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Distributed Practical Consensus in Multi-agent Networks with Communication Constrains

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***Abstract*—This paper deals with the multi-agent consensus problem subject to communication constrains. Two types of communication constrains are discussed in this paper: i) each agent can only exchange quantized data with its neighbors and ii) each agent can only obtain the delayed information from its neighbors. Solutions of the resulting system are defined in the Filippov sense. For the consensus protocol which only considers quantization effect, we prove that Filippov solutions converge to a practical consensus set in a finite time. For the consensus protocol considering quantization and time delay simultaneously, it is shown that Filippov solutions converge to a practical consensus set asymptotically. Moreover, we also present how initial state of the agents, time delay and quantization parameter affect the final practical consensus set. Numerical examples are provided to demonstrate the effectiveness of the obtained theoretical results.**

***Index Terms*—Multi-agent networks, consensus, time delay, quantization.**

I. Introduction

The concept of consensus originates from the cooperative control, which means all agents reach an agreement on certain quantities of interest. Recently, consensus control has become one of the most focused problems in distributed coordination control of multi-agent networks due to its broad applications in the fields of unmanned aerial vehicles(UAVs), clusters of satellites, automated highway systems and congestion control in communication networks (see, e.g., [1]–[5]).

Due to the physical location, communication bandwidth and unavoidable information losses, only limited information can be sent, transmitted and received by agents in real systems. Hence, consideration of the communication constrains is necessary and important for the design of control strategy or algorithm (see, e.g., [6]–[11]). Two important phenomena in communication are signal quantization and time delay. Unlike the error-free information exchange, the signals in real-world systems are required to be quantized before transmission when high data rate are not available. Some related results about network control system with quantized data have been reported in [8], [11]. In [8], robust H_∞ estimation problems for uncertain systems subject to quantization are investigated. In [11], H_∞ filter is designed for a class of nonlinear discrete time-varying stochastic systems with quantization effects. On the other hand, consensus problems with quantized data are also challenging and desired to be investigated. Discrete-time consensus protocols with quantization have been extensively studied (see [7], [9], [10], [12]–[14]).

Besides quantization, another significant communication constraint in multi-agent networks is the time delay, which is usually caused by an agent waiting to send out message via a busy channel, or by a signal processing and propagation [15]–[17]. It has been shown that conventional consensus protocol with time delay may lead to unexpected results [16].

From the viewpoint of both mathematics and engineering, it is worth noting that the quantization will lead to a system with no solutions in classical sense. Hence, considering solutions in a more general sense is necessary. A ground work have been laid in [18], which extend the quantized consensus model to the continuous-time case and quantized consensus results have been obtained for the network model. In this paper, we shall further extend the previous results of [18] by using different methods. Moreover, we shall address the consensus problems with consideration of the time delay and quantization simultaneously.

The main contributions of this paper are presented as follows:

- Two consensus protocols considering different communication constraints are proposed in this paper. The convergence analysis of the proposed protocols are discussed in detail.
- The effect of the time delay and quantization on the practical consensus set is obtained in this work.

The organization of the remaining part is given as follows. In Section II, some preliminaries about algebra graph theory and discontinuous differential equations are summarized. In Section III and IV, consensus analysis of the proposed protocol are presented in detail. In Section V, two numerical simulation examples are given to show the effectiveness of the theoretical results. In Section VI, concluding remarks are drawn.

$Notations$: The standard notations will be used in this paper. Throughout this paper, $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ represent n-dimensional Euclidean space and the set of $m\times n$ real matrices, respectively. $\mathbb{Z}$ denotes a set containing all integer numbers. The superscript "T" represents the transpose. $\lfloor\cdot\rfloor$ denotes the lower integer function. $[a, b]$ means the closed interval with endpoints a and b. Let $\mathcal{C}([-\tau, 0]; \mathbb{R})$ denote the family of all continuous $\mathbb{R}$-valued functions $g(s)$ on $[-\tau, 0]$ with the norm $\|g(.)\| = sup_{-\tau\leq s\leq 0}|g(s)|$.

