Chemical Engineering Science, Vol. 42, No. 7, pp. 1765–1780, 1987.
Printed in Great Britain.

0009–2509/87 $3.00 + 0.00

DESIGN OF RESILIENT PROCESSING PLANTS—IX. EFFECT OF MODEL UNCERTAINTY ON DYNAMIC RESILIENCE

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(*Received* 11 *September* 1985; *accepted* 4 *November* 1986)

Abstract—The achievable quality of control for a particular system (its dynamic resilience) is limited by the nonminimum phase characteristics of the plant, constraints on the manipulated variables and model uncertainty. Model uncertainty requires that the controller be detuned and performance be sacrificed. The goal of this paper is to quantify this well-known qualitative statement. The closed-loop system must remain stable for all possible plants as defined by the uncertainty description. This robust stability requirement is used to derive simple bounds on the nominal performance for some specific cases. These bounds are relatively easy to evaluate and should be effective tools for screening alternative designs in terms of their resilience characteristics. The RGA and the minimized condition number are accurate measures with respect to *element* uncertainty, provided the relative errors of the transfer matrix elements are independent (uncorrelated) and have similar magnitude bounds.

1. INTRODUCTION

Most chemical plants are designed on the basis of steady state considerations, and the control system is designed separately in a subsequent stage of the project. This separation is acceptable provided that there exist suitable design-stage methods which can assess the "controllability" of the plant. That is, it must be determined *a priori* whether the design of a control system offering "reasonable" closed-loop response will subsequently be feasible. Until recently, such methods were not available. As a result, the expected performance often was not achieved in the operating plant. In some instances, a minor change at the initial design stage could have resulted in a "controllable" plant.

Previously, the controllability assessment has been based on simulations. This approach is complex and requires a complete dynamic model of the plant. Usually a number of case studies are performed with different choices of inputs, disturbances, operating conditions, controller structures and controller parameters. All those choices could bias the controllability assessment in an erroneous manner.

Morari (1983) suggested making the problem of controllability assessment independent of the controller selection problem. This is done by finding a plant's best achievable closed-loop control performance for all possible constant parameter linear controllers. This target, the upper bound on the achievable closed loop performance, is defined as the plant's *dynamic resilience*. Thus, "dynamic resilience" is an expression of the *plant's inherent* limitation on the closed-loop system's dynamic response which is not biased by specific choices of controllers.

The limitations imposed by non-minimum phase elements and constraints have been discussed in quantitative detail by Morari (1983) and Holt and Morari (1985a, b). Fundamentally, perfect control can only be achieved if the plant is invertible. *Non-minimum phase* elements [Right Half Plane (RHP) zeros and time delays] make it impossible to invert the plant and retain (internal) stability of the closed-loop system. The effect of *constraints* on performance is also related to a plant's closeness to singularity. If the minimum singular value of a plant $\bar{P}$, ($\underline{\sigma}(\bar{P})$) is small then the plant is nearly singular. This means that the plant has a very small gain for a particular input direction. To achieve tight control, the controller would have to provide very large input signals in this direction, possibly violating input size constraints.

The objective of this paper is to study the effects of *model uncertainty* on dynamic resilience. Model uncertainty requires that the controller be detuned and performance be sacrificed. The primary goal is to quantify this well-known qualitative statement by deriving expressions relating achievable closed-loop performance and uncertainty.

The first (and most important) step is to quantify the model uncertainty. This is usually not a trivial problem, and very misleading results may arise if an inappropriate uncertainty description is used. Another goal of this paper is to demonstrate some of these pitfalls. Therefore, the design engineer encounters a difficult situation: simple achievable performance bounds may be obtained with a crude uncertainty description but such bounds are often misleading. On the other hand, a detailed description of the model uncertainty is needed to find more meaningful bounds. Such descriptions are normally not available. A first step in resolving this dilemma is to identify for specific problem classes (e.g. distillation columns) the sources of model uncertainty which are likely to cause complications. The engineer can then concentrate on these when quantifying the uncertainty. Some of the examples in this paper will be helpful in this respect.

II. UNCERTAINTY, STABILITY AND PERFORMANCE

(1) *Model uncertainty: causes and definition*

The linear time invariant models used throughout this paper describe the actual plant dynamics only approximately.

1. All real processes are nonlinear. In this paper, linear transfer functions are used to represent the plant and some "uncertainty" is introduced by *linearizing* the nonlinear plant at various operating points. This may lead to a linear model with "uncertain" coefficients.
2. In other cases the process may be represented quite accurately by linear models. However, *different operating conditions* can lead to changes of the parameters in the linear model. For example, increased throughput/flowrates usually result in smaller deadtimes and time constants.
3. Consequently, in many cases parts of the "uncertainty" are known accurately. However, there will always exist "*true*" *uncertainties* even though the underlying process is essentially linear: The model parameters are never known exactly and, at high frequencies, even the model order is unknown.

Definition of model uncertainty. We assume the plant P is linear and time invariant, but that its exact mathematical description is unknown. However, it is known to be in a specified "neighborhood" of the "nominal" system, whose mathematical "model" $\tilde{P}$ is available. This neighborhood will be denoted the "uncertainty set"; it defines the "set of possible plants" Π. In some cases the uncertainty set Π may include a finite number of plants. However, in most cases we will define Π in terms of norm-bounded perturbations on $\tilde{P}$, and the set Π becomes infinite.

(2) *The effect of model uncertainty*

Before discussing how uncertainty limits the achievable performance (dynamic resilience), a digression on why feedback is used for control is of interest. Obviously, for stable plants in the absence of uncertainty, *feedforward* control would be sufficient. *Feedback* is used to control a plant despite unmeasured disturbances and model uncertainty. One particular example is the application of integral action in order to achieve perfect steady state control. Without exact knowledge of the steady state gain, perfect control may be achieved through feedback.

However, even though high gain feedback can be used to reduce the effect of uncertainty, it is intuitively obvious that there must be a limit to the extent that uncertainty can be tolerated before the system must be detuned and performance sacrificed. Thus uncertainty may impose limitations on the achievable performance (dynamic resilience). Here, quantitative effects of uncertainty on closed-loop performance will be found. First, additional terminology is required:

Performance. "Performance" is the quality of the closed-loop response. Typically, the error signal (e) should be small for the expected disturbances (d) and reference signals (r) (Fig. 1). The sensitivity function (S) describes the relationship between r, d and e

$$e = S(r-d), \quad S = (I+PC)^{-1}. \tag{1}$$

In order to have "good" performance, S has to be "small". In this paper, the magnitude of S is measured using the singular value $\bar{\sigma}$. At a given frequency ω, $\bar{\sigma}(S(j\omega))$ represents the "worst" amplification ($\|e\|_2/\|r-d\|_2$) of $(r-d)$. By "worst" we mean that $r-d$ is in the direction giving rise to the largest amplification. A typical *performance specification* is

$$\bar{\sigma}(S) \leq 1/|w_P| \,\forall\, \omega \tag{2}$$

where $w_P(s)$ is a weight which is used to define what responses are acceptable. The complementary sensitivity function H will also be used to measure performance. H is defined by

$$H = I - S$$

or

$$H = PC(I+PC)^{-1}. \tag{3}$$

H relates the output y to the reference signal r

$$y = Hr. \tag{4}$$

It is desirable to have $H \approx I$. $\bar{\sigma}(H) < 1$ at some frequency implies that tight control ($H \approx I$) is not possible. $\tilde{S}$ and $\tilde{H}$ are used to denote the nominal ($P = \tilde{P}$) sensitivity and complementary sensitivity functions.

Nominal stability (*N.S.*). The nominal closed loop system (with no uncertainty) is stable.

Nominal performance (*N.P.*). The nominal closed loop system $\tilde{S}$ (with no uncertainty) satisfies the performance specification (2).

Robust stability (*R.S.*). The closed loop system is stable for *all* plants in the "uncertainty set".

Robust performance (*R.P.*). The closed loop system satisfies the performance specification (2) for *all* plants in the "uncertainty set".

In the context of 'how' uncertainty affects performance, there are at least three problems of interest:

Problem 1. The effect of the robust *stability* requirement on *nominal performance*: How does the stability requirement for all plants in the uncertainty set limit the nominal performance?

Problem 2. The effect of the robust *performance* specification on *nominal performance*: If we specify that

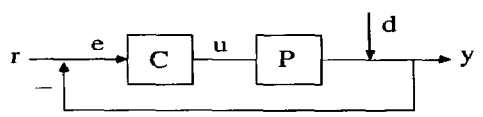

Fig. 1. Feedback system with controller C and plant P.

some particular performance requirement has to be satisfied for all plants in the uncertainty set, how does this bound the nominal performance?

Problem 3. Achievable *robust performance*: Design the best possible controller; what is the best achievable performance by *all* plants in the uncertainty set?

In Problem 1 and 2, a "lower bound" on *robust* performance is specified (for Problem 1 this "lower bound" is simply the requirement of stability), and we are considering effect on the *nominal* performance. The goal is to derive some simple bounds on the *nominal* system which, when satisfied, give the desired robust performance. These bounds are intended to assist the engineer in designing a controller for the nominal system such that the specified performance for all plants in the uncertainty set is achieved.

In Problem 3, there is no in particular concern for the performance of the nominal system. In this case, the problem is to find the "upper bound" on robust performance using any linear controller. This situation is addressed by Doyle (1984) and actually involves finding the optimal controller; this "μ-synthesis" is a complicated mathematical and numerical problem which will not be addressed here.

This paper will be concerned mainly with Problem 1. Problem 2 may formulate trivially as a special case of Problem 1 provided that the appropriate norm ($\bar{\sigma}$) is used to define performance (Doyle *et al.*, 1982; Doyle, 1984). Problem 1 is important in itself for the case when the plant is "operating" most of the time close to its nominal point, but with occasional plant perturbations. In this case performance may not be important when perturbations occur provided that the system remains stable. Furthermore, for Problem 1 it will be possible to derive reasonably simple bounds on the achievable nominal performance. Simplicity is desired in order for the engineer to gain insight into 'why' a particular design is sensitive to uncertainty.

Two approaches may be taken in order to find bounds on nominal performance imposed by robust stability (Problem 1):

1. A performance related transfer function which is to be bounded [for example, $\bar{\sigma}(\tilde{H})$ or $\bar{\sigma}(\tilde{S})$] is selected. This requires that the uncertainty be expressed in terms of a specific single perturbation ("unstructured" uncertainty) as discussed in Section III. The bounds derived using unstructured uncertainty are generally *conservative* since the actual uncertainty rarely "fits" into a single norm-bounded perturbation.
2. A reasonably "tight" description of the uncertainty is chosen. This is done by identifying more precisely where the uncertainty occurs in the system, such as by considering uncertainty in the model parameters. This generally leads to an uncertainty description with multiple perturbations (Δ_i's). By assuming norm bounds [e.g. $\bar{\sigma}(\Delta_i) \leqslant 1$] on these uncertainties, it is possible to derive *non-conservative* conditions for robust stability using the Structured Singular Value, μ. This approach is due to Doyle and coworkers (1982, 1984) and is considered in Section IV. One disadvantage of this procedure is that the resulting conditions are not in terms of a simple bound on $\bar{\sigma}(\tilde{H})$ or $\bar{\sigma}(\tilde{S})$, but involve $\mu(N)$ where N may be a complicated function of $\tilde{S}$ and $\tilde{H}$.

A number of conditions in this paper are stated as both necessary and sufficient for robust stability, but it is stressed that necessity is only meaningful if the assumed uncertainty is an accurate ("tight") description of the true uncertainty.

