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Output Consensus of Linear Multi-agent Systems

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***Abstract*—This study concerns output consensus problems of linear multi-agent systems with fixed topologies and agents described by homogeneous or heterogeneous linear systems. Based on generalized inverse, graph and linear system theory, conditions are investigated for output consensusability with respect to a set of admissible consensus protocols. The given output consensus conditions depend on both topologies within linear multi-agent systems and structure properties of each agent's dynamic. The designed output feedback control law can guarantee output consensusability of the considered linear multi-agent systems with respect to a given admissible set. A provided example illustrates applicability of the proposed approach.**

I. Introduction

In recent years, consensus problems in multi-agent systems have received intensive attention from many fields including physics, biology and control theory and engineering due to their numerous and various applications in these areas [1]–[3]. The consensus problem is one of the most fundamental distributed coordination control problems of multi-agent networks. Roughly speaking, consensus means that multiple agents reach an agreement on a common value which might be, for example, the altitude in multi-spacecraft alignment, heading direction in flocking behavior, or average in distributed computation [4]. Numerous results have been obtained in consensus problems from different perspectives [5]–[14]. For phase transition of a group of self-driven particles, a simple and popular model has been proposed by using graph theory [2]. Numerically it has been demonstrated that all agents move in the same direction eventually. Attitude alignment has been studied for the network of agents with an undirected graph in which each agent has a discrete-time integrator dynamic [5]. Moreover, [5] has provided a theoretical explanation for the model in [2]. Many researchers have applied reinforcement learning to multi-agent systems, and these results show that reinforcement learning can perform well in multi-agent systems [6]. For first-order discrete-time multi-agent systems with time-varying topologies and stochastic communication noises, average-consensus conditions for distributed stochastic approximation type protocols have been proposed by using probability limit theory and algebraic graph theory [7]. Several dynamic consensus algorithms for second-order multi-agent systems and sufficient conditions for state consensus of the system have been proposed in [8] and [9]. Based on matrix theory, algebraic graph theory and Lyapunov control approach, [10] has derived some sufficient conditions to achieve second-order consensus for multi-agent systems with directed topology and nonlinear dynamic. A class of nonlinear high-order multi-agent systems have been studied and achieved consensus even though the communication graph which has no spanning tree [11]. To solve the consensus problem of multi-agent systems with a time-invariant communication topology and agents described by linear time-invariant systems, the necessary and sufficient condition has been given, which can guarantee the existence of an observer-type protocol solving the consensus problem, meanwhile, under this condition, an unbounded consensus region has been yielded if and only if each agent is both stabilizable and detectable [12]. [15] has provided necessary and sufficient state consensus conditions with respect to a set of admissible consensus protocols for descriptor multi-agent systems with fixed topologies.

In fact, most of existing results have investigated in consensus problems for multi-agent systems described by homogeneous agents. Comparing the discussion of consensus problems for homogeneous multi-agent systems with that for heterogeneous ones, it is easy to see that the later one has more significant due to heterogeneous multi-agent systems have more extensive description in the real world. However, rare works have been published to deal with consensus problems for multi-agent systems consisting of heterogeneous agents [16]–[18]. [16] has given state consensus conditions using tools of graph, algebra and descriptor linear system theory for descriptor multi-agent systems with agents described by homogeneous or heterogeneous descriptor systems. For heterogeneous and nonlinear multi-agent systems with topologies changing in an intermittent and arbitrary way, the necessary and sufficient condition of cooperative controllability has been proposed by using matrix-theoretical approach [17]. [18] has studied output consensus problems for a class of heterogeneous multi-agent systems consisting of uncertain linear SISO systems. Thus, this paper investigates output consensus problem for a class of multi-agent systems consisting of homogeneous or heterogeneous linear systems. Based on graph, generalized inverse and linear system theory, output consensus conditions are proposed. And designing the output feedback control law can guarantee that the studied system is output consensusable with respect to an admissible set.