II. Preliminaries and problem formulation

A. Basic graph theory

Let $\mathcal{G}(\mathcal{V},\, \mathcal{E},\, \mathcal{A})$ be a *weighted directed graph* with the set of nodes $\mathcal{V} = \{v_1,\, v_2,\, \ldots, v_N\}$, the set of edges $\mathcal{E} \subset \mathcal{V}\times\mathcal{V}$ and a weighted adjacency matrix $\mathcal{A} = [\bar{a}_{ij}]$ with nonnegative adjacency elements $\bar{a}_{ij}$. An edge of $\mathcal{G}$ is denoted by $e_{ij} = (v_i,\, v_j)$, where v_i and v_j are called the parent and child vertices, respectively. For adjacency matrix $\mathcal{A}$, $(v_i,\, v_j) \in \mathcal{E} \Longleftrightarrow \bar{a}_{ji} > 0$. For the undirected graph, one has $\bar{a}_{ij} = \bar{a}_{ji}$. The set of *neighbors* of v_i in $\mathcal{G}$ is denoted by $\mathcal{N}_i(v_i) = \{v_j : (v_j,\, v_i) \in \mathcal{E}\}$. Let $A = \mathcal{A} - \Delta = [a_{ij}]$, where $\Delta = [\Delta_{ij}]_{N\times N}$ is a diagonal matrix with $\Delta_{ii} = \sum_{j=1}^{N} \bar{a}_{ij}$. Then, we have $a_{ij} = \bar{a}_{ij} \geq 0$ for $i \neq j$ and $a_{ii} = -\sum_{j=1,\, j\neq i}^{N} a_{ij}$. The matrix $-A$ is called the *Laplacian* of the directed graph. A *path* in a digraph is an ordered sequence of vertices such that any two consecutive vertices are an directed edge of the digraph. A directed graph $\mathcal{G}$ is *strongly connected* if there is a path for any pair of distinct vertices in $\mathcal{G}$.

B. Discontinuous differential equations

For differential equations with discontinuous right-hand sides we understand the solutions in terms of differential inclusions following Filippov (1988).

Now we introduce the concept of Filippov solution. Consider the following system

$$\frac{d\mathbf{x}(t)}{dt} = f(\mathbf{x}(t)), \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$ is Lebesgue measurable and locally essentially bounded.

Definition 1: A set-valued map is defined as

$$\mathcal{K}(f(\mathbf{x})) = \bigcap_{\delta>0}\bigcap_{\mu(N)=0} \bar{co}[f(B(\mathbf{x},\delta)\setminus N)], \tag{2}$$

where $\bar{co}(\Omega)$ is the closure of the convex hull of set Ω, $B(\mathbf{x},\delta) = \{y : \|y-\mathbf{x}\| \leq \delta\}$, and $\mu(N)$ is Lebesgue measure of set N. A solution in the sense of Filippov of the Cauchy problem for equation (1) with initial condition $\mathbf{x}(0) = \mathbf{x}_0$ is an absolutely continuous function $\mathbf{x}(t), t \in [0, T]$, which satisfies $\mathbf{x}(0) = \mathbf{x}_0$ and differential inclusion:

$$\frac{d\mathbf{x}}{dt} \in \mathcal{K}(f(\mathbf{x})), \quad a.e.\ t \in [0, T], \tag{3}$$

where $\mathcal{K}(f(\mathbf{x})) = (\mathcal{K}[f_1(\mathbf{x})], \cdots, \mathcal{K}[f_n(\mathbf{x})])$.

Let $h : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function and S_h be the set of points where h fails to be differentiable. Clarke generalized gradient of h at $\mathbf{x}$ is the set $\partial_c h(\mathbf{x}) = co\{\lim_{i\to+\infty} \nabla h(\mathbf{x}^{(i)}) : \mathbf{x}^{(i)} \to \mathbf{x}, \mathbf{x}^{(i)} \notin S \cup S_h\}$, where S can be any set of zero measure [19]. A Filippov solution to (3) is a maximal solution if it cannot be extended further in time.