III. SINGLE PERTURBATIONS (UNSTRUCTURED UNCERTAINTY)

In this section, the uncertainty which occurs at different parts of the system will be lumped into one single perturbation L. In most cases this will correspond to "unstructured" uncertainty. (More precisely, "unstructured" uncertainty means that several sources of uncertainty are described with a *single* perturbation which is a "full" matrix of the *same size* as the plant P).

Let $P \in \Pi$ be any member of the set of possible plants Π, and let $\tilde{P} \in \Pi$ denote the nominal model of the plant. To describe unstructured uncertainty the following four single perturbations are commonly used: additive (L_A), multiplicative input (L_I), multiplicative output (L_O) and inverse multiplicative output (L_S) perturbations (Fig. 2)

$$P = \tilde{P} + L_A \text{ or } L_A = P - \tilde{P} \tag{5a}$$

$$P = \tilde{P}(I + L_I) \text{ or } L_I = \tilde{P}^{-1}(P - \tilde{P}) \tag{5b}$$

$$P = (I + L_O)\tilde{P} \text{ or } L_O = (P - \tilde{P})\tilde{P}^{-1} \tag{5c}$$

$$P = (I - L_S)^{-1}\tilde{P} \text{ or } L_S = (P - \tilde{P})P^{-1}. \tag{5d}$$

Additional examples are given by Doyle *et al.* (1982). The conditions derived for robust stability will be different depending on which single perturbation is chosen to describe the uncertainty.

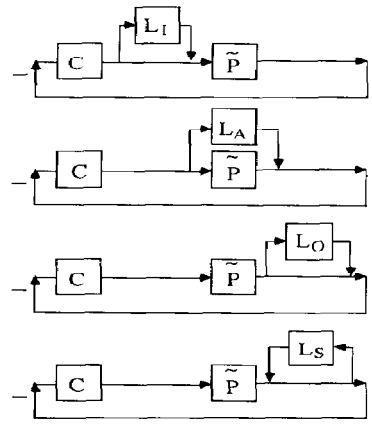

Fig. 2. Four common uncertainty descriptions involving single perturbations: input multiplicative uncertainty (L_I); additive uncertainty (L_A); output multiplicative uncertainty (L_O); output inverse multiplicative uncertainty (L_S).

(1) *Simple bounds on* $\bar{\sigma}(\tilde{H})$, $\bar{\sigma}(\tilde{H}_I)$ *and* $\bar{\sigma}(\tilde{S})$

In each of the cases above the magnitude of the perturbation L may be measured in terms of a bound on $\bar{\sigma}(L)$

$$\bar{\sigma}(L) \leq l(\omega) \quad \forall \omega \tag{6}$$

where

$$l(\omega) = \max_{P \in \Pi} \bar{\sigma}(L).$$

The bound $l(\omega)$ can also be interpreted as a scalar weight on a normalized perturbation $\Delta(s)$

$$L(s) = l(s)\Delta(s), \quad \bar{\sigma}(\Delta) \leq 1 \ \forall \omega. \tag{7}$$

The magnitude bound $l(\omega)$ will *not* generally constitute a tight description of the "real" uncertainty. This means that the set of plants satisfying (7) will be larger than the original set Π.

Output multiplicative uncertainty. The sensitivity function S has to be stable for all $P \in \Pi$. Using the identity

$$S = \tilde{S}(I + (P - \tilde{P})\tilde{P}^{-1}\tilde{H})^{-1} = \tilde{S}(I + L_O\tilde{H})^{-1} \tag{8}$$

and the Nyquist stability condition, the following robust stability bound is derived.

Theorem 1. *Bound in terms of* $\bar{\sigma}(\tilde{H})$ (*Doyle and Stein, 1981; Postlethwaite and Foo, 1985*) Assume the nominal system is closed loop stable, that is, assume in particular that $\tilde{H}$ is stable. Let Π be *any* set of plants such that P and $\tilde{P}$ have the same number of RHP (unstable) poles. Then robust stability is guaranteed

$$\text{if} \quad \bar{\sigma}(\tilde{H}) \leq 1/l_O(\omega) \quad \text{where} \quad l_O(\omega) = \max_{P \in \Pi} \bar{\sigma}(L_O) \tag{9}$$

[Condition (9) is necessary and sufficient for robust stability if it is assumed that *all* plants satisfying $\bar{\sigma}(L_O) \leq l_O(\omega)$ actually occur (Doyle and Stein, 1981).]

The robust stability condition (9) can always be satisfied for open loop stable systems since $\tilde{H} = 0$ (no feedback) is always possible. However, good disturbance rejection and good command following require $\tilde{H} \approx I$ (i.e. $\bar{\sigma}(\tilde{H}) \approx 1$). Condition (9) says that the system has to be "detuned" ($\bar{\sigma}(\tilde{H}) < 1$) at frequencies where $l_O(\omega) \geq 1$. This is reasonable since $l_O(\omega) > 1$ for some ω implies that the plant can have zeros on both sides of the imaginary axis; it is well known that RHP zeros limit the achievable performance.

Input multiplicative uncertainty. In this case a theorem similar to Theorem 1 is obtained, but with $\tilde{H}$ replaced $\tilde{H}_I$ (Postlethwaite and Foo, 1985):

$$\text{R.S.} \quad \text{if} \quad \bar{\sigma}(\tilde{H}_I) \leq 1/l_I(\omega), \quad l_I(\omega) = \max_{P \in \Pi} \bar{\sigma}(L_I) \tag{10}$$

where

$$\tilde{H}_I = C(I + \tilde{P}C)^{-1}\tilde{P} = \tilde{P}^{-1}\tilde{H}P. \tag{11}$$

$\tilde{H}_I$ is the nominal closed-loop transfer function as seen from the *input* of the plant. It is desirable to have this transfer function close to I in order to reject disturbances affecting the inputs to the plant. However, since performance is usually measured at the output of the plant it may be of interest to use (10) in order to derive a bound in terms of $\tilde{H}$. To derive this bound the inequality

$$\bar{\sigma}(\tilde{H}_I) = \bar{\sigma}(\tilde{P}^{-1}\tilde{H}\tilde{P}) \leq \bar{\sigma}(\tilde{P}^{-1})\bar{\sigma}(\tilde{H})\sigma(\tilde{P}) = \gamma(\tilde{P})\bar{\sigma}(\tilde{H})$$

is used; the bound is:

$$\text{R.S.} \quad \text{if} \quad \bar{\sigma}(\tilde{H}) \leq \frac{1}{\gamma(\tilde{P})}\frac{1}{l_I(\omega)} \ \forall \omega. \tag{12}$$

Here $\gamma(\tilde{P}) = \bar{\sigma}(\tilde{P})/\sigma(\tilde{P})$ is the condition number of the plant. Equation (12) has been used to introduce the condition number as a *stability* sensitivity measure with respect to input uncertainty (Morari, 1983), but this is misleading. The condition number enters the stability condition (12) mainly as the result of the conservative step introduced by going from an input [eq. (10)] to an output uncertainty description [eq. (12)]. For $\gamma(\tilde{P})$ large, (12) may be arbitrarily conservative even though the uncertainty is tightly described in terms of a norm-bounded input uncertainty such that (10) is both necessary and sufficient. However, even though (12) is misleading and the system is stable, input uncertainty usually *does* cause control problems when $\gamma(\tilde{P})$ is large. As shown by Morari and Doyle (1986), *robust performance* (measured at the output of the plant) may be poor in such cases (even though the *nominal* performance may be excellent). Output uncertainty does not lead to the same performance problems, and this indicates why input uncertainty is of more concern than output uncertainty for ill-conditioned plants.

Inverse multiplicative output uncertainty. Using the identity

$$S = \tilde{S}(I - L_S\tilde{S})^{-1}\tilde{P}P^{-1} \tag{13}$$

and the inverse Nyquist stability condition, the following theorem may be derived.

Theorem 2. *Bound in terms of* $\bar{\sigma}(\tilde{S})$ (*Postlethwaite and Foo, 1985*) Assume the nominal system is closed loop stable, that is, assume in particular that $\tilde{S}$ is stable. Let Π be *any* set of plants such that P and $\tilde{P}$ have the same number of RHP zeros. Then robust stability is guaranteed if

$$\bar{\sigma}(\tilde{S}) \leq \frac{1}{l_S(\omega)} \quad \text{where} \quad l_S(\omega) = \max_{P \in \Pi} \bar{\sigma}(L_S). \tag{14}$$

For minimum phase systems (no time delays or RHP zeros), the nominal sensitivity function $\tilde{S}$ may be arbitrary small ("perfect control") and (14) can always be satisfied. Therefore, condition (14) seems to imply that for minimum phase systems arbitrarily good performance ($\tilde{S}$ small) is possible regardless of how large the uncertainty is. This is claimed by Postlethwaite, but is not quite true. The pitfall is that any real system has to be strictly proper, and $S = I$ and $\tilde{S} = I$ as $\omega \to \infty$ must be required. Consequently, to satisfy (14) it is necessary that $\bar{\sigma}(L_S) = \bar{\sigma}((P - \tilde{P})P^{-1})$

$\leqslant 1$ as $\omega \to \infty$ for all possible P. This condition is generally violated in practice, because the order of the actual plant is higher than that of the model.

Theorems 1 and 2 prescribe two fundamentally different ways of handling uncertainty: To guarantee robust stability Theorem 1 prescribes that the system be detuned (low gain), while Theorem 2 prescribes that the control be tightened (high gain). In practice, it is desirable to combine the two approaches: By tightening the control at low frequencies better performance is obtained. Eventually, at higher frequencies, the system has to be detuned to guarantee robust stability. In fact, Postlethwaite and Foo (1985) has shown that it is possible to combine Theorem 1 and 2 over different frequency ranges. However, the bounds are still conservative since there is no "tight" description of the uncertainty. A better approach is to derive tighter uncertainty descriptions in the first place and then derive robust stability bounds (Section IV).

(2) *Input uncertainty for distillation column (example)*

Conditions (9) and (10) indicate that the system has to be detuned such that $\bar{\sigma}(\tilde{H}) < 1/l_O(\omega)$ [or $\bar{\sigma}(\tilde{H}_I) < (1/l_I(\omega))$] in order to guarantee robust stability. However, because of the conservativeness introduced by using unstructured uncertainty, these conditions are generally only sufficient for robust stability; the detuning indicated may be much larger than what is actually necessary. This is illustrated conveniently through an example.

