathbf{CG})^{-1}) \leq 1/|w_I| \quad \forall\, w \tag{29}$$

where the structured singular value μ is computed with respect to a diagonal matrix. Condition 29 is satisfied for the controllers used as shown graphically in Figure 2.

For each of these four systems, the responses to two set-point changes are shown:

$$\boldsymbol{y}_{s1} = \begin{pmatrix}1\\0\end{pmatrix} \quad \boldsymbol{y}_{s2} = \begin{pmatrix}0.4\\0.6\end{pmatrix}$$

The set-point change $\boldsymbol{y}_{s1}$ has a large component in the "bad" direction corresponding to the low plant gain ($\gamma_d(\mathbf{G})$ = 110.7 for the LV configuration and $\gamma_d(\mathbf{G})$ = 54.9 for the DV configuration). $\boldsymbol{y}_{s2}$ has the same direction as a feed flow disturbance and has $\gamma_d(\mathbf{G})$ = 11.8 and 4.3 for the two configurations (Table III).

The responses are shown both for the nominal case (Δ_I = 0) and with 20% relative uncertainty on each manipulated input

$$\Delta_I = \begin{pmatrix}0.2 & 0\\ 0 & -0.2\end{pmatrix}$$

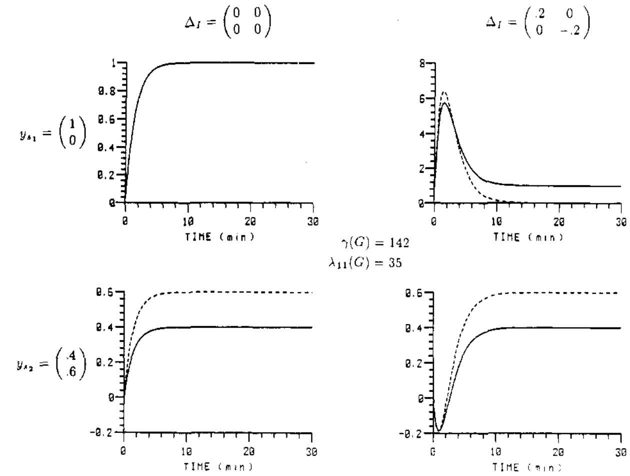

Figure 3. LV configuration. Closed-loop responses $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ for inverse-based controller. $\mathbf{C}(s) = (0.7/s)\mathbf{G}_{LV}^{-1} = (0.7/s)\begin{pmatrix}39.94 & -31.49\\ 39.43 & -32.00\end{pmatrix}$. Responses are shown for two different set-point changes, $\boldsymbol{y}_{s1}$ and $\boldsymbol{y}_{s2}$, both for the nominal case with no uncertainty (left) and with 20% error on the manipulated inputs ΔL and ΔV (right). The simulations illustrate that an inverse-based controller (e.g., a decoupler) should never be used for plants with large RGA elements because of the sensitivity to input uncertainty.

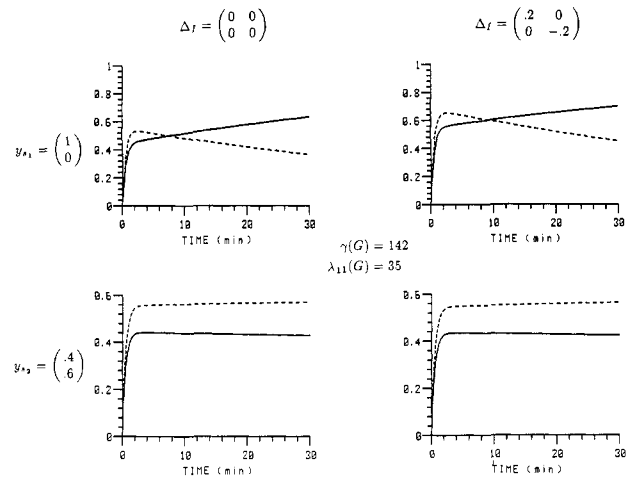

Figure 4. LV configuration. Closed-loop responses, $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$, for diagonal controller $\mathbf{C}(s) = (1/s)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$. The plant has large RGA elements, and a diagonal controller yields responses which are strongly dependent on the disturbance (or set-point) direction. The responses to $\boldsymbol{y}_{s2}$ are acceptable, but the response to the set-point change $\boldsymbol{y}_{s1}$ is extremely sluggish.