III. Finite-time Practical Consensus under the effects of quantization

Consider the following multi-agent system with dynamics

$$\dot{x}_i(t) = u_i(t), \quad i = 1, \ldots, N,$$

where $x_i(t) \in R$ is the state of the agent i, and $u_i(t)$ is called the consensus protocol. The following consensus protocol

$$u_i(t) = \sum_{j\in\mathcal{N}_i} a_{ij}(x_j(t) - x_i(t)), \quad i = 1, \ldots, N, \tag{4}$$

has been proposed in [1], which requires that each agent receives information from its neighbors timely and accurately.

Due to the communication bandwidth constraints in many real multi-agent networks, the agents can only use the quantized information of the neighboring agents. The following consensus protocol will be studied in this part.

$$\frac{dx_i(t)}{dt} = \sum_{j\in\mathcal{N}_i} a_{ij}[q_\mu(x_j(t)) - q_\mu(x_i(t))], \quad i = 1, \ldots, N, \tag{5}$$

where $q_\mu(z)$ denotes one-parameter family of uniform quantizers defined by $q_\mu(z) = \lfloor \frac{z}{\Delta\mu} + \frac{1}{2} \rfloor \mu$. Here, μ and Δ are called the *quantization parameters* and *error bound* of the quantizer, respectively. Moreover, if $\mathbf{x} = (x_1,\, x_2,\, ...,\, x_N)^T \in \mathbb{R}^N$, we denote $q_\mu(\mathbf{x}) = (q_\mu(x_1),\, q_\mu(x_2),\, ...,\, q_\mu(x_N))^T$.

We know that system (5) may not have the global solution in the sense of $Carath\acute{e}odory$ due to the discontinuous of function $q_\mu(.)$ [18]. Hence, we shall consider solutions in a more general sense, i.e, the Filippov solution of system (5). The concept of the Filippov solution to the differential equation (5) is as follows:

Definition 2: A absolutely continuous function $\mathbf{x} : [0,\, T) \to R^N$ is a solution in the sense of Filippov for the discontinuous system (5) if $\mathbf{x}$(t) satisfy that

$$\frac{dx_i(t)}{dt} \in \mathcal{K}[\sum_{j\in\mathcal{N}_i} a_{ij}(q_\mu(x_j(t)) - q_\mu(x_i(t)))], \quad i = 1, \ldots, N.$$

Based on the measurable selection theorem of set-valued function (see [20], p. 308, Th. 8.1.3), if $\mathbf{x}(t)$ is a Filippov solution of system (5), then there exists a measurable function $\gamma(t) \in \mathcal{K}[q_\mu(\mathbf{x}(t)))]$ such that for almost all $t \in [0,\, T)$, the following equation is true:

$$\frac{dx_i(t)}{dt} = \sum_{j=1,\, j\neq i}^{N} a_{ij}(\gamma_j(t) - \gamma_i(t)), \quad i = 1, \ldots, N. \tag{6}$$

Any function γ as in (6) is called an *output* associated to the solution $\mathbf{x}$.

Remark 1: Due to the introduction of the quantization effect, complete consensus cannot be ensured by the proposed protocol, but only *practical consensus* can be achieved. That is, the final consensus values lie within an interval, so called *practical consensus set*, as discussed in [18] and [21].

Lemma 1: Suppose $x(t)$ be a Filippov solution to (5). Let $\mathcal{N} = \{1, \ldots, N\}$, $M(t) = \max_{i \in \mathcal{N}} x_i(t)$ and $m(t) = \min_{i \in \mathcal{N}} x_i(t)$. Then, $M(t)$ is a non-increasing function for t and $m(t)$ is a non-decreasing function for t.

Proof: The proof is similar to the one of Lemma 3, [22]. We omit here due to the length limit. ■

Let $\xi = \{\xi_1, \xi_2, \ldots, \xi_N\}$ be the normalized left eigenvector of matrix A with respect to the zero eigenvalue satisfying $\sum_{i=1}^{N} \xi_i = 1$. It can be obtained that $\xi_i > 0$ from Perron-Frobenius theorem (see [23]). Denote $\mathcal{N} = \{1, \ldots, N\}$ in the following part.