Consider the distillation column described in Table 1 where the overhead composition is to be controlled at $y_D = 0.99$ and the bottom composition at $x_B = 0.01$ using the distillate D and boilup V as manipulated inputs. By linearizing the nonlinear model at steady state and by assuming that the dynamics may be approximated by a first order response with time constant $\tau = 75$ min, the following linear model is derived (Skogestad and Morari, 1986):

$$\tilde{P} = \frac{1}{\tau s + 1}\begin{bmatrix} -0.878 & 0.014 \\ -1.082 & -0.014 \end{bmatrix}.$$

A simple decentralized control system with two PI controllers is chosen

$$C(s) = \frac{1+\tau s}{s}\begin{bmatrix} -0.15 & 0 \\ 0 & -7.5 \end{bmatrix}. \tag{15}$$

This controller gives acceptable nominal performance, and can be shown [Section IV(2)] to give robust stability when there is relative uncertainty of magnitude $w_I(s)$ on each manipulated variable:

$$w_I(s) = 0.2\frac{5s+1}{0.5s+1}.$$

This implies a relative uncertainty of up to 20% in the low frequency range which increases at high frequencies, reaching a value of 1 at $\omega \approx 1\ \text{min}^{-1}$. This increase with frequency allows for a time delay of about 1 min, and may represent the effect of the flow dynamics which were neglected when developing the model. This relative uncertainty can be written in terms of two *scalar* multiplicative perturbations Δ_D and Δ_V.

$$\begin{aligned} \mathrm{d}D &= (1 + w_I(s)\Delta_D)\,\mathrm{d}D_c, \quad |\Delta_D| \leqslant 1 \ \forall\omega \\ \mathrm{d}V &= (1 + w_I(s)\Delta_V)\,\mathrm{d}L_c, \quad |\Delta_V| \leqslant 1 \ \forall\omega \end{aligned} \tag{16}$$

($\mathrm{d}D$ and $\mathrm{d}V$ are the actual inputs, while $\mathrm{d}D_c$ and $\mathrm{d}V_c$ are the desired values of the flow rates as computed by the controller). Equation (16) can be approximated by an "unstructured" single perturbation $L_I = w_I\Delta_I$ (Δ_I is a "full" 2×2 matrix)

$$\begin{bmatrix} \mathrm{d}D \\ \mathrm{d}V \end{bmatrix} = (I + w_I(s)\Delta_I)\begin{bmatrix} \mathrm{d}D_c \\ \mathrm{d}V_c \end{bmatrix}, \quad \bar{\sigma}(\Delta_I) \leqslant 1\ \forall\omega \tag{17}$$

with $l_I(\omega) = |w_I(j\omega)|$. Equation (10) indicates that robust stability is guaranteed if $\bar{\sigma}(\tilde{H}_I) \leqslant 1/l_I(\omega)\ \forall\omega$. However, from Fig. 3 it is seen that this condition is violated over a wide frequency range, despite the fact that the system is known to be robustly stable. The

Table 1. Data for distillation column example

Binary separation, constant molar flows, feed liquid	
Relative volatility	$\alpha = 1.5$
No. of theoretical trays	$N = 40$
Feed tray location	$N_F = 21$
Feed rate and composition	$F = 1$ kmol/min, $z_F = 0.5$
Product compositions	$y_D = 0.99$, $x_B = 0.01$
Product rates	$D = B = 0.5$ kmol/min
Computed from steady state model	
Reflux rate	$L = 2.71$ kmol/min ($1.39\ L_{\min}$)

Linearized steady state gains

LV-configuration:

$$\begin{bmatrix} \mathrm{d}y_D \\ \mathrm{d}x_B \end{bmatrix} = \begin{bmatrix} 0.878 & -0.864 \\ 1.082 & -1.096 \end{bmatrix}\begin{bmatrix} \mathrm{d}L \\ \mathrm{d}V \end{bmatrix}$$

DV-configuration:

$$\begin{bmatrix} \mathrm{d}y_D \\ \mathrm{d}x_B \end{bmatrix} = \begin{bmatrix} -0.878 & 0.014 \\ -1.082 & -0.014 \end{bmatrix}\begin{bmatrix} \mathrm{d}D \\ \mathrm{d}V \end{bmatrix}$$

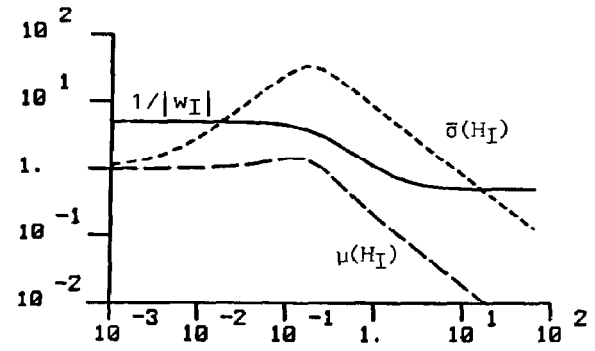

Fig. 3. Robust stability for the distillation column with diagonal input uncertainty is guaranteed since $\mu(H_I) \leqslant 1/|w_I|\ \forall\omega$. The use of unstructured uncertainty and $\bar{\sigma}(H_I)$ is conservative, and would require the system to be detuned to guarantee robust stability.

reason for the conservativeness of condition (10) in this instance is that the use of unstructured uncertainty (17) includes plants not included in the "true" uncertainty description (16). These problems may be avoided by using the structured singular value $\mu(\tilde{H}_I)$ as discussed in Section IV(2).

(3) *Integral control and robust stability*

Because of the importance of integral control in the context of process control we will derive specifically conditions under which controllers with integral action can be designed in the presence of uncertainty. We will keep the uncertainty as general as possible. To this end define Π_A as the set of plants which is generated by a single weighted additive norm bounded perturbation (Fig. 4)

$$\Pi_A = \{P: P = \tilde{P} + L_A\}, \quad L_A = W_2 \Delta_A W_1, \quad \bar{\sigma}(\Delta_A) \leq 1, \forall \omega. \tag{18}$$

This is a generalization of (5) and (7) because the weights W_1 and W_2 are allowed to be matrices. (Note, for example, that the *multiplicative* input uncertainty $L_I = l_I \Delta_I$, may be written in this *additive* form by choosing $W_2 = \tilde{P} l_I$, $W_1 = I$.)

A necessary and sufficient condition for "perfect control" and robust stability will be stated first. Note that "perfect control" ($\tilde{H} = I$, $\forall \omega$) is clearly not possible for real systems which must be strictly proper (i.e. $\tilde{H} \to 0$ as $\omega \to \infty$), but the notion of "perfect control" is nevertheless useful.

Theorem 3. *Perfect control* ($\Pi = \Pi_A$). Assume $\tilde{P}$ is minimum phase and that all plants have the same number of unstable poles. Robust stability and "perfect control" ($\tilde{H} = I$) may be achieved

$$\text{iff} \quad \det(P\tilde{P}^{-1}) \neq 0 \ \forall \omega, \ \forall P \in \Pi_A. \tag{19}$$

This theorem implies that perfect control is possible if and only if none of the plants P in the set Π_A have zeros on the $j\omega$-axis (i.e. $\det P \neq 0$). The necessity of condition (19) is obvious since perfect control ($S = \tilde{S} = 0$) is never possible for plants with RHP zeros. Regarding sufficiency, it is clear from Theorem 2 that for minimum phase plants, perfect control is always possible in principle. The search for zeros is restricted to the $j\omega$-axis as the result of the particular norm bounded uncertainty assumed; it implies that zeros cannot appear in the RHP without crossing the $j\omega$-axis. Theorem 3 offers little that is new; it is stated mainly as a means to prove Theorem 7 in Section V. The following conditions for integral control are more interesting.

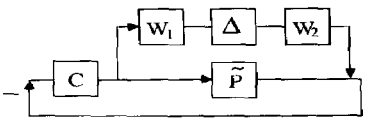

Fig. 4. System with weighted additive uncertainty. Rearranging this system to fit Fig. 6 gives $M = W_1 C(I + \tilde{P}C)^{-1} W_2$.

Theorem 4*A*. *Integral control.* Let Π be *any* set of plants such that P and $\tilde{P}$ have the same number of RHP (unstable) poles. Also, assume that PC and $\tilde{P}C$ are strictly proper. Then robust stability and integral control ($\tilde{H}(0) = I$) may be achieved

$$\text{only if} \quad \det(P(0)\tilde{P}(0)^{-1}) > 0 \ \forall P \in \Pi. \tag{20}$$

Theorem 4A implies that for stable plants, integral control is never possible if the sign of the plant, expressed in terms of $\det P(0)$, changes. This is a direct generalization of the result for SISO systems. Note that Theorem 4A does not apply to cases where the pole may cross the $j\omega$-axis. As an example, let $\tilde{P} = 1/(s + a)$ and $P = 1/(s - a)$ ($a > 0$). These plants may be stabilized using a single controller with integral action (e.g. $C = k(s + a)/s$, $k > a$) despite of the fact that they do not satisfy condition (20).

For the special case when Π is of the norm-bounded form Π_A (18), condition (20) is both necessary and sufficient:

Theorem 4*B*. *Integral control* ($\Pi = \Pi_A$). Assume all plants $P \in \Pi_A$ are stable and that PC and $\tilde{P}C$ are strictly proper. Then robust stability and integral control may be achieved

$$\text{iff} \quad \det(P(0)\tilde{P}(0)^{-1}) > 0 \ \forall P \in \Pi_A. \tag{21}$$

The sufficiency of condition (21) follows mainly from the assumed norm-bounded additive perturbation, and also from the fact that at frequencies $\omega \neq 0$, the robust stability condition [similar to (10)] may always be satisfied by detuning the system (provided the plant is stable). Note that condition (21) does *not* imply robust stability if Π is not on the form Π_A. As an example, consider the set Π consisting of the two plants $\tilde{P} = \text{diag}\{1, 1\}$ and $P = \text{diag}\{-1, -1\}$. Since this corresponds to two SISO plants where the gains change sign, integral control is not possible.

IV. MULTIPLE PERTURBATIONS ("STRUCTURED" UNCERTAINTY)

In this section, we will describe the uncertainty in a "structured" manner by actually trying to identify the sources and locations of uncertainty in the system. This usually leads to an uncertainty description with multiple perturbations (Δ_i). These perturbations may correspond to uncertainty in the model parameters, uncertainty with respect to the manipulated variables (input or actuator uncertainty) and the outputs (measurement uncertainty), etc. By using such a mechanistic approach, we can norm-bound each perturbation (e.g. $\|\Delta_i\| < 1$) without introducing too much additional conservativeness and get a "tight" description of the uncertainty set.

However, we should not necessarily describe the uncertainty as rigorously as possible. Rather, we should take the engineer's approach and describe the uncertainty as rigorously as necessary. This means some of the sources of uncertainty (occurring at different places of the system) should be lumped into a "un-

structured" multiplicative perturbation, for example, if this does not add too much conservativeness. This leads to a *practical uncertainty description*: some sources of uncertainty are described in a "structured" manner (e.g. parametric uncertainty), while the rest (usually uncertain high-frequency dynamics) is lumped into a single "unstructured" perturbation (see reactor example below).

The main objective in this section is to familiarize the reader with the work of Doyle (1982, 1984). The results are presented without further motivation; subsequent examples illustrate how these results may be used in practice.

(1) *General theory*

Consider the uncertainty as perturbations on the nominal system. Each perturbation Δ_i is assumed to be a stable and *norm-bounded* transfer matrix

$$\bar{\sigma}(\Delta_i) < 1 \ \forall \omega. \tag{22}$$

Weighting matrices are used to normalize the uncertainty such that the bound is one at all frequencies; that is, the actual perturbation L_i is written

$$L_i = W_2 \Delta_i W_1. \tag{23}$$

If Δ_i represents a real parameter variation we may restrict Δ_i to be real, but in general Δ_i may be any stable rational transfer matrix satisfying (22). The choice of the singular value $\bar{\sigma}$ as the norm for bounding Δ_i is not arbitrary, but is needed to obtain the necessity in the theorems which follow.

The perturbations (uncertainties) which may occur at different places in the feedback system (e.g. Fig. 5), can be collected and placed into one large block diagonal perturbation matrix

$$\Delta = \text{diag}\{\Delta_1, \ldots, \Delta_n\} \tag{24}$$

for which we have

$$\bar{\sigma}(\Delta) < 1 \ \forall \omega. \tag{25}$$

The blocks Δ_i in (24) can have any size and may also be repeated. For example, repetition is needed in order to handle correlations between the uncertainties in different elements. The nominal closed-loop system with no uncertainty ($\Delta = 0$) is assumed to be stable. The perturbations (uncertainty) give rise to stability problems because of the "additional" feedback paths created by the uncertainty. This is shown explicitly by writing the uncertainty as perturbations on the nominal system in the form ($M\Delta$-structure) shown in Fig. 6. M is the nominal closed-loop system "as seen from" the various uncertainties, and is stable since the nominal system is assumed stable. More precisely, M is

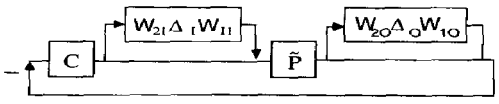

Fig. 5. System with weighted multiplicative input and output uncertainty. Rearranging this system to fit Fig. 6 gives M as in eq. (29).