Notation: For the given vector or matrix X, $\|X\|$ represents the Euclidean norm of X. Let $\sigma(A)$ be the set of all eigenvalues of the square matrix A. $\mathbb{C}^-$ represents the open left-half complex plane.

II. Preliminaries and Problem Formulation

In general, information exchange between agents in a multi-agent system can be modeled by directed or undirected graphs [19]. Similar to [20], let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a weighted digraph with the set of vertices $\mathcal{V} = \{1, 2, \cdots, N\}$ and the set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. In $\mathcal{G}$, the i-th vertex represents the i-th agent, and a directed edge from i to j is denoted as an ordered pair $(i, j) \in \mathcal{E}$, which means that agent j can directly receive information from agent i. In this case, the vertex i is called the parent vertex and the vertex j is called the child vertex. The set of neighbors of the i-th agent is denoted by $N_i = \{j \in \mathcal{V} | (j, i) \in \mathcal{E}\}$. And the weighted adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ with nonnegative elements, and $a_{ii} = 0, a_{ij} > 0 \Leftrightarrow j \in N_i$, otherwise $a_{ij} = 0$.

Definition *2.1:* [21] Let $A \in \mathbb{C}^{m\times n}$. Then the matrix $X \in \mathbb{C}^{n\times m}$ satisfying the following four equations (usually called the Penrose conditions)

$$AXA = A,\ XAX = X,\ (AX)^* = AX,\ (XA)^* = XA$$

is called the Moore-Penrose inverse of A, and is denoted by $X = A^\dagger$.

Lemma *2.1:* [21] Let $A \in \mathbb{C}^{m\times n}$. The generalized inverse X satisfying the Penrose conditions is existent and unique. Moreover, if $\text{rank}A = n$, then $A^\dagger A = I_n$; if $\text{rank}A = m$, then $AA^\dagger = I_m$.

Consider a linear multi-agent system consisting of N agents indexed by $1, 2, \cdots, N$, respectively. The dynamics of the i-th agent is described by homogeneous or heterogeneous linear systems

$$\dot{x_i}(t) = A_i x_i(t) + B_i u_i(t), \tag{1a}$$

$$y_i(t) = C_i x_i(t), \tag{1b}$$

$i = 1,\ 2, \cdots\ N$, where $x_i(t) \in \mathbb{R}^{n_i}$, $u_i(t) \in \mathbb{R}^{r_i}$ and $y_i(t) \in \mathbb{R}^m$ are the state, control input and control output of the i-th agent, respectively; $A_i \in \mathbb{R}^{n_i\times n_i}$, $B_i \in \mathbb{R}^{n_i\times r_i}$, $C_i \in \mathbb{R}^{m\times n_i}$, and $\text{rank}C_i = n_i$. System (1) is said to be homogeneous if $A_1 = A_2 = \ldots = A_N$, $B_1 = B_2 = \ldots = B_N$ and $C_1 = C_2 = \ldots = C_N$, otherwise, system (1) is said to be heterogeneous. Regarding the above N agents as vertices, the topology relationship among them can be conveniently described by a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ with $\mathcal{V} = \{1, 2, \cdots, N\}$ and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$.

Comparing output feedback with state feedback, the former one would have more practical significant in consensus protocols. Because of constraints on measurement or economic costs in practice, it is sometimes hard to directly measure the relative information of all states. However, only the relative information of all outputs is available. Consensus protocols based on outputs or states are equivalent if all the states are measurable. Hence, adopt two kinds of output consensus protocols with the output feedback case as follows, respectively.