which give the following error terms (eq 18) for $\mathbf{GC}$ when an inverse-based controller is used:

$$(\mathbf{G\Delta_I G^{-1}})_{LV} = \begin{pmatrix}35.1\Delta_1 - 34.1\Delta_2 & -27.7(\Delta_1 - \Delta_2)\\ 43.2(\Delta_1 - \Delta_2) & -34.1\Delta_1 + 35.1\Delta_2\end{pmatrix} = \begin{pmatrix}13.8 & -11.1\\ 17.2 & -13.8\end{pmatrix}$$

$$(\mathbf{G\Delta_I G^{-1}})_{DV} = \begin{pmatrix}0.45\Delta_1 + 0.55\Delta_2 & 0.45(\Delta_1 - \Delta_2)\\ -0.55(\Delta_1 - \Delta_2) & 0.55\Delta_1 + 0.45\Delta_2\end{pmatrix} = \begin{pmatrix}-0.02 & 0.18\\ -0.22 & 0.02\end{pmatrix}$$

Conclusion. The simulations illustrate the following points.

An inverse-based controller gives poor response when the plant has large RGA elements (λ_{11} is large) and there is input uncertainty (Figure 3).

A diagonal controller cannot correct for the strong directionality of a plant with large RGA elements (recall eq 25). This results in responses which are strongly dependent on the disturbance (or set-point) direction (Figure 4). The response to $\boldsymbol{y}_{s2}$ (disturbance in F) which has $\gamma_d(\mathbf{G})$ = 11.8 is acceptable, but the response to the set-point change $\boldsymbol{y}_{s1}$ is extremely sluggish. This system may be acceptable, despite the large value of λ_{11}, provided set-point changes are not important.

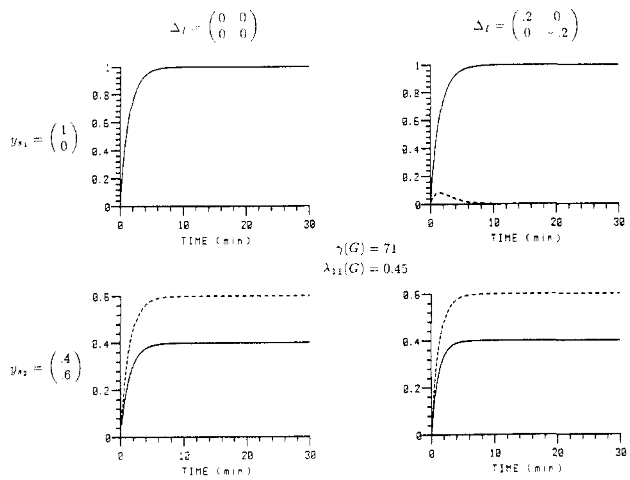

Figure 5. *DV* configuration. Closed-loop response, y_1 and y_2, for inverse-based controller $\mathbf{C}(s) = (0.7/s)\mathbf{G}_{DV}^{-1} = (0.7/s)\begin{pmatrix} -0.5102 & -0.5102 \\ 39.43 & -32.00 \end{pmatrix}$. An inverse-based controller may give very good response for an ill-conditioned plant, even in the presence of input uncertainty, provided the RGA elements are small.