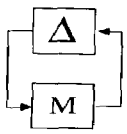

Fig. 6. Interconnection structure for studying effect of uncertainty on stability. $\Delta = \text{diag}\{\Delta_1, \ldots, \Delta_n\}$.

the interconnection matrix giving the nominal transfer functions from the output of the perturbations Δ_i to their inputs. Constructing M is conceptually straightforward, but may be tedious for specific problems.

We want to derive conditions on M in order to guarantee *robust stability*. It may be shown (Doyle *et al.*, 1982) that for a nominally ($\Delta = 0$) stable system, robust stability is equivalent to the stability of the $M\Delta$-structure in Fig. 6. This system is stable if and only if $\det(I + \Delta M)$ does not encircle the origin as s traverses the Nyquist D contour for all possible Δ. Because the perturbations are norm bounded [i.e. all Δ's satisfying (25) are allowed] this is equivalent to

$$\begin{aligned} &\det(I + \Delta M) \neq 0 \quad \forall \omega, \ \forall \Delta, \ \bar{\sigma}(\Delta) < 1 \\ &\Leftrightarrow \rho(\Delta M) \leqslant 1 \quad \forall \omega, \ \forall \Delta, \ \bar{\sigma}(\Delta) < 1. \end{aligned} \tag{26}$$

Condition (26) by itself is not very useful since it is only a yes/no condition which must be tested for all possible perturbations Δ. What is desired is a condition on the matrix M, preferably on some norm of M. This is supplied by the following theorem.

Theorem 5. Necessary and sufficient condition for robust stability (Doyle *et al.*, 1982). Assume the nominal system ($\Delta = 0$) is stable. Then the closed loop system (Fig. 6) is stable for all Δ, $\bar{\sigma}(\Delta) < 1$ if and only if

$$\mu(M) \leqslant 1 \ \forall \omega. \tag{27}$$

Theorem 5 may be interpreted as a "generalized small gain theorem" applied to (26) which also takes the *structure* of Δ into account. The function μ, called the *Structured Singular Value* (SSV), is defined in order to get the tightest possible bound on M such that (26) is satisfied. A more precise definition of μ and some of its properties are given in Appendix 1. It is important to note that $\mu(M)$ depends *both* on the matrix M *and* on the *structure* of the perturbations Δ. $\mu(M)$ is a generalization of the spectral radius $\rho(M)$ and the maximum singular value $\bar{\sigma}(M)$ in that $\mu(M) = \rho(M)$ when the perturbation Δ is totally structured ($\Delta = \delta I$, $|\delta| \leqslant 1$), and $\mu(M) = \bar{\sigma}(M)$ when the perturbation is unstructured (Δ is a full matrix). Note that the matrix M is a function of the *nominal* system only, and the condition $\mu(M) \leqslant 1$ limits the possible nominal transfer functions.

At this point, it is not apparent that the uncertainty description (22)–(25), does indeed provide a useful framework for handling uncertainty. Furthermore, it is not clear how to find the matrix M. Hopefully this will become clearer through the examples below.

(2) *Input uncertainty for distillation column (example)*

It is now possible to derive a less conservative robust stability test for the distillation column example. Previously, we assumed that the input uncertainty was unstructured, but now Δ_I in (17) may be restricted to be a *diagonal* matrix which results in a tight description of the uncertainty. The interconnection matrix $M = w_I(s)\tilde{H}_I$ and, from Theorem 5,

$$\text{R.S.} \quad \text{iff} \quad \mu(\tilde{H}_I) \leqslant 1/|w_I(j\omega)| = 1/l_I(\omega) \; \forall \omega$$

where $\mu(\tilde{H}_I)$ is computed with respect to the diagonal matrix Δ_I. From Fig. 3 we see that this condition is satisfied and robust stability is guaranteed with the chosen controller (15).

(3) *Simultaneous input and output multiplicative uncertainty*

Consider the system in Fig. 5 which has both input and output multiplicative uncertainty with respect to the model of the plant $\tilde{P}$. The possible plants are given by

$$\begin{aligned} P &= (I + L_O)\tilde{P}(I + L_I) \\ L_I &= W_{2I}\Delta_I W_{1I}, \quad \bar{\sigma}(\Delta_I) < 1 \; \forall \omega \\ L_O &= W_{2O}\Delta_O W_{1O}, \quad \bar{\sigma}(\Delta_O) < 1 \; \forall \omega. \end{aligned} \tag{28}$$

The perturbation block Δ_I represents the multiplicative input uncertainty. If its source is uncertainty with respect to the manipulated variables, then

$$\Delta_I: \quad \text{diagonal}, \; W_{1I} = \text{diag}\{w_{Ii}\}, \quad W_{2I} = I$$

where w_{Ii} represents the relative uncertainty on each manipulated input.

The block Δ_O represents the multiplicative output uncertainty. If its source is uncertainty or neglected deadtimes involved in one or more of the measurements, then

$$\Delta_O: \text{ diagonal}, \; W_{1O} = \text{diag}\{w_{Oi}\}, \; W_{2O} = I.$$

w_{Oi} represents the relative uncertainty for each measurement. These sources of input and output uncertainty are present in any plant. Δ_I and Δ_O are restricted to be *diagonal* matrices, since there is little reason to assume that the actuators or measurements influence each other. However, some of the unmodelled dynamics in the plant $\tilde{P}$ itself, which has cross terms, may be approximated by lumping them into Δ_I or Δ_O, thus making either one of them a "full" matrix.

To examine the constraints on the nominal system imposed by the robust stability requirement for this uncertainty description, let $\Delta = \text{diag}\{\Delta_I, \Delta_O\}$ and rearrange the system in Fig. 5 into the form in Fig. 6. The interconnection matrix M becomes:

$$\begin{aligned} M &= \begin{bmatrix} -W_{1I}C\tilde{P}(I + C\tilde{P})^{-1}W_{2I} & -W_{1I}C(I + \tilde{P}C)^{-1}W_{2O} \\ W_{1O}\tilde{P}(I + C\tilde{P})^{-1}W_{2I} & -W_{1O}\tilde{P}C(I + \tilde{P}C)^{-1}W_{2O} \end{bmatrix} \\ &= \begin{bmatrix} W_{1I} & \\ & W_{1O} \end{bmatrix} \begin{bmatrix} -\tilde{P}^{-1}\tilde{H}\tilde{P} & -\tilde{P}^{-1}\tilde{H} \\ \tilde{S}\tilde{P} & -\tilde{H} \end{bmatrix} \begin{bmatrix} W_{2I} & \\ & W_{2O} \end{bmatrix} \end{aligned} \tag{29}$$

and robust stability is guaranteed for all Δ such that $\bar{\sigma}(\Delta) < 1$

$$\text{iff} \quad \mu(M) \leqslant 1, \forall \omega.$$

μ is computed with respect to the structure of Δ which in turn depends on the structure assumed for Δ_I and Δ_O. Note that conditions (9) and (10) in Section IV are special cases of (29) when the weights are assumed to be scalar, Δ_I and Δ_O are "full" matrices, and either $\Delta_I = 0$ or $\Delta_O = 0$. However, this only applies to stable Δ's; an unstable Δ_O (or Δ_I) may be allowed in (9) [or (10)], while only stable Δ's were allowed when deriving (29).

(4) *Simultaneous parametric and unstructured uncertainty (reactor example)*

Consider a perfectly mixed batch reactor where an exothermic reaction is taking place. The reaction temperature T is controlled using the temperature T_c of the fluid in the cooling jacket (the fluid in the cooling jacket may be boiling, and T_c may be adjusted by changing the pressure). A heat balance for the batch reactor gives

$$C_p\dot{T} = (-\Delta H_r)r - UA(T - T_c)$$

where

T: reactor temperature (K)
T_c: coolant temperature (K)
r: reaction rate (function of T) (mol/s)
ΔH_r: heat of reaction (negative constant) (J/mol)
C_p: total heat capacity of fluid in reactor (J/K)
UA: overall heat transfer coefficient (J/s K).

Linearizing the reaction rate

$$r = r^0 + k_T \mathrm{d}T$$

results in a linear transfer function from T_c to T

$$\mathrm{d}T(s) = \frac{UA/C_p}{s + a}\mathrm{d}T_c(s) \tag{30}$$

where

$$a = \frac{UA - (-\Delta H_r)k_T}{C_p}. \tag{31}$$

Two sources of uncertainty will be considered for the linear model (30):

(1) The effect of nonlinearity expressed as uncertainty in the pole location a.
(2) Neglected high-frequency dynamics.

Pole uncertainty (Δ_s). Most of the terms in (31) are nearly constant, except for $k_T = \partial r/\partial T$ which is a strong function of temperature (operating point). From (31) we see that the reactor may be open loop stable ($a > 0$) at low temperatures where k_T is small, and unstable at high temperatures where the reaction is more temperature sensitive. To describe the effect temperature has on a, let

$$|a - \tilde{a}| < r_a \tilde{a}$$

where $\tilde{a}$: nominal pole location,
r_a: relative "uncertainty" in a (real constant).

If $r_a > 1$, the plant may change between stability and unstability. Equivalently, the possible a's may be written in terms of a norm-bounded perturbation Δ_s

$$a = \tilde{a}(1 + r_a\Delta_s), \quad |\Delta_s| < 1, \quad \Delta_s \text{ real} \qquad (32)$$

and this may be written as an inverse multiplicative perturbation $(I + w_s\Delta_s)^{-1}$ on the plant

$$\frac{1}{s+a} = \frac{1}{s+\tilde{a}} \frac{1}{1 + w_s(s)\Delta_s}, \quad w_s(s) = \frac{r_a}{1 + s/\tilde{a}}. \qquad (33)$$

Neglected dynamics [*modelled as unstructured output uncertainty* (Δ_O)]. A plant always has some unknown uncertainty, mainly at higher frequencies, which cannot be modelled in a "structured" manner (using parametric uncertainty, etc.). These sources are most conveniently modelled as multiplicative uncertainty; in this case we choose to use output multiplicative uncertainty $(I + w_O\Delta_O)$. Physically, this uncertainty may include neglected (and unknown) dynamics for changing the cooling temperature T_c (if T_c is manipulated indirectly with pressure), neglected actuator dynamics (the valve used to control pressure) and neglected dynamics introduced by the heat capacity of the walls. A conservative choice for $w_O(s)$ is found by approximating the neglected dynamics as an effective time delay, and choosing $|w_O(j\omega)| \approx 1$ at the frequency where the phase lag represented by the neglected dynamics reaches 60° ($|1 - e^{j\theta}| = 1$ for $\theta = 60°$).

A block diagram representation of the uncertainty is depicted in Fig. 7. Note that both blocks (Δ_s and Δ_O) are in general needed: We *cannot* lump the pole uncertainty (Δ_s) into the output uncertainty (Δ_O) if the pole is allowed to cross the $j\omega$-axis. This would result in $|w_O(j\omega)| \to \infty$ at $\omega = 0$. Similarly, we *cannot* lump the output uncertainty into the pole uncertainty. The reason is that the inverse multiplicative uncertainty description (Δ_s) cannot be used to model neglected or uncertain RHP zeros (this would require an unstable perturbation Δ_s). It is therefore not suited for handling neglected high frequency dynamics which most certainly include RHP zeros (one simple example is the "dead band" on any valve).

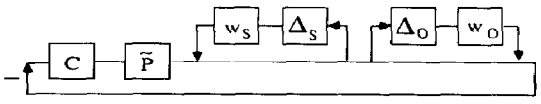

Fig. 7. Block diagram representation of uncertainty for reactor example. Δ_O represents the neglected (and uncertain) high-frequency dynamics. Δ_s represents the pole uncertainty (changes between stability and instability are possible if $|w_s(j\omega)| > 1$ at some frequency). Rearranging this system to fit Fig. 6 gives M as in eq. (34).