Case One: Since each agent has limited capability of collecting information, output consensus protocols for each agent in multi-agent systems are distributed and only depend on the information of the agent itself and its neighbors. Adopt output consensus protocols of the i-th agent are given by

$$u_1(t) = \sum_{j\in N_1} a_{1j} K_{1j}[y_j(t) - y_1(t)], \tag{2a}$$

$$\begin{aligned} u_i(t) =& D_i(C_1 A_1 C_1^\dagger - C_i A_i C_i^\dagger) y_i(t) \\ &+ \sum_{j\in N_i} a_{ij} K_{ij}[y_j(t) - y_i(t)], \\ & t \geq 0, i = 2, 3, \cdots, N, \end{aligned} \tag{2b}$$

where $D_i \in \mathbb{R}^{r_i\times m}$ such that $C_i B_i D_i = I$ and $K_{ij} \in \mathbb{R}^{r_i\times m}$, $i, j = 1, 2, \cdots, N$, are weighted constant matrices which will be designed.

Applying the property of a_{ij}, (2) is equivalent to

$$u_1(t) = \sum_{j=1}^{N} a_{1j} K_{1j}[y_j(t) - y_1(t)], \tag{3a}$$

$$\begin{aligned} u_i(t) =& D_i(C_1 A_1 C_1^\dagger - C_i A_i C_i^\dagger) y_i(t) \\ &+ \sum_{j=1}^{N} a_{ij} K_{ij}[y_j(t) - y_i(t)], \\ & t \geq 0, i = 2, 3, \cdots, N. \end{aligned} \tag{3b}$$

Let

$$u(t) = [u_1^T(t)\ u_2^T(t)\ \cdots\ u_N^T(t)]^T.$$

Define an admissible control set:

$$\begin{aligned} \mathcal{U}_1 =& \{u(t) : [0, \infty) \to \mathbb{R}^{rN} | u_1(t) = \sum_{j=1}^{N} a_{1j} K_{1j}[y_j(t) \\ & - y_1(t)],\ u_i(t) = D_i(C_1 A_1 C_1^\dagger - C_i A_i C_i^\dagger) y_i(t) \\ & + \sum_{j=1}^{N} a_{ij} K_{ij}[y_j(t) - y_i(t)], \\ & i = 2, 3, \cdots, N,\ t \geq 0,\ K_{ij} \in \mathbb{R}^{r\times m}, \\ & i, j = 1, 2, \cdots, N\}. \end{aligned}$$

Case Two: Adopt output consensus protocols of the i-th agent are given by

$$\begin{aligned} u_i(t) =& \sum_{j\in N_i} a_{ij} K_{ij}[y_j(t) - y_i(t)], \\ & t \geq 0, i = 1, 2, \cdots, N, \end{aligned} \tag{4}$$

where $K_{ij} \in \mathbb{R}^{r_i\times m}$, $i, j = 1, 2, \cdots, N$, are weighted constant matrices which will be designed.

Similar to Case One, define the following admissible control set:

$$\begin{aligned} \mathcal{U}_2 =& \{u(t) : [0, \infty) \to \mathbb{R}^{rN} | u_i(t) = \sum_{j=1}^{N} a_{ij} K_{ij}[y_j(t) \\ & - y_i(t)],\ i = 1, 2, \cdots, N, t \geq 0,\ K_{ij} \in \mathbb{R}^{r\times m}, \\ & i, j = 1, 2, \cdots, N\}. \end{aligned}$$

The admissible control set $\mathcal{U}_k (k = 1, 2)$ covers a relatively large class of distributed output consensus protocols. Therefore, we want to know under what conditions, the multi-agent system consisting of homogeneous or heterogeneous linear time-invariant systems is output consensusable with respect to such a kind of admissible control set. To solve this problem, a definition of the output consensusability of a multi-agent system with respect to an admissible control set $\mathcal{U}_k(k=1,2)$ is given as follows.

Definition *2.2:* For linear multi-agent system (1), if there exists a $u(t) \in \mathcal{U}_1$ ($\mathcal{U}_2$) such that for any initial value $x_i(0)$,

$$\lim_{t\to\infty} \|y_j(t) - y_i(t)\| = 0,\ i,j = 1,2,\cdots,N,$$

then the system (1) is said to be output consensusable with respect to $\mathcal{U}_1$ ($\mathcal{U}_2$).