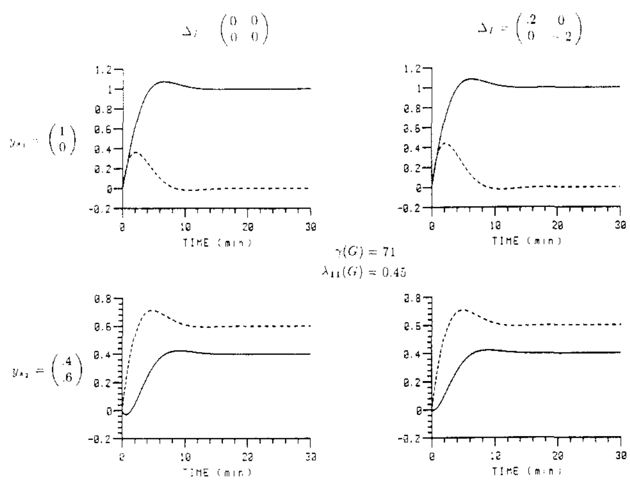

Figure 6. *DV* configuration. Closed-loop responses, y_1 and y_2, for diagonal controller $\mathbf{C}(s) = (-0.2/s)\Lambda^{-1} = (0.2/s)\begin{pmatrix} -0.718 & 0 \\ 0 & -50.8 \end{pmatrix}$. A diagonal controller performs satisfactory ill-conditioned plants with $\mathbf{V} \approx \mathbf{I}$ (which implies small RGA elements). However, "interactions" are still present because $\mathbf{U} = \begin{pmatrix} -0.63 & 0.78 \\ -0.78 & -0.63 \end{pmatrix}$ is not diagonal.

An inverse-based controller may give very good response for an ill-conditioned plant with diagonal input uncertainty, provided λ_{11} is small (Figure 5).

A diagonal controller may remove most of the directionality in an ill-conditioned plant if $\mathbf{V} \approx \mathbf{I}$. However, "interactions" are still present because

$$\mathbf{U} = \begin{pmatrix} -0.63 & 0.78 \\ -0.78 & -0.63 \end{pmatrix}$$

is not diagonal (Figure 6).

Nomenclature

$\mathbf{C}(s)$ = transfer matrix of controller

$\mathbf{G} = \mathbf{U}\Sigma\mathbf{V}^H$, singular value decomposition (Klema and Laub, 1980)

$\mathbf{G}(s) = \{g_{ij}\}$, transfer matrix of the plant

$\mathbf{G}_p(s)$ = perturbed plant (with uncertainty)

$\mathbf{K}(s)$ = diagonal transfer matrix of single-loop controllers

Greek Symbols

Δ_{ij} = relative element uncertainty, $g_{p_{ij}} = g_{ij}(1 + \Delta_{ij})$, $|\Delta_{ij}| < r$

Δ_i = relative uncertainty on input i

$\Delta_I = \mathrm{diag}\,\{\Delta_i\}$, matrix of relative input uncertainties, $\mathbf{G}_p = \mathbf{G}(\mathbf{I} + \Delta_I)$

$\Lambda(\mathbf{M}) = \mathbf{M}(s) \times \mathbf{M}^{-1}(s)^T$, RGA of the transfer matrix $\mathbf{M}$ ($\times$ denotes element by element multiplication)

Λ, $\Lambda(\mathbf{G})$ = RGA of plant

$\Lambda(\mathbf{C})$ = RGA of controller

$\|\Lambda\|_{i1} = \max_j \sum_{i=1}^n |\lambda_{ij}|$, induced 1-norm ("maximum column sum")

$\|\Lambda\|_{i\infty} = \max_i \sum_{j=1}^n |\lambda_{ij}|$, induced ∞-norm ("maximum row sum")

$\|\Lambda\|_m = 2\max\{\|\Lambda\|_{i1}, \|\Lambda\|_{i\infty}\}$

$\|\Lambda\|_1 = \sum_{i,j} |\lambda_{ij}|$, 1-norm (sum of element magnitudes)

$\Sigma(\mathbf{G}) = \mathrm{diag}\,\{\sigma_{max}(\mathbf{G}), ..., \sigma_{min}(\mathbf{G})\}$ – singular values

$\sigma_{max}(\mathbf{G})$ = maximum singular value

$\sigma_{min}(\mathbf{G})$ = minimum singular value

$\gamma(\mathbf{G}) = \sigma_{max}(\mathbf{G})/\sigma_{min}(\mathbf{G})$, condition number

$\gamma^*(\mathbf{G}) = \min_{\mathbf{S}_1,\mathbf{S}_2} \gamma(\mathbf{S}_1\mathbf{G}\mathbf{S}_2)$, minimized scaled condition number ($\mathbf{S}_1$ and $\mathbf{S}_2$ are diagonal matrices with real, positive entries)

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