Combining the two scalar perturbations into one block perturbation $\Delta = \text{diag}\{\Delta_s, \Delta_O\}$ and rearranging Fig. 7 to match Fig. 6 gives the following interconnection matrix:

$$M = \begin{bmatrix} w_s\tilde{S} & -w_O\tilde{H} \\ w_s\tilde{S} & -w_O\tilde{H} \end{bmatrix}. \qquad (34)$$

From Theorem 5, robust stability is guaranteed

$$\text{iff} \quad \mu(M) \leqslant 1 \; \forall \omega$$

$$\text{or} \qquad \text{iff} \quad |w_s\tilde{S}| + |w_O\tilde{H}| \leqslant 1 \forall \omega. \qquad (35)$$

Because of the identity $\tilde{H} + \tilde{S} = 1$, this bound is *impossible* to satisfy if $|w_s|$ and $|w_O|$ are both "large" (that is, close to one or larger) over the same frequency range. For $r_a > 1$ the pole may cross the $j\omega$-axis, and $|w_s| > 1$ for $\omega < \omega^* = \tilde{a}\sqrt{r_a^2 - 1}$ and $|w_s| < 1$ for $\omega > \omega^*$. In that situation, robust stability is guaranteed only if the "unstructured" relative uncertainty given in terms of $|w_O(j\omega)|$ reaches one at a frequency *higher* than ω^*.

If pole uncertainty were the only source of uncertainty ($w_O = 0$), the robust stability bound would be $|\tilde{S}| \leqslant 1/|w_s|$. Since the plant is minimum phase, this bound could always be satisfied by increasing the gain and making $\tilde{S}$ small, regardless of the size of r_a.

In summary, the uncertainty regarding the pole location is handled by "tightening" the control at low frequencies. Indeed, $\tilde{S}$ small ("tight" control) is needed in order to stabilize an unstable plant. However, to realize robust stability in face of the uncertain high-frequency dynamics, it is necessary to detune the system and make $\tilde{H}$ small ($\tilde{S} \approx I$) at frequencies where $w_O(\omega)$ is larger than one. The implication of this result for process design is that we cannot stabilize an unstable plant if there are RHP zeros or model uncertainty in the same frequency range as the location of the unstable pole.

(5) *Independent uncertainty in the transfer matrix elements*

In many cases the uncertainty is most easily described in terms of uncertainties on the individual transfer matrix elements. This kind of uncertainty description may arise from an experimental identification of the system. *No* claim is made that this uncertainty description is a good representation of how the uncertainty actually occurs, but it is included as a possibly useful description in some cases.

The simplest form of element uncertainty arises from the assumption that each element p_{ij} in the plant P is independent, but confined to a disk with radius

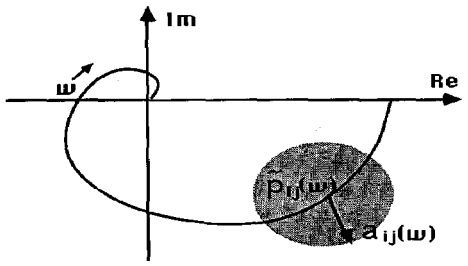

Fig. 8. Additive element uncertainty: $|p_{ij} - \tilde{p}_{ij}| < a_{ij}(\omega)$. The disc represents the set of possible $p_{ij}(j\omega)$ at a given frequency.

$a_{ij}(\omega)$ around $\tilde{p}_{ij}$ in the Nyquist plane (Fig. 8), i.e.

$$|p_{ij} - \tilde{p}_{ij}| < a_{ij}(\omega) \; \forall \omega. \tag{36}$$

This corresponds to treating each element as an independent SISO plant with additive uncertainty of size $a_{ij}(\omega)$. Multiplicative (relative) uncertainty r_{ij} on the elements may also be written in the form (36) by using

$$a_{ij}(\omega) = r_{ij}(\omega)|\tilde{p}_{ij}|. \tag{37}$$

The two main limitations of the uncertainty description (36) are

(1) The disk shape is potentially conservative, such as for pure time delay error.
(2) Correlations between the elements cannot be handled [potentially *very* conservative as shown for the distillation example in Section V(3)].

Defining the complex perturbation, Δ_{ij}, (36) becomes

$$p_{ij} - \tilde{p}_{ij} = \Delta_{ij} a_{ij}, \quad |\Delta_{ij}| < 1. \tag{38}$$

Or equivalently, on matrix form

$$P - \tilde{P} = \begin{bmatrix} \Delta_{11}a_{11} & \Delta_{12}a_{12} & \cdots \\ \Delta_{21}a_{21} & & \\ & & \Delta_{nn}a_{nn} \end{bmatrix}. \tag{39}$$

Introducing weighting matrices E and L it is possible to rewrite (39) in terms of the "large" diagonal perturbation matrix Δ_E

$$P - \tilde{P} = E\Delta_E L \tag{40}$$

where $E \in R^{n \times n^2}$, $L \in R^{n^2 \times n}$ and $\Delta_E \in C^{n^2 \times n^2}$ are defined as

$$E = [I \; I \; \ldots \; I], \quad L = \begin{bmatrix} \mathbf{a}_1 & & \\ & \mathbf{a}_2 & \\ & & \ddots \\ & & & \mathbf{a}_n \end{bmatrix}, \quad \mathbf{a}_i = \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ni} \end{bmatrix} \tag{41}$$

$$\Delta_E = \text{diag}\{\Delta_{11}, \Delta_{21}, \ldots, \Delta_{nn}\}, \quad |\Delta_{ij}| < 1.$$

A block diagram representation of (40) is given by Fig. 4 with $W_2 = E$ and $W_1 = L$. This system may be rearranged into the form in Fig. 6 with the interconnection matrix $M = LC(I + \tilde{P}C)^{-1}E = L\tilde{P}^{-1}\tilde{H}E$. From Theorem 5 follows the *necessary and sufficient* condition for robust stability:

$$\mu(L\tilde{P}^{-1}\tilde{H}E) \leq 1 \; \forall \omega \tag{42}$$

where μ is computed with respect to the diagonal matrix Δ_E. In principle, this condition may be used to generate all nominal closed loop transfer matrices $\tilde{H}$ for which the closed loop system is robustly stable. Alternatively, it may be used to check whether a particular design meets the robust stability requirement. However, at the design stage (when dynamic resilience is to be determined), $\tilde{H}$ is not known, but rather the restrictions on $\tilde{H}$ as imposed by the uncertainty are of interest. In order to obtain an explicit bound on $\tilde{H}$ from (42), assume that $\tilde{H} = \tilde{h}I$, that is, assume the nominal response is decoupled with *identical* responses. From (42) follows:

$$\text{R.S. } (\tilde{H} = \tilde{h}I) \quad \text{iff} \quad \bar{\sigma}(\tilde{H}) = |\tilde{h}| \leq \frac{1}{\mu(L\tilde{P}^{-1}E)} \; \forall \omega. \tag{43}$$

Again, this bound shows that the system has to be detuned and performance be sacrificed when the uncertainty is large, that is, in this case when $\mu(L\tilde{P}^{-1}E) > 1$. $\mu(L\tilde{P}^{-1}E)$ is a measure of the dynamic resilience which takes into account both the size of the uncertainty and the sensitivity of the plant to uncertainty. In Section V(3), a numerical example incorporating condition (43) is provided.

Upper bounds for $\mu(L\tilde{P}^{-1}\tilde{H}E)$. Alternative *sufficient* robust stability conditions for this uncertainty description have been derived by Kouvaritakis and Latchman (1985) and by Kantor and Andres (1983). Kouvariakis and Latchman's (1985) condition provides a tight upper bound on μ

$$\mu(L\tilde{P}^{-1}\tilde{H}E) \leq \min_{D_1, D_2} \frac{\bar{\sigma}(D_1 A D_2)}{\underline{\sigma}(D_1\tilde{H}^{-1}\tilde{P}D_2)} \equiv k^*(A, \tilde{H}^{-1}\tilde{P}). \tag{44}$$

Here $A = \{a_{ij}\}$ and D_1 and D_2 are diagonal matrices with real, positive entries. Kouvaritakis and Latchman (1985) claim that (44) is an equality, but their proof is wrong (Doyle, 1986). However, the bound is tight in most cases and is useful since it is easier to compute than $\mu(L\tilde{P}^{-1}\tilde{H}E)$. Another upper bound which is even easier to compute, but is more conservative, is given by Kantor and Andres (1983)

$$\mu(L\tilde{P}^{-1}\tilde{H}E) \leq \rho(A|\tilde{P}^{-1}\tilde{H}|). \tag{45}$$

The spectral radius of the positive matrix $A|\tilde{P}^{-1}\tilde{H}|$ is easily computed as the Perron-Frobenius root of the matrix.

Special case: equal relative uncertainty. Consider the special case when all the elements have the same relative uncertainty r, i.e.

$$A = r|\tilde{P}|. \tag{46}$$

Assuming $\tilde{H} = \tilde{h}I$ and using (44) and (45), the robust

stability condition (42) becomes

$$\text{R.S.} \quad \text{if} \quad |\tilde{h}| < 1/r\,k^*(|\tilde{P}|, \tilde{P})\ \forall\omega \tag{47}$$

$$\text{if} \quad |\tilde{h}| < 1/r\rho(|\tilde{P}||\tilde{P}^{-1}|)\ \forall\omega. \tag{48}$$

These conditions obviously also hold for the case when the relative uncertainties are different provided that r is replaced by the largest relative uncertainty in any element, $r_{\max}$

$$r_{\max}(\omega) = \max_{ij} r_{ij}(\omega) \tag{49a}$$

where

$$r_{ij} = \max_{P\in\Pi}\left|\frac{p_{ij}-\tilde{p}_{ij}}{\tilde{p}_{ij}}\right|. \tag{49b}$$

Note that $k^*(|\tilde{P}|, \tilde{P})$ can be viewed as a minimized condition number. A more thorough discussion of this result appears in Section V.

V. THE CONDITION NUMBER AS A SENSITIVITY MEASURE

This section discusses the use of the condition number $\gamma(\tilde{P})$ as a sensitivity measure with respect to uncertainty. A plant is called ill-conditioned if the condition number $\gamma(\tilde{P})$ is high. Physically, this means that the gain of the plant is strongly dependent on the input direction (see Notation). We give two interpretations to the condition number:

(1) The minimized condition number, $\gamma^*(\tilde{P})$ is a *stability* sensitivity measure with respect to independent uncertainty on the elements with similar relative magnitude.
(2) $\gamma(\tilde{P})$ is a robust *performance* sensitivity measure with respect to input uncertainty [as discussed following eq. (12)].

(1) $\gamma^*(\tilde{P})$ *as a sensitivity measure*

It has been argued previously in a somewhat qualitative manner (Grosdidier *et al.*, 1985) that for robust stability the minimized condition number $\gamma^*(\tilde{P})$ is a measure of sensitivity with respect to model uncertainty. Furthermore, there is a direct relationship between large elements in the Relative Gain Array (RGA) and $\gamma^*(\tilde{P})$ (Grosdidier *et al.*, 1985), and large elements in the RGA are often claimed to indicate sensitivity to model uncertainty. It will be shown that the minimized condition number $\gamma^*(\tilde{P})$ and the RGA are useful measures with respect to *element* uncertainty, but *only* if the relative errors of the transfer matrix elements are independent and have similar magnitude bounds. This proves to be a restrictive assumption in many cases.

Express the uncertainty in terms of the largest relative uncertainty, $r_{\max}$, in any of the transfer matrix elements (49). This uncertainty description is independent of scaling.