Remark *2.1:* Due to heterogeneity of the agents, to achieve the state consensus (i.e., $\lim_{t\to\infty} \|x_j(t) - x_i(t)\| = 0$) is impossible generally.

Remark *2.2:* If $A_1 = A_i$ in the output consensus protocols (3), it is not necessary to assume that $C_iB_iD_i = I$.

Remark *2.3:* Definition 2.2 is different from the state consensus definition in [22], where the states of all the agents are required to converge to the same one constant value. Similar to [20], here only the output differences between different agents are required to tend to zero, no matter whether the outputs themselves converge or not.

Remark *2.4:* When $\sigma(A_i) \subseteq \mathbb{C}^-,\ i = 1,2,\cdots,N$, take $K_{i,j} = 0,\ i,j = 1,2,\cdots,N$ in output consensus protocols (3) or (4). Then it follows that $y_i,\ i = 1,2,\cdots,N$, converges to zero exponentially. Hence multi-agent system (1) is naturally output consensusable with respect to $\mathcal{U}_k(k=1,2)$. Thus, assume that the eigenvalues of the matrices $A_i,\ i=1,2,\cdots,N$ are not all in the open left-half plane in this paper.

III. Output Consensusability Conditions for Linear Multi-agent Systems

Lemma *3.1:* [23]For $A, B \in \mathbb{R}^{n\times n}$, if (A,B) is stabilizable, then Riccati equation

$$A^TP + PA - PBB^TP + I_n = 0$$

has a unique nonnegative definite solution P, furthermore, $\sigma(A - BB^TP) \subseteq \mathbb{C}^-$.

Theorem *3.1:* System (1) is output consensusable with respect to $\mathcal{U}_1$ if the following conditions hold:

(i) $(C_1A_1C_1^\dagger,\ C_1B_1 + C_2B_2)$ and $(C_1A_1C_1^\dagger,\ C_iB_i)$, $i = 3,4,\cdots,N$, are stabilizable;

(ii) $a_{12} \neq 0$, $a_{21} \neq 0$ and there exists at least one $j \in \{1,2,\cdots,i-1\}$ such that $a_{ij} \neq 0,\ i = 3,4,\cdots N$.

Proof:

Using Definition 2.2, system (1) is output consensusable with respect to $\mathcal{U}_1$ if and only if there exist matrices $K_{ij} \in \mathbb{R}^{r\times m}$ $i,j = 1,2,\cdots N$, and output consensus protocols (3) such that for any $i \neq j$,

$$\lim_{t\to\infty} \|y_j(t) - y_i(t)\| = 0.$$

Combining (1a) and (1b) with $\mathrm{rank}C_i = n_i$, it follows that

$$\begin{aligned}\dot{y}_i(t) &= C_i\dot{x}_i(t)\\ &= C_iA_iC_i^\dagger C_ix_i(t) + C_iB_iu_i(t)\\ &= C_iA_iC_i^\dagger y_i(t) + C_iB_iu_i(t).\end{aligned} \tag{5}$$

Then output consensusability of system (1) is equivalent to state consensusability of system (5) with respect to $\mathcal{U}_1$, which implies there exist matrices $K_{ij} \in \mathbb{R}^{r\times m}$ $i,j = 1,2,\cdots N$, and output consensus protocols (3) such that for any $i \neq j$,

$$\lim_{t\to\infty} \|y_j(t) - y_i(t)\| = 0 \tag{6}$$

for system (5).

Let $\delta_i(t) \triangleq y_1(t) - y_i(t),\ i = 2,3,\cdots,N$. One obtains (6) is equivalent to $\lim_{t\to\infty} \|\delta_i(t)\| = 0,\ i = 2,3,\cdots,N$.