Theorem 1: Consider the multi-agent network (5) with a strongly connected graph G. The initial conditions associated with (5) are given as $x_i(0)$, $(i = 1, 2, \ldots, N)$. Let $k = \lfloor \frac{\sum_{i=1}^{N} \xi_i x_i(0)}{\mu\Delta} + \frac{1}{2} \rfloor$. Then $x_i(t)$ will converge to the set $\mathcal{D} = [(k - \frac{1}{2})\Delta\mu, (k + \frac{1}{2})\mu\Delta]$ for any $i \in \mathcal{N}$ in a finite time, where μ and Δ are quantization parameters and error bound of the quantizer.

Proof: The proof of Theorem 1 is divided into two parts.

Part (I) we shall take three steps to prove that each agent in the network will converge to a set $[(k - \frac{1}{2})\mu\Delta, (k + \frac{1}{2})\mu\Delta]$ in finite time.

Step 1. Consider the Lyapunov functional as

$$V(t) = \sum_{i=1}^{N} \xi_i \int_0^{x_i(t)} q_\mu(s) ds. \tag{7}$$

Note that $cq_\mu(c) \geq 0$ for any $c \in \mathbb{R}$, we have $V(t) \geq 0$.

Notice that for $p_i(s) = \int_0^s q_\mu(u) du$, we have $\partial p_i(s) = \{v \in \mathbb{R} : q_\mu^-(s) \leq v \leq q_\mu^+(s)\}$. Based on the chain rule (for details, see Proposition 6, [24]), $V(t)$ is differentiable for a.e. $t \geq 0$. Differentiating $V(t)$ along the solution of (6) gives that

$$\begin{aligned} \frac{dV(t)}{dt} &= \sum_{i=1}^{N} \xi_i \gamma_i(t) \sum_{j=1, j \neq i}^{N} a_{ij}[\gamma_j(t) - \gamma_i(t)] \\ &= \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} \xi_i a_{ij}[2\gamma_i(t)\gamma_j(t) - 2\gamma_i^2(t)]. \end{aligned} \tag{8}$$

Notice that

$$\begin{aligned} &\sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} \xi_i a_{ij} \gamma_i^2(t) = \sum_{i=1}^{N} (-a_{ii}) \xi_i \gamma_i^2(t) \\ &= \sum_{j=1}^{N} (-a_{jj}) \xi_j \gamma_j^2(t) = \sum_{j=1}^{N} \sum_{i=1, i \neq j}^{N} \xi_i a_{ij} \gamma_j^2(t) \\ &= \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} \xi_i a_{ij} \gamma_j^2(t), \end{aligned} \tag{9}$$

we have

$$\frac{dV(t)}{dt} = -\frac{1}{2} \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} \xi_i a_{ij} (\gamma_i(t) - \gamma_j(t))^2. \tag{10}$$

Step 2. Let $\Phi = \{\mathbf{x}(t) \in \mathbb{R}^N :\mid \gamma_i(t) - \gamma_j(t) \mid < \frac{\mu}{N+1}, \forall i, j \in \mathcal{N}, i \neq j, a_{ij} \neq 0\}$. We claim that the agents in the network converge to the set Φ in finite time.

Let $J = \{t \geq 0 : \mathbf{x}(t) \notin \Phi\}$. For $\mathbf{x}(t) \in \mathbb{R}^N$ and $t \in J$, there exist $i, j \in \{1, 2, \ldots, N\}$, $i \neq j$ and $a_{ij} \neq 0$ such that $\mid \gamma_i(t) - \gamma_j(t) \mid \geq \frac{\mu}{N+1}$. Hence, for $a.e.\, t \in J$

$$\begin{aligned} \dot{V}(t) &\leq -\frac{1}{2} \xi_i a_{ij} (\frac{\mu}{N+1})^2 \\ &\leq -\frac{1}{2} \zeta \mu^2, \end{aligned} \tag{11}$$

where $\zeta = \min_{1 \leq i, j \leq N, a_{ij} > 0} [(\frac{1}{N+1})^2 a_{ij} \xi_i]$. Further, we can obtain

$$\begin{aligned} 0 &\leq V(t) \\ &\leq V(0) - \frac{1}{2} \zeta \mu^2 t, \; t \geq 0. \end{aligned} \tag{12}$$

It follows from $V(0) \geq 0$ that (12) holds if and only if $t \leq \frac{2V(0)}{\zeta\mu^2}$. Therefore, $\mathbf{x}(t)$ will arrive to the set Φ in finite time.