Theorem 6. *Condition number criterion.* Let Π be any set of plants such that P and $\tilde{P}$ have the same number of RHP (unstable) poles, and define $r_{\max}$ as in (49). Assume the nominal response is decoupled, $\tilde{H} = \text{diag}\{\tilde{h}_i\}$ and assume the system is nominally stable. Then robust stability is guaranteed

$$\text{if} \quad |\tilde{h}_i| < \frac{1}{r_{\max}\gamma_a^*(\tilde{P})}\ \forall\omega,\ \forall i \tag{50}$$

which is satisfied

$$\text{if} \quad |\tilde{h}_i| < \frac{1}{r_{\max}\sqrt{n}\gamma^*(\tilde{P})}\ \forall\omega,\ \forall i. \tag{51}$$

$\gamma^*(\tilde{P})$ is the minimized condition number and $\gamma_a^*(\tilde{P})$ is the minimized "absolute" condition number as defined in the Notation. The minimized condition numbers $\gamma^*(\tilde{P})$ and $\gamma_a^*(\tilde{P})$ are similar in magnitude since (Lemma 2, Appendix 2)

$$\gamma_a^*(\tilde{P})/\sqrt{n} \leq \gamma^*(\tilde{P}) \leq \gamma_a^*(\tilde{P}).$$

Condition (50) in Theorem 6 is very similar to condition (47) involving $k^*(|\tilde{P}|, \tilde{P})$, but there are two differences:

(1) Condition (50) also holds the when the decoupled nominal responses are not identical.
(2) Condition (50) is less conservative since $\gamma_a^*(\tilde{P}) \leq k^*(|\tilde{P}|, \tilde{P})$ (use $\bar{\sigma}(|D_1\tilde{P}D_2|) \leq \bar{\sigma}(D_1|\tilde{P}|D_2)$).

By comparing (47), (48), (50) and (51) the following chain of inequalitites is obtained

$$\gamma^*(\tilde{P}) \leq \gamma_a^*(\tilde{P}) \leq k^*(|\tilde{P}|, \tilde{P}) \leq \rho(|\tilde{P}||\tilde{P}^{-1}|). \tag{52}$$

Condition (50) is clearly conservative if the individual relative uncertainties on the elements, r_{ij}, are different in magnitude. However, from the discussion following eq. (44), the bound is expected to be tight when the relative error bounds are equal; in fact, the bound is the tightest possible for 2×2 plants.

Theorem 7. (2×2). Assume $\tilde{H} = \tilde{h}I$ and $r_{ij} = r_{\max}$. Then condition (50) in Theorem 6 is necessary and sufficient for robust stability.

In particular, Theorem 7 implies that, for the case of equal relative element uncertainty [using the nomenclature from Section IV(5)],

$$(2\times 2): \quad \mu(L\tilde{P}^{-1}E) = r\gamma_a^*(\tilde{P}) \tag{53}$$

and for higher order systems $r\gamma_a^*(\tilde{P})$ is expected to give a tight upper bound on $\mu(L\tilde{P}^{-1}E)$.

Improved condition at steady state. The uncertainty description above assumes that each transfer matrix element is given by

$$p_{ij} = \tilde{p}_{ij}(1 + r_{ij}\Delta_{ij}), \quad |\Delta_{ij}| < 1$$

where Δ_{ij} is a *complex* scalar. This may be reasonable at non-zero frequencies, but does not make any physical sense at steady state ($\omega = 0$) where $\tilde{P}$, P and Δ_{ij} are *real*. Theorem 7 may therefore be conservative at $\omega = 0$ where complex perturbations cannot occur. Fortunately, for 2×2 systems it turns out that we can derive a tight condition by replacing $\gamma_a^*(\tilde{P})$ by $\gamma^*(\tilde{P})$.

Indeed, for the case of equal relative uncertainty,

$$(2 \times 2): \quad \mu_{\text{real}}(L\tilde{P}^{-1}E) = r\gamma^*(\tilde{P}) \quad (\omega = 0). \quad (54)$$

Theorem 4B and (54) may be combined into the following theorem.

Theorem 8. (2×2). *Integral Control* $(\tilde{H}^{(0)} = I)$. Assume the uncertainties of the elements in $\tilde{P}(0)$ are independent and *real* and have equal relative magnitude bounds r. Then for open loop stable systems, robust stability and integral control may be achieved

$$\text{iff} \quad \gamma^*(\tilde{P}(0)) < 1/r. \quad (55)$$

If the magnitude bounds on the relative uncertainties are not equal, and r is replaced by $r_{\max}$, Theorem 8 provides a sufficient condition for robust stability and integral control. Theorem 8 is unique to 2×2 systems; numerical examples show no such relationship for systems of higher dimensions.

Theorems 7 and 8 give very clear interpretations of the minimized condition numbers as sensitivity measures: $\gamma^*(\tilde{P}(0))$ and $\gamma_a^*(\tilde{P}(j\omega))$ are accurate measures of sensitivity only if the plant uncertainties are given in terms of *independent* (uncorrelated) norm-bounded elements with *equal relative error bounds*. For other uncertainty structures the minimized condition number may be a very misleading sensitivity measure, and bounds on the uncertainties such as (55) may be arbitrarily conservative. This will be illustrated by a subsequent example.

(2) *Relationship to the RGA*

A relationship between $\gamma^*(\tilde{P})$ and the induced 1- and ∞-norms of the RGA has been conjectured by Grosdidier *et al.* (1985):

$$\gamma^*(\tilde{P}) \leq 2 \max [\|\text{RGA}\|_{i1}, \|\text{RGA}\|_{i\infty}]. \quad (56)$$

Numerical examples show that this bound does *not* hold for systems of dimension 4×4 or higher. However, for 2×2 systems (56) holds even with $\gamma_a^*(\tilde{P})$, and a stronger result is:

Theorem 9. (2×2). $\gamma_a^*(\tilde{P}) \leq \|\text{RGA}\|_1$. (57)

Note that for 2×2 systems $\|\text{RGA}\|_1 = 2\|\text{RGA}\|_{i1} = 2\|\text{RGA}\|_{i\infty}$. Numerical examples for 3×3 and 4×4 systems support the following extension to systems with higher dimensions:

Conjecture 1 $(n \times n)$: $\gamma_a^*(\tilde{P}) \leq \|\text{RGA}\|_1 + k(n)$ (58)

with $k(2) = 0$, $k(3) \approx 1$ and $k(4) \approx 2$.

The use of the function $k(n)$ was suggested by Nett (1986). For real matrices and high condition numbers, $\|\text{RGA}\|_1$ approaches $\gamma_a^*(\tilde{P})$. The bound (58) appears to be most conservative for small condition numbers. Note that these relationships also hold for the frequency dependent RGA if it is defined as in the Notation.

Theorems 6 and 7 and Conjecture 1 provide at least a partial explanation of why ill-conditioned multivariable systems with large RGA should already be avoided at the design stage: When γ_a^* (or γ^*) or equivalently $\|\text{RGA}\|_1$ is large, then the performance measured in terms of $|\tilde{h}|$ is very restricted [cf (50)] even if the model uncertainty $r_{\max}$ is small.

(3) *Integral control of high purity distillation column (example)*

This example will illustrate that the stability bounds (50) and (55) can be extremely conservative if the element uncertainties are not independent. Once again consider the distillation column of Table 1, but this time with reflux L and boilup V as the manipulated inputs. The steady state gain matrix is

$$\tilde{P}(0) = \begin{bmatrix} 0.878 & -0.864 \\ 1.082 & -1.096 \end{bmatrix}$$

and

$$(\text{RGA})_{11} = 35.07, \quad \|\text{RGA}\|_1 = 138.275,$$

$$\gamma^*(\tilde{P}) = \gamma_a^*(\tilde{P}) = 138.268, \ \gamma(\tilde{P}) = 141.7.$$

From the high condition number, $\gamma^*(\tilde{P})$, one might conclude that the plant may become singular for very small perturbations. This would be true if the uncertainty had the form of independent element errors, but not necessarily otherwise. To illustrate this point consider conditions for using integral control $(\tilde{H}(0) = I)$ under two different assumptions about the uncertainty.

Case 1. The elements are assumed independent and norm bounded with equal relative error r. From Section IV(5), eq. (43) implies that robust stability with integral control may be achieved

$$\text{iff} \quad \mu(L\tilde{P}^{-1}E) < 1 \quad (\omega = 0)$$

where μ is computed with respect to the *real* perturbation matrix Δ_E. Here:

$$E = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad L = r \begin{bmatrix} 0.878 & 0 \\ 1.082 & 0 \\ 0 & 0.864 \\ 0 & 1.096 \end{bmatrix}$$

$$L\tilde{P}^{-1}E = r \begin{bmatrix} 35.07 & -27.65 & 35.07 & -27.65 \\ 34.07 & -27.65 & 34.07 & -27.65 \\ 43.22 & -34.07 & 43.22 & -34.07 \\ 43.22 & -35.07 & 43.22 & -35.07 \end{bmatrix}$$

which gives

$$\mu_{\text{real}}(L\tilde{P}^{-1}E) = 138.268r$$

which is equal to $r\gamma^*(P)$ as expected from (54). The upper bound $\rho(|\tilde{P}||\tilde{P}^{-1}|)$ on $\gamma^*(\tilde{P})$ (52) happens to give the same result, i.e. $\rho(|\tilde{P}||\tilde{P}^{-1}|) = 138.268$. Consequently, robust stability with integral action is possible

$$\text{iff} \quad r < \frac{1}{\gamma^*(P)} = 0.0072.$$

In practice, the variation in each element (mainly due to nonlinearities) is much larger than 0.7%, and

integral control does not seem to be possible for this distillation column.

Case 2. A more realistic uncertainty description for this high purity distillation column is the following additive uncertainty (Skogestad and Morari, 1986)

$$P - \tilde{P} = \begin{bmatrix} d & -d \\ -d & d \end{bmatrix}$$

which may be written in terms of one real scalar Δ-block

$$P - \tilde{P} = W_2 \Delta W_1, \quad W_2 = |d| \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad W_1 = [1 \ \ -1], \quad |\Delta| < 1.$$

This highly structured uncertainty is mainly due to the material balance constraints which cannot be violated. Using Theorem 5, robust stability and integral control ($\tilde{H}(0) = I$) are possible iff

$$\mu_{\text{real}}(W_1 \tilde{P}^{-1} W_2) \leqslant 1 \quad (\omega = 0).$$

Here

$$W_1 \tilde{P}^{-1} W_2 = 0 \cdot |d|.$$

Consequently, robust stability and integral control are possible for any value of d and the elements may even change sign without causing stability problems. Thus, despite the high condition number, the system is not at all sensitive to this physically-motivated model error.

VI. CONCLUSIONS

To guarantee robust stability, model uncertainty requires feedback controllers be detuned and performance be sacrificed. To what extent detuning proves necessary depends on the size of the uncertainty as well as the sensitivity of the plant.

(I) *General case*

The Structured Singular Value $\mu(M)$ is by definition the best measure of the effect of uncertainty on performance:

$$\text{robust stability iff} \quad \mu(M) \leqslant 1 \ \forall \omega. \tag{27}$$

However, here the issue is not control system design but rather process design. From this viewpoint, systems whose closed-loop stability and performance are very sensitive to model errors are undesirable because they are either impossible to control or require that enormous effort be put into the design of the control system. Condition (27) assumes that a control system has already been designed and is therefore unsuitable for screening purposes at the design stage. If additional assumptions are made on the type of model uncertainty and the control structure, achievable performance can be related directly to characteristics of the system itself. In the following summary it is assumed that the nominal closed-loop system is decoupled ($\tilde{H} = \tilde{h}I$) with identical responses. This proves to be a reasonable assumption at low frequencies, and leads to the least conservative bounds.