Notice that

$$\begin{aligned}\dot{\delta}_i(t) =& C_1A_1C_1^\dagger\delta_i(t) - C_1B_1\sum_{j=1}^N a_{1j}K_{1j}\delta_j(t)\\ &+ C_iB_i\sum_{j=1}^N a_{ij}K_{ij}[\delta_j(t) - \delta_i(t)]\\ =&(C_1A_1C_1^\dagger - C_iB_i\sum_{j=1}^N a_{ij}K_{ij})\delta_i(t)\\ &+\sum_{j=1}^N (C_iB_ia_{ij}K_{ij} - C_1B_1a_{1j}K_{1j})\delta_j(t)\end{aligned}$$

$i = 2,3,\cdots,N$. Let

$$\delta(t) = [\delta_2^T(t)\ \delta_3^T(t)\ \cdots\ \delta_N^T(t)]^T.$$

One obtains

$$\dot{\delta}(t) = \bar{A}\delta(t), \tag{7}$$

where

$$\bar{A} = \begin{pmatrix} \bar{A}_{22} & \bar{A}_{23} & \cdots & \bar{A}_{2N}\\ \bar{A}_{32} & \bar{A}_{33} & \ddots & \vdots\\ \vdots & \ddots & \ddots & \bar{A}_{(N-1)N}\\ \bar{A}_{N2} & \cdots & \bar{A}_{N(N-1)} & \bar{A}_{NN}\end{pmatrix},$$

$$\bar{A}_{22} = C_1A_1C_1^\dagger - C_2B_2(\sum_{j=1}^{N} a_{2j}K_{2j}) - a_{12}C_1B_1K_{12},$$
$$\bar{A}_{23} = a_{23}C_2B_2K_{23} - a_{13}C_1B_1K_{13},$$
$$\bar{A}_{2N} = a_{2N}C_2B_2K_{2N} - a_{1N}C_1B_1K_{1N},$$
$$\bar{A}_{32} = a_{32}C_3B_3K_{32} - a_{12}C_1B_1K_{12},$$
$$\bar{A}_{33} = A_1 - C_3B_3(\sum_{j=1}^{N} a_{3j}K_{3j}) - a_{13}C_1B_1K_{13},$$
$$\bar{A}_{(N-1)N} = a_{(N-1)N}C_{N-1}B_{N-1}K_{(N-1)N} - a_{1N}C_1B_1K_{1N},$$
$$\bar{A}_{N2} = a_{N2}C_NB_NK_{N2} - a_{12}C_1B_1K_{12},$$
$$\bar{A}_{N(N-1)} = a_{N(N-1)}C_NB_NK_{N(N-1)} - a_{1(N-1)}C_1B_1K_{1(N-1)},$$
$$\bar{A}_{NN} = C_1A_1C_1^\dagger - C_NB_N(\sum_{j=1}^{N} a_{Nj}K_{Nj}) - a_{1N}C_1B_1K_{1N}.$$

In order to prove that the system (1) is output consensusable with respect to $\mathcal{U}_1$, it suffices to show that there exist matrices $K_{ij},\ i,j = 1,2,\cdots,N$, such that system (7) is stable which is equivalent to $\sigma(\bar{A}) \subseteq \mathbb{C}^-$.

In output consensus protocols (3), choose

$$K_{11} = K_{13} = K_{14}\cdots = K_{1N} = 0,$$

$$K_{ij} = 0,\ j \geq i,\ i = 2,3,\cdots,N.$$

Then $\bar{A}$ turns into

$$\bar{A} = \begin{pmatrix} \tilde{A}_{22} & 0 & \cdots & 0 \\ \tilde{A}_{32} & \tilde{A}_{33} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \tilde{A}_{N2} & \cdots & \tilde{A}_{N(N-1)} & \tilde{A}_{NN} \end{pmatrix},$$