Step 3. we shall prove that there exists $k \in \mathbb{R}$ such that $x_i(t)$ will converge to the set $[(k - \frac{1}{2})\mu\Delta, (k + \frac{1}{2})\mu\Delta]$ in finite time for every $i \in \mathcal{N}$.

Note that $\gamma_i(t) \in \mathcal{K}[q_\mu(x_i(t))]$ and $\gamma_j(t) \in \mathcal{K}[q_\mu(x_j(t)]$, we can get that there exist $k_{ij} \in \mathbb{R}$ such that $x_i(t)$ and $x_j(t)$ belong to the set $[(k_{ij} - \frac{1}{2})\mu\Delta, (k_{ij} + \frac{1}{2})\mu\Delta]$ if $\mid \gamma_i(t) - \gamma_j(t) \mid < \frac{\mu}{N+1}$. Hence, based on the proof of step 2, there exists a $T_0 \geq 0$ such that $\forall i, j \in \mathcal{N}$, $i \neq j$, $a_{ij} \neq 0$, $x_i(T_0)$ and $x_j(T_0)$ belong to the set $[(k_{ij} - \frac{1}{2})\mu\Delta, (k_{ij} + \frac{1}{2})\mu\Delta]$. Due to the network is strongly connected, there exists a $k \in \mathbb{R}$ such that $k_{ij} = k$.

It follows from Lemma 1 that $x_i(t) \in [(k - \frac{1}{2})\mu\Delta, (k + \frac{1}{2})\mu\Delta]$ for $\forall i \in \mathcal{N}$, $t \geq T_0$.

Part (II) Estimate the value of k.

Up till now, we have proved the first part of the Theorem 1. Next, we shall give the value of k which is shown to be depending on the initial values of the multi-agent network. Let $\eta(t) = \sum_{i=1}^{N} \xi_i x_i(t)$. We can calculate the derivative of $\eta(t)$ as follows:

$$\begin{aligned} \dot{\eta}(t) &= \sum_{i=1}^{N} \xi_i \sum_{j=1, j \neq i}^{N} a_{ij}[\gamma_j(t) - \gamma_i(t)] \\ &= \sum_{i=1, i \neq j}^{N} \xi_i a_{ij} \sum_{j=1}^{N} \gamma_j(t) - \sum_{i=1}^{N} \xi_i \gamma_i(t) \sum_{j=1, j \neq i}^{N} a_{ij} \\ &= -\sum_{j=1}^{N} \xi_j a_{jj} \gamma_j(t) + \sum_{i=1}^{N} \xi_i a_{ii} \gamma_i(t) \\ &= 0. \end{aligned}$$

Due to $\dot{\eta}(t) = 0$ for a.e. $t \in [0, \infty)$ and the continuity of $\eta(t)$, it can be easily obtained that $\eta(t)$ is a constant. That is, $\eta(t) = \eta(0) = \sum_{i=1}^{N} \xi_i x_i(0)$.

Next, we need to estimate the value of k. Let $\mathcal{D} = [(k - \frac{1}{2})\mu\Delta, (k + \frac{1}{2})\mu\Delta]$, we have proved that there exists a T_0 such that $x_i(t) \in \mathcal{D}, \quad \forall t \geq T_0, \quad \forall i \in \mathcal{N}$.

It follows from $\sum_{i=1}^{N}\xi_i = 1$ that $\sum_{i=1}^{N}\xi_i x_i(t) \in \mathcal{D}, \quad \forall t \geq T_0$. Thus, $\sum_{i=1}^{N}\xi_i x_i(0) \in [(k-\frac{1}{2})\mu\Delta,\, (k+\frac{1}{2})\mu\Delta]$.