(II) *Uncorrelated element uncertainty*

$$P - \tilde{P} = E \Delta L \tag{40}$$

$$\Delta = \text{diag}\{\Delta_{ij}\}, \quad \bar{\sigma}(\Delta_{ij}) < 1$$

robust stability iff

$$|\tilde{h}| < \frac{1}{\mu(L\tilde{P}^{-1}E)} \ \forall \omega. \tag{43}$$

(III) *Uncorrelated element uncertainty with similar relative errors*

Each element: $p_{ij} = \tilde{p}_{ij}(1 + r_{ij}\Delta_{ij})$, $|\Delta_{ij}| < 1$.
Largest relative error: $r_{\max} = \max_{ij} r_{ij}$.

(1) Robust stability if

$$|\tilde{h}| < \frac{1}{r_{\max}\gamma_a^*(\tilde{P})} \ \forall \omega. \tag{50}$$

(2) 2×2 systems, $r_{ij} = r \ \forall i, j$, *complex* Δ_{ij}: robust stability iff (Theorem 7)

$$|\tilde{h}| < \frac{1}{r\gamma_a^*(\tilde{P})} \ \forall \omega.$$

(3) 2×2 systems, $r_{ij} = r \ \forall i, j$, *real* Δ_{ij}: integral control and robust stability may be achieved iff

$$|\tilde{h}(0)| = 1 < \frac{1}{r\gamma^*(\tilde{P}(0))}. \tag{55}$$

The minimized condition number γ^* (or γ_a^*) or equivalently the RGA is a reliable indicator of closed-loop sensitivity to element uncertainty only if the relative errors of the transfer matrix elements are independent (uncorrelated) and have similar magnitude bounds.

Acknowledgements—We are thankful to Pierre Grosdidier and John C. Doyle for numerous useful discussions and remarks. Financial support from the National Science Foundation, the Department of Energy and Norsk Hydro is gratefully acknowledged.

NOTATION

$C(s)$	rational transfer matrix of fixed-parameter controller
$P(s)$	$n \times n$ square rational transfer matrix of actual plant $= \{p_{ij}\}$
$\tilde{P}(s)$	$n \times n$ square rational transfer matrix of nominal plant $= \{\tilde{p}_{ij}\}$
Π	set of all possible plants, i.e. $P \in \Pi$ and $\tilde{P} \in \Pi$
$\|G\|$	matrix G with all elements replaced by their absolute value
$\|\|G\|\|_1$	$= \sum_{i,j} \|g_{ij}\|$; 1-norm of matrix G
$\|\|G\|\|_2$	$= \left[\sum_{i,j} \|g_{ij}\|^2\right]^{1/2}$; 2-norm or Frobenius norm of matrix G
$\|\|G\|\|_{i1}$	$= \max_j \sum_{i=1}^{n} \|g_{ij}\|$; induced 1-norm ("max column sum")
$\|\|G\|\|_{i\infty}$	$= \max_i \sum_{j=1}^{n} \|g_{ij}\|$; induced ∞-norm ("max row sum")

RGA$(G) = G \times (G^{-1})^T$ where $\times$ denotes element-by-element multiplication (also called the Schur or Haddemard product)

$\rho(G)$ spectral radius of G, i.e. magnitude of largest eigenvalue

$\bar{\sigma}(G)$ maximum singular value or spectral norm of the transfer matrix G, which at each frequency is equal to the induced 2-norm

$$\bar{\sigma}(G(j\omega)) = \max_u \frac{\|Gu\|_2}{\|u\|_2}(j\omega) = \|G\|_{i2}$$

$\underline{\sigma}(G)$ minimum singular value

$$\underline{\sigma}(G(j\omega)) = \min_u \frac{\|Gu\|_2}{\|u\|_2}(j\omega)$$

we have the property $\sigma(G) = 1/\bar{\sigma}(G^{-1})$

$\gamma(G)$ $= \bar{\sigma}(G)/\underline{\sigma}(G)$; condition number

$\gamma_a(G)$ $= \bar{\sigma}(|G|)/\underline{\sigma}(G)$; absolute condition number

$\gamma^*(G)$ minimized condition number, $\gamma^*(G) = \min_{D_1, D_2} \gamma(D_1 G D_2)$, where D_1 and D_2 are diagonal matrices with real, positive entries. For G, 2×2 and real:

$$\gamma^*(G) = \begin{cases} \dfrac{1+\kappa^{1/2}}{|1-\kappa^{1/2}|} & \kappa > 0 \\ 1 & \kappa \leqslant 0 \end{cases}$$

(Grosdidier, 1985)

$\gamma_a^*(G)$ minimized absolute condition number,

$$\gamma_a^*(G) = \min_{D_1, D_2} \frac{\bar{\sigma}(|D_1 G D_2|)}{\sigma(D_1 G D_2)}$$

(2×2): $\gamma_a^*(G) = \dfrac{1+|\kappa|^{1/2}}{|1-\kappa^{1/2}|}$ (Appendix 2)

$\kappa(G)$ Rijnsdorps interaction measure for 2×2 plant

$$\kappa(G) = \frac{g_{12} g_{21}}{g_{11} g_{22}}$$

$\mu(G)$ structured singular value (see Appendix 1).

The Laplace variable s or $j\omega$ is omitted in most cases.

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APPENDIX 1. THE SSV μ AND ITS PROPERTIES

Definition (Doyle, 1982)

The function $\mu(M)$, called the structured singular value (SSV) is defined at each frequency such that $\mu^{-1}(M)$ is equal to the *smallest* $\bar{\sigma}(\Delta)$ needed to make $(I + \Delta M)$ singular, i.e.

$$\mu^{-1}(M) = \min_{\delta} \{\delta | \det(I + \Delta M) = 0 \text{ for some } \Delta, \bar{\sigma}(\Delta) \leqslant \delta(\omega)\}. \quad \text{(A1-1)}$$

Δ is a *block* diagonal perturbation matrix. $\mu(M)$ depends on the matrix M *and* the structure of the perturbations Δ. The definition of μ may be extended by restricting Δ to a smaller set, e.g. Δ real. The above definition is not in itself useful for computing μ since the optimization problem implied by it does not appear to be easily solvable. Fortunately, Doyle (1982) has proven several properties of μ which makes it more useful for applications.

Properties of μ (Doyle, 1982)

(1) The following bounds exist for μ:

$$\rho(M) \leqslant \mu(M) \leqslant \bar{\sigma}(M) \quad \text{(A1-2)}$$

$\mu(M) = \rho(M)$ in the case $\Delta = \delta I$. $\mu(M) = \bar{\sigma}(M)$ in the case Δ is "unstructured", i.e. Δ is a full matrix.

(2) Let $\mathcal{U}$ be the set of all unitary matrices with the same structure as Δ, then

$$\max_{U \in \mathcal{U}} \rho(MU) = \mu(M). \quad \text{(A1-3)}$$

This optimization problem is in general not convex.

(3) Let $\mathcal{D}$ be the set of real positive diagonal matrices $D = \text{diag}\{d_i I_i\}$ where the size of each block (size of I_i) is equal to the size of the blocks Δ_i. Then for 3 or fewer blocks

$$\min_{D \in \mathcal{D}} \bar{\sigma}(DMD^{-1}) = \mu(M). \quad \text{(A1-4)}$$

For four or more blocks numerical evidence suggests that (A1-4) gives a tight upper bound on $\mu(M)$. A good estimate for the scaling matrix D is found by minimizing $\|DMD^{-1}\|_2$ (the Frobenius norm).

(4) $\mu(\alpha M) = |\alpha| \mu(M)$, α is a scalar.

(5) For real matrices M with real, non-repeated perturbations, the search in (A1-3) may be performed with real matrices U only, and only the cornerpoints ("± 1") need to be considered. For (20) and (21) in Theorem 4 this implies that only cornerpoints for the possible $P(0)$'s need to be checked.

APPENDIX 2. PROOF OF THEOREMS

Proof of Theorem 3

The proof uses condition (26) from Section IV which applies to any stable norm bounded perturbation. The interconnection matrix M for the norm bounded additive set of plants, Π_A, is (rearrange Fig. 4 to get Fig. 6)

$$M = W_1C(I+\tilde{P}C)^{-1}W_2 = W_1\tilde{P}^{-1}\tilde{H}W_2. \quad \text{(A2-1)}$$

Using $P-\tilde{P} = W_2\Delta_A W_1$ we find $\det(I+\Delta_A M) = \det(I+\Delta_A W_1\tilde{P}^{-1}\tilde{H}W_2) = \det(I+W_2\Delta_A W_1\tilde{P}^{-1}\tilde{H}) = \det(I+(P-\tilde{P})\tilde{P}^{-1}\tilde{H})$, and assuming nominal stability and using (26) it is found that robust stability is guaranteed

$$\text{iff} \quad \det(I+(P-\tilde{P})\tilde{P}^{-1}\tilde{H}) \neq 0 \ \forall P\in\Pi_A. \quad \text{(A2-2)}$$

Theorem 3 follows from (A2-2) by assuming $\tilde{H} = I$.

Proof of Theorem 4*A*

Applying the Nyquist stability condition to (8) we see that closed loop stability requires that the image of

$$\det(I+(P-\tilde{P})\tilde{P}^{-1}\tilde{H}) \quad \text{(A2-3)}$$

does not encircle the origin as a transverses the Nyquist D contour for any $P\in\Pi$. For the case of integral control, the image starts from $(\omega = 0)$ $\det P(0)\tilde{P}(0)^{-1}$. Using the strictly-proper assumption, the image ends at $(\omega=\infty)$ $\det I = 1$. The image of (A2-3) will therefore always encircle the origin if it starts on the negative real axis and the system will be unstable.

Proof of Theorem 4*B*

Necessity: Follows from Theorem 4A.

Sufficiency: For this uncertainty description robust stability is guaranteed (Theorem 5)

$$\text{iff} \quad \mu(W_1\tilde{P}^{-1}\tilde{H}W_2) \leqslant 1 \ \forall\omega \quad \text{(A2-4)}$$

$$\text{iff} \quad \det(I+(P-\tilde{P})\tilde{P}^{-1}\tilde{H}) \neq 0 \ \forall\omega, \ \forall P\in\Pi_A \quad \text{(A2-2)}$$

$\omega = 0$: (A2-2) with $\tilde{H}(0) = I$ is satisfied if (21) is satisfied.

$\omega > 0$: For stable plants, it is always possible to select a controller such that $\tilde{H} = \tilde{h}I$ and $|\tilde{h}| \leqslant 1/\mu(W_2\tilde{P}^{-1}W_1)\ \forall\omega$, i.e. such that (A2-4) holds. This proves that by assuming (21), (A2-4) is satisfied for all ω, and robust stability can always be achieved.