where

$$\tilde{A}_{22} = C_1A_1C_1^\dagger - a_{21}C_2B_2K_{21} - a_{12}C_1B_1K_{12},$$
$$\tilde{A}_{32} = a_{32}C_3B_3K_{32} - a_{12}C_1B_1K_{12},$$
$$\tilde{A}_{33} = C_1A_1C_1^\dagger - C_3B_3(a_{31}K_{31} + a_{32}K_{32}),$$
$$\tilde{A}_{N2} = a_{N2}C_NB_NK_{N2} - a_{12}C_1B_1K_{12},$$
$$\tilde{A}_{N(N-1)} = a_{N(N-1)}C_NB_NK_{N(N-1)},$$
$$\tilde{A}_{NN} = C_1A_1C_1^\dagger - C_NB_N(\sum_{j=1}^{N-1} a_{Nj}K_{Nj}).$$

In this case, system (7) is stable if and only if there exist K_{12} and $K_{ij},\ j < i$ such that

$$\sigma\left(C_1A_1C_1^\dagger - a_{21}C_2B_2K_{21} - a_{12}C_1B_1K_{12}\right) \subseteq \mathbb{C}^-,$$

$$\sigma\left(C_1A_1C_1^\dagger - C_iB_i(\sum_{j=1}^{i-1} a_{ij}K_{ij})\right) \subseteq \mathbb{C}^-,\ i = 3,4\cdots,N.$$

Since condition (ii) holds, it is concluded from Lemma 3.1 that

$$(C_1A_1C_1^\dagger)^TX + X^T(C_1A_1C_1^\dagger) - X^T(C_1B_1 + C_2B_2)(C_1B_1 + C_2B_2)^TX + I_n = 0 \quad (8)$$

has the unique admissible solution P_2 and

$$(C_1A_1C_1^\dagger)^TX_i + X_i^T(C_1A_1C_1^\dagger) - X_i^TC_iB_i(C_iB_i)^TX_i + I_n = 0 \quad (9)$$

has the unique admissible solution $P_i,\ i = 3,4,\cdots,N$. Hence

$$\sigma\left(C_1A_1C_1^\dagger - (C_1B_1 + C_2B_2)(C_1B_1 + C_2B_2)^TP_2\right) \subseteq \mathbb{C}^-,$$

$$\sigma\left(C_1A_1C_1^\dagger - C_iB_i(C_iB_i)^TP_i\right) \subseteq \mathbb{C}^-,\ i = 3,4,\cdots,N.$$

Let

$$a_{12}K_{12} = a_{21}K_{21} = (C_1B_1 + C_2B_2)^TP_2 \quad (10)$$

and

$$\sum_{j=1}^{i-1} a_{ij}K_{ij} = (C_iB_i)^TP_i,\ i = 3,4,\cdots,N. \quad (11)$$

Under the condition (iii), it is obtained that equations (10) and (11) have at least one group solutions K_{12} and $K_{ij},\ j < i,\ i = 2,3,\cdots,N$, such that

$$\sigma\left(C_1A_1C_1^\dagger - a_{21}C_2B_2K_{21} - a_{12}C_1B_1K_{12}\right) \subseteq \mathbb{C}^-,$$

$$\sigma\left(C_1A_1C_1^\dagger - C_iB_i(\sum_{j=1}^{i-1} a_{ij}K_{ij})\right) \subseteq \mathbb{C}^-,\ i = 3,4\cdots,N.$$

Combining the above results, one obtains multi-agent system (1) is output consensusable with respect to $\mathcal{U}_1$. ■

Theorem *3.2:* System (1) is output consensusable with respect to $\mathcal{U}_2$ if the following conditions hold:

(i) $C_1A_1C_1^\dagger = C_2A_2C_2^\dagger = \cdots = C_NA_NC_N^\dagger$;

(ii) $(C_1A_1C_1^\dagger,\ C_1B_1 + C_2B_2)$ and $(C_1A_1C_1^\dagger,\ C_iB_i)$, $i = 3,4,\cdots,N$, are stabilizable;

(iii) $a_{12} \neq 0$, $a_{21} \neq 0$ and there exists at least one $j \in \{1,2,\cdots,i-1\}$ such that $a_{ij} \neq 0,\ i = 3,4,\cdots N$.