We consider the following two cases:

Case 1: If there exists a $k_0 \in \mathbb{Z}$ such that $\sum_{i=1}^{N}\xi_i x_i(0) = (k_0 - \frac{1}{2})\mu\Delta$, then $k = k_0$ or $k_0 - 1$. Since $x_i(t) \in \mathcal{D}, \quad \forall t \geq T_0, \quad \forall i \in \mathcal{N}$, we have $x_i(t) = (k_0 - \frac{1}{2})\mu\Delta, \quad \forall t \geq T_0$. In this case, we can select $k = k_0$. That is, $k = \frac{1}{\mu\Delta}\sum_{i=1}^{N}\xi_i x_i(0) + \frac{1}{2} = \lfloor \frac{1}{\mu\Delta}\sum_{i=1}^{N}\xi_i x_i(0) + \frac{1}{2}\rfloor$.

Case 2: If $\sum_{i=1}^{N}\xi_i x_i(0) \neq (k_0 - \frac{1}{2})\mu\Delta$ for any $k \in \mathbb{Z}$, then, $\sum_{i=1}^{N}\xi_i x_i(0) \in ((k-\frac{1}{2})\mu\Delta,\, (k+\frac{1}{2})\mu\Delta)$. Hence, $k = \lfloor \frac{1}{\mu\Delta}\sum_{i=1}^{N}\xi_i x_i(0) + \frac{1}{2}\rfloor$.

Therefore, $k = \lfloor \frac{1}{\mu\Delta}\sum_{i=1}^{N}\xi_i x_i(0) + \frac{1}{2}\rfloor$. This completes the proof of this theorem. ■

Remark 2: As discussed in the Theorem 2 and Proposition 4 of [18], the practical consensus results for the model (5) with $\Delta = 1$ are investigated via LaSalle invariance principle of differential inclusions. While in this paper, using different methods, we extend the previous results from the following three aspects:

- We do not assume the network is undirected or balanced.
- We show that the Filippov solutions of (5) reach set $\mathcal{D} = [(k-\frac{1}{2})\mu,\, (k+\frac{1}{2})\mu]$ in a finite time even if $x_{ave}(0) = \frac{1}{N}\sum_{i=1}^{N} x_i(0) = (k_0 + \frac{1}{2})\mu$ for some $k_0 \in \mathbb{Z}$.
- We present an explicit relationship between the practical consensus set and initial conditions.

Corollary 1: Consider the multi-agent network (5) with a strongly connected graph G. The initial conditions associated with (5) are given as $x_i(0), (i = 1, 2, \ldots, N)$. Let $k = \lfloor \frac{1}{\mu\Delta}\sum_{i=1}^{N}\xi_i x_i(0) + \frac{1}{2}\rfloor$. Then $x_i(t)$ will converge to the set $\Omega = [\sum_{i=1}^{N}\xi_i x_i(0) - \mu\Delta,\, \sum_{i=1}^{N}\xi_i x_i(0) + \mu\Delta]$ in a finite time, where μ and Δ are quantization parameters and error bound of the quantizer.

Proof: According to Theorem 1, $x_i(t)$ converges to the set $[(k-\frac{1}{2})\mu\Delta,\, (k+\frac{1}{2})\mu\Delta]$ in a finite time, where $k = \lfloor \frac{1}{\mu\Delta}\sum_{i=1}^{N}\xi_i x_i(0) + \frac{1}{2}\rfloor$. It follows from $\sum_{i=1}^{N}\xi_i x_i(0) - \frac{1}{2}\mu\Delta \leq k\mu\Delta \leq \sum_{i=1}^{N}\xi_i x_i(0) + \frac{1}{2}\mu\Delta$ that $x(t)$ will converge to the set $\Omega = [\sum_{i=1}^{N}\xi_i x_i(0) - \mu\Delta,\, \sum_{i=1}^{N}\xi_i x_i(0) + \mu\Delta]$ in a finite time. ■

Remark 3: From Corollary 1, how the initial condition of the agents and quantization parameter μ affect the practical consensus set Ω can be observed explicitly. It is interesting to observe that the size of the practical consensus set can be made arbitrarily small by decreasing the quantization parameter μ.