Proof of Theorem 6

Consider *any* set of plants Π, such that all $P\in\Pi$ have the same number of RHP poles. Assuming nominal stability, the Nyquist stability condition applied to (8) implies that robust stability is guaranteed if and only if the image of $\det(I+(P-\tilde{P})\tilde{P}^{-1}\tilde{H})$ does not encircle the origin as s traverses the Nyquist D-contour for all $P\in\Pi$. A sufficient condition for robust stability using the small gain theorem is therefore

$$\rho((P-\tilde{P})\tilde{P}^{-1}\tilde{H}) < 1 \ \forall\omega, \ \forall P\in\Pi. \quad \text{(A2-5)}$$

Here the spectral radius ρ is invariant under similarity transformations. In particular, let D_1 and D_2 be real diagonal "scaling" matrices. Then for *any* $P\in\Pi$ we have

$$\begin{aligned}\rho((P-\tilde{P})\tilde{P}^{-1}\tilde{H}) &= \rho(D_1(P-\tilde{P})D_2D_2^{-1}\tilde{P}^{-1}D_1^{-1}D_1\tilde{H}D_1^{-1})\\ &\leqslant \bar{\sigma}(D_1(P-\tilde{P})D_2D_2^{-1}\tilde{P}^{-1}D_1^{-1}D_1\tilde{H}D_1^{-1})\\ &\leqslant \frac{\bar{\sigma}(D_1(P-\tilde{P})D_2)\bar{\sigma}(D_1\tilde{H}D_1^{-1})}{\underline{\sigma}(D_1\tilde{P}D_2)}\\ &= \frac{\bar{\sigma}(D_1(P-\tilde{P})D_2)}{\underline{\sigma}(D_1\tilde{P}D_2)}\bar{\sigma}(\tilde{H})\\ &\qquad \text{(for } \tilde{H} = \text{diag}\{\tilde{h}_i\}) \quad \text{(A2-6)}\\ &\leqslant r_{\max}\frac{\bar{\sigma}(|D_1\tilde{P}D_2|)}{\underline{\sigma}(D_1\tilde{P}D_2)}\bar{\sigma}(\tilde{H}) \quad \text{(Lemma 1).}\end{aligned}$$

(A2-7)

Combining (A2-5) and (A2-7) and choosing the scalings D_1 and D_2 to get the least conservative bound, R.S. is guaranteed

$$\text{if} \quad \min_{D_1,D_2}\bar{\sigma}(\tilde{H})r_{\max}\frac{\bar{\sigma}|D_1\tilde{P}D_2|}{\underline{\sigma}(D_1\tilde{P}D_2)} < 1 \ \forall\omega \quad (\tilde{H} = \text{diag}\{\tilde{h}_i\}) \quad \text{(A2-8)}$$

which is equivalent to

$$\bar{\sigma}(\tilde{H}) < \frac{1}{r_{\max}\gamma_a^*(\tilde{P})} \quad (\tilde{H} = \text{diag}\{\tilde{h}_i\}). \quad (50)$$

This proves condition (50). Condition (51) follows directly from (50) by applying Lemma 2 below.

Lemma 1.

Consider any set of plants Π. Then,

$$\max_{P\in\Pi}(\bar{\sigma}(D_1(P-\tilde{P})D_2) \leqslant r_{\max}\bar{\sigma}(|D_1\tilde{P}D_2|) \quad \text{(A2-9)}$$

and equality applies if the set Π is norm bounded with independent elements and all elements have the same relative uncertainty.

Proof.

Let A be the matrix which bounds each element in $P-\tilde{P}$

$$|P-\tilde{P}| \leqslant A(\omega) \ \forall P\in\Pi. \quad \text{(A2-10)}$$

Then

$$\max_{P\in\Pi}\bar{\sigma}(P-\tilde{P}) \leqslant \bar{\sigma}(A) \leqslant r_{\max}\bar{\sigma}(|\tilde{P}|). \quad \text{(A2-11)}$$

[The first inequality is an equality if all P satisfying (A2-10) may occur in practice.] The last inequality follows trivially since $A \leqslant r_{\max}|\tilde{P}|$. It will be an equality if the relative uncertainty bounds of the elements are equal. To derive (A2-9), note that the relative errors r_{ij} and $r_{\max}$ are unchanged by applying the *diagonal* scalings D_1 and D_2 to the plant.

Comment.

Note that the bound involves $|D_1\tilde{P}D_2|$ and not $D_1|\tilde{P}|D_2$. The last would be more conservative since for D_1 and D_2 real and positive $\bar{\sigma}(|D_1\tilde{P}D_2|) \leqslant \bar{\sigma}(D_1|\tilde{P}|D_2)$.

Lemma 2.

Let G be a matrix of size $n\times n$. Then

$$\bar{\sigma}(G) \leqslant \bar{\sigma}(|G|) \leqslant \sqrt{n}\bar{\sigma}(G) \quad \text{(A2-12)}$$

Proof.

The following property is proved by Stone (1962)

$$\frac{1}{\sqrt{n}}\|G\|_2 \leqslant \bar{\sigma}(G) \leqslant \|G\|_2.$$

Using the obvious property $\|G\|_2 = \||G|\|_2$ we get

$$\bar{\sigma}(|G|) \leqslant \|G\|_2 \leqslant \sqrt{n}\bar{\sigma}(G). \quad \text{QED}$$

Proof of Theorem 7

Let $\tilde{P}$ be a nonsingular 2×2 transfer matrix and consider the case of independent elements with equal relative errors r.

$$P = \begin{bmatrix}\tilde{p}_{11}(1+r\Delta_{11}) & \tilde{p}_{12}(1+r\Delta_{12})\\ \tilde{p}_{21}(1+r\Delta_{21}) & \tilde{p}_{22}(1+r\Delta_{22})\end{bmatrix}, \quad |\Delta_{ij}| < 1 \ \forall\omega. \quad \text{(A2-13)}$$

Comparing (50) with (43) we see that Theorem 7 holds if it can be proved for this uncertainty description that

$$(2\times 2): \quad \mu(L\tilde{P}^{-1}E) = r\gamma_a^*(\tilde{P}). \quad (53)$$

Since both $\mu(L\tilde{P}^{-1}E)$ and $r\gamma_a^*(\tilde{P})$ scale linearly with r, (53) is equivalent to the following statement

$$\mu(L\tilde{P}^{-1}E) \leqslant 1 \Leftrightarrow r\gamma_a^*(\tilde{P}) \leqslant 1.$$

Note from (43) that $\mu(L\tilde{P}^{-1}E) \leqslant 1$ is a condition for having "perfect control" $(\tilde{H} = I)$. Then using Theorem 3, which

applies to robust stability and "perfect control", we get

$$\mu(L\tilde{P}^{-1}E) \leq 1 \Leftrightarrow \det P(j\omega)) \neq 0, \ \forall P.$$

Theorem 7 will therefore be correct if for each frequency

$$r\gamma_a^*(\tilde{P}) \leq 1 \Leftrightarrow \det (P(j\omega)) \neq 0, \ \forall P \qquad \text{(A2-14)}$$

i.e. if it can prove the following statement: "At each frequency the smallest r which makes

$$\det P = 0 \text{ is } r = 1/\gamma_a^*(\tilde{P})". \qquad \text{(A2-15)}$$

Define at each frequency

$$\kappa = \frac{\tilde{p}_{12}\tilde{p}_{21}}{\tilde{p}_{11}\tilde{p}_{22}} = |\kappa| e^{j\phi}$$

and use

$$\det P = 0 \quad \text{iff} \quad (1+r\Delta_{11})(1+r\Delta_{22}) = \kappa(1+r\Delta_{12})(1+r\Delta_{21}). \qquad \text{(A2-16)}$$

The smallest r which satisfies (A2-16) is found for $\Delta_{11} = \Delta_{22} = \Delta_1$, $\Delta_{12} = \Delta_{21} = \Delta_2$ and

$$(1+r\Delta_1)^2 = \kappa(1+r\Delta_2)^2.$$

Introduce $\Delta_1 = e^{j\phi_1}$, $\Delta_2 = e^{j\phi_2}$, where ϕ_1 and ϕ_2 are free to be chosen, to find:

$$1 - |\kappa|^{1/2} e^{j\phi/2} = r|\kappa|^{1/2} e^{j(\frac{\phi}{2}+\phi_2)} - re^{j\phi_1}.$$

The left-hand side is fixed. Using geometrical arguments it is evident that the smallest r satisfying this expression is found when ϕ_1 and ϕ_2 are chosen such that the two terms on the right hand side are aligned and in the direction of the left hand side:

$$|1 - |\kappa|^{1/2} e^{j\phi/2}| = |1 - \kappa^{1/2}| = r(|\kappa|^{1/2} + 1)$$

$$r = \frac{|1-\kappa^{1/2}|}{1+|\kappa|^{1/2}}. \qquad \text{(A2-17)}$$

The derivation of the expression for $\gamma_a^*(\tilde{P})$ is very tedious but straightforward and follows the derivation for $\gamma^*(\tilde{P})$ (Grosdidier *et al.*, 1985). This derivation shows that r given in (A2-17) is equal to $1/\gamma_a^*(\tilde{P})$ which proves (A2-15) and thus proves the theorem.

Proof of Theorem 8

The proof is similar to that of Theorem 7. The set of plants is again given by (A2-13) but the perturbations are assumed to be real ($-1 < \Delta_{ij} < 1$) and all the elements in $\tilde{P}$ are also assumed to be real. As for the proof of Theorem 7, (54) is proved if we can prove the following statement is proved: "The smallest r which makes

$$\det P = 0 \ (\tilde{P} \text{ and } \Delta_{ij} \text{ real}) \text{ is } r = 1/\gamma^*(\tilde{P})". \qquad \text{(A2-18)}$$

It is necessary to find the smallest r which satisfies (A2-16) when Δ_{ij} is real.

Case 1: $\kappa < 0$. In this case (A2-16) cannot be satisfied for any $r < 1$, but it may clearly be satisfied if $r = 1$ (e.g. choose $\Delta_{12} = -1$ and $\Delta_{11} = -1$). Consequently, the smallest r which makes $\det P = 0$ in this case is $r = 1$, and since $\gamma^*(\tilde{P}) = 1$ for $\kappa < 0$ (Grosdidier, 1985) we have $r = 1/\gamma^*(\tilde{P})$.

Case 2*a*: $\kappa > 1$. Only cornerpoints of (A2-16) need to be checked (see Appendix 1). Then it is obvious that the smallest r which satisfies (A2-16) for $\kappa > 1$ is the solution of (choose $\Delta_{11} = \Delta_{22} = 1$, $\Delta_{12} = \Delta_{21} = -1$)

$$(1+r)^2 = \kappa(1-r)^2$$

which has as its smallest root $r = (\sqrt{\kappa}-1)/(\sqrt{\kappa}+1) = |1 - \sqrt{\kappa}|/(1+\sqrt{\kappa})$.

Case 2*b*: $0 < \kappa < 1$. The smallest r which satisfies (A2-16) in this case is a solution of

$$(1-r^2) = \kappa(1+r)^2$$

which has as its smallest root $r = (1-\sqrt{\kappa})/(1+\sqrt{\kappa}) = |1 - \sqrt{\kappa}|/(1+\sqrt{\kappa})$. $\gamma^*(\tilde{P})$ is given in the Notation. From this it is evident that r is equal to $1/\gamma^*(\tilde{P})$ also for $\kappa > 0$ and this proves statement (A2-18). QED

Proof of Theorem 9

For 2×2 systems the RGA becomes

$$\text{RGA}(\tilde{P}) = \begin{bmatrix} \lambda_{11} & 1-\lambda_{11} \\ 1-\lambda_{11} & \lambda_{11} \end{bmatrix},$$

$$\lambda_{11} = \frac{\tilde{p}_{11}\tilde{p}_{22}}{\tilde{p}_{11}\tilde{p}_{22} - \tilde{p}_{12}\tilde{p}_{21}} = \frac{1}{1-\kappa} \qquad \text{(A2-19)}$$

$$\|\text{RGA}\|_1 = 2(|\lambda_{11}| + |1-\lambda_{11}|) = 2\frac{1+|\kappa|}{|1-\kappa|}. \qquad \text{(A2-20)}$$

Using the expression for γ_a^* (inverse of eq. A2-17)

$$\gamma_a^* = \frac{1+|\kappa|^{1/2}}{(|\kappa| - 2|\kappa|^{1/2}\cos\phi/2 + 1)^{1/2}}$$

$$\leq \frac{1+2|\kappa|^{1/2}+|\kappa|}{|1-\kappa|} \leq \|\text{RGA}\|_1. \quad \text{QED}$$