Proof: Here, this proof is omitted since it is very similar to that of Theorem 3.1. ■

Based on Theorem 3.1, give the following algorithm to design the output feedback control law $K_{ij},\ i,j = 1,2,\cdots N$, which can guarantee that system (1) is output consensusable with respect to $\mathcal{U}_1$ under the precondition of Theorem 3.1.

Algorithm *3.1:* Input: the matrices $A_i \in \mathbb{R}^{n_i\times n_i}$, $B_i \in \mathbb{R}^{n_i\times r_i}$, $C_i \in \mathbb{R}^{m\times n_i}$, $i = 1,2,\cdots,N$, and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$;

Output: the gain matrix $K_{ij},\ i,j = 1,2,\cdots,N$.

(a) Compute Moore-Penrose inverse of C_i, and it is denoted by $C_i^\dagger,\ i = 1,2,\cdots,N$;

(b) Solve Ricatti equations (8) and (9), and the admissible solutions are denoted by P_2 and $P_i,\ i = 3,4,\cdots,N$, respectively;

(c) Choose $K_{11}=K_{13}=K_{14}\cdots=K_{1N}=0,\ K_{ij}=0,\ j\geq i,\ i=2,3,\cdots,N$;
(d) Solve an arbitrary group of solutions of matrix equations (10) and (11), and denote them by K_{12} and $K_{ij},\ j<i,\ i=2,3,\cdots,N$, respectively.

Thus, the required matrices $K_{ij},\ i,j=1,2,3$, are designed.

IV. A Numerical Example

A numerical example is provided to demonstrate the application of Algorithm 3.1. Consider linear multi-agent system (1) consisting of $N=3$ heterogeneous agents with

$$A_1=\begin{bmatrix}-2 & 0.5\\ 1 & -1\end{bmatrix},\ A_2=\begin{bmatrix}0.5 & 0\\ -1 & 1\end{bmatrix},$$

$$A_3=\begin{bmatrix}0 & 1\\ -2 & 1\end{bmatrix},\ B_1=\begin{bmatrix}0.6 & 1\\ -5 & 1\end{bmatrix},$$

$$B_2=\begin{bmatrix}4 & 1.5\\ -1 & 0.9\end{bmatrix},\ B_3=\begin{bmatrix}0.3 & -1\\ 2 & 5\end{bmatrix},$$

$$C_1=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\ C_2=\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix},\ C_3=\begin{bmatrix}1 & 3\\ 1 & 2\end{bmatrix},$$

and the topology $\mathcal{G}=(\mathcal{V},\mathcal{E},\mathcal{A})$, where $\mathcal{V}=\{1,2,3\}$,

$$\mathcal{E}=\{(1,2),\ (1,3),\ (2,1),\ (2,3),\ (3,1),\ (3,2)\}$$

and

$$\mathcal{A}=\begin{bmatrix}0 & 0.5 & 2\\ 0.2 & 0 & 3\\ 1 & 0.4 & 0\end{bmatrix}.$$

According to the steps in Algorithm 3.1, the following can be obtained:

(a)

$$C_1^\dagger=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\ C_2^\dagger=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix},$$

$$C_3^\dagger=\begin{bmatrix}-2 & 3\\ 1 & -1\end{bmatrix};$$

(b)

$$P_2=\begin{bmatrix}0.2 & 0.1\\ 0.1 & 0.2\end{bmatrix},\ P_3=\begin{bmatrix}0.1 & 0.1\\ -0.1 & 0.2\end{bmatrix};$$