IV. Practical Consensus under Quantization and Time Delay

Considering time delay as another very important communication constraint in the process of information exchange, we propose the following practical consensus protocol:

$$\frac{dx_i(t)}{dt} = \sum_{j\in\mathcal{N}_i} a_{ij}[q_\mu(x_j(t-\tau)) - q_\mu(x_i(t))], \quad i = 1, \ldots, N, \tag{13}$$

where τ is the communication delay from agent j to agent i and $q(.)$ is the same as in model (5). The initial conditions associated with (13) are given as $x_i(s) = \phi_i(s) \in \mathcal{C}([-\tau, 0],\, R) \quad (i = 1, 2, \ldots, N)$.

Next, we will present the the Filippov solution of system (13).

Definition 3: A function $\mathbf{x} : [-\tau,\, T) \to R^N$ (T might be ∞) is a solution in the sense of Filippov for the discontinuous system (13) on $[-\tau,\, T)$, if

1) $\mathbf{x}$ is continuous on $[-\tau,\, T)$ and absolutely continuous on $[0,\, T)$;
2) $\mathbf{x}$(t) satisfy that

$$\frac{dx_i(t)}{dt} \in \mathcal{K}[\sum_{j\in\mathcal{N}_i} a_{ij}(q_\mu(x_j(t-\tau)) - q_\mu(x_i(t)))], \quad i = 1, \ldots, N. \tag{14}$$

It follows from Theorem 1 in [25] that

$$\begin{aligned} &\mathcal{K}[\sum_{j\in\mathcal{N}_i} a_{ij}(q_\mu(x_j(t-\tau)) - q_\mu(x_i(t)))] \\ \subseteq &\sum_{j\in\mathcal{N}_i} a_{ij}(\mathcal{K}[q_\mu(x_j(t-\tau))] - \mathcal{K}[q_\mu(x_i(t))]). \end{aligned} \tag{15}$$

Similar to (6), if $\mathbf{x}(t)$ is the solution of system (13), then there exists a measurable function $\gamma(t) \in \mathcal{K}[q_\mu(\mathbf{x}(t)))]$ such that for a.e. $t \in [0,\, T)$, the following equation is true:

$$\frac{dx_i(t)}{dt} = \sum_{j=1}^{N} a_{ij}(\gamma_j(t-\tau) - \gamma_i(t)), \quad i = 1, \ldots, N. \tag{16}$$

Now, we shall present the definition of an initial value problem associated to (13).

Definition 4: For any continuous function $\phi : [-\tau,\, 0] \to \mathbb{R}^N$ and any measurable selection $\psi : [-\tau,\, 0] \to \mathbb{R}^N$, such that $\psi(s) \in \mathcal{K}[q_\mu(\phi(s))]$ for a.e. $s \in [-\tau,\, 0]$, an absolute continuous function $\mathbf{x}(t) = \mathbf{x}(t,\, \phi,\, \psi)$ is said to be a solution of Cauchy problem for system (13) on $[0,\, T)$ with initial value $(\phi,\, \psi)$, if

$$\begin{cases} \dot{x}(t) = \sum_{j=1}^{N} a_{ij}(\gamma_j(t-\tau) - \gamma_i(t)) \\ for\ \ a.e.\ \ t \in [0,\, T), \quad i = 1, \ldots, N, \\ \mathbf{x}(s) = \phi(s), \quad \forall s \in [-\tau,\, 0], \\ \gamma(s) = \psi(s) \quad a.e.\ \ s \in [-\tau,\, 0]. \end{cases} \tag{17}$$

Note that the solution of the system (17) depends on the initial function ϕ and also on the selection of the output $\psi(s) \in \mathcal{K}[q_\mu(\phi(s))]$. In the following part, we shall show that the global solution for system (17) exists. The proof is omitted here due to the length limit.

Theorem 2: For any initial function ϕ and the selection of the output $\psi(s) \in \mathcal{K}[q_\mu(\phi(s))]$, there exists the global solution for system (17).

Next, we shall present the consensus result under the effects of quantization and time delay simultaneously. The proof will be presented in the full-length version of the paper. The initial conditions associated with (5) are given as $x_i(s) = \phi_i(s) \in \mathcal{C}([-\tau, 0],\, \mathbb{R}), \quad (i = 1, 2, \ldots, N)$. The Filippov