(c)

$$K_{11}=K_{13}=K_{22}=K_{23}=K_{33}=0_{n\times n};$$

(d)

$$K_{12}=\begin{bmatrix}0.9 & -1.3\\ 1.0 & 1.1\end{bmatrix},\ K_{21}=\begin{bmatrix}2.2 & -3.2\\ 2.5 & 2.7\end{bmatrix},$$

$$K_{31}=\begin{bmatrix}0.7 & 1\\ -1 & 2\end{bmatrix},\ K_{32}=\begin{bmatrix}-1.1 & -1.7\\ 4.1 & -3.4\end{bmatrix}.$$

Then the matrices $K_{ij},\ i,j=1,2,3$, are designed.

In this case, the closed-loop system made of (1) and (2) is formulated:

$$\dot{x}(t)=A_cx(t), \tag{12a}$$

$$y(t)=Cx(t), \tag{12b}$$

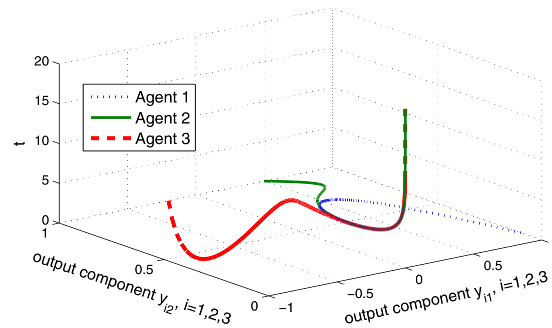

Fig. 1. the output responses of system (12)

where

$$A_c=\begin{bmatrix}A_{11} & a_{12}B_1K_{12}C_2 & a_{13}B_1K_{13}C_3\\ a_{21}B_2K_{21}C_1 & A_{22} & a_{31}B_2K_{31}C_3\\ a_{31}B_3K_{31}C_1 & a_{32}B_3K_{32}C_2 & A_{33}\end{bmatrix},$$

$$x=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix},\ y=\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix},\ C=\begin{bmatrix}C_1 & 0 & 0\\ 0 & C_2 & 0\\ 0 & 0 & C_3\end{bmatrix},$$

$$A_{11}=A_1-a_{12}B_1K_{12}C_1-a_{13}B_1K_{13}C_1,$$
$$A_{22}=B_2D_2C_1A_1C_1^\dagger C_2-a_{21}B_2K_{21}C_2-a_{31}B_2K_{31}C_2,$$
$$A_{33}=B_3D_3C_1A_1C_1^\dagger C_3-a_{31}B_3K_{31}C_3-a_{32}B_3K_{32}C_3.$$

Choose the initial states of system (1) arbitrarily as follows:

$$x_1(0)=\begin{bmatrix}1\\ 0.1\end{bmatrix},\ x_2(0)=\begin{bmatrix}1.5\\ 1\end{bmatrix},\ x_3(0)=\begin{bmatrix}3.4\\ -1.2\end{bmatrix}.$$

Denote

$$y_i=\begin{bmatrix}y_{i1}\\ y_{i2}\end{bmatrix},\ i=1,2,3.$$

The simulation of system (12) is presented in Figure 1. It shows that the output responses of the system (12) indicates multi-agent system (1) is output consensusable with respect to the given admissible set $\mathcal{U}_1$.

V. Conclusion

For linear multi-agent systems with fixed topologies and agents consisting of general linear systems, the conditions of output consensusability have been provided using generalized inverse, graph and linear system theory. The designed output feedback control law can guarantee that the studied multi-agent systems are output consensusable with respect to a given admissible set. Moreover, the simulation results have successfully demonstrated the applicability of the proposed approach in this paper.

The study of output consensus conditions for linear multi-agent systems with fixed topologies is a basic problem which only serves as a stepping stone to study more complicated agent dynamics. Future research will be on multi-agent systems with time delays, switching topology or time-varying topology, and so on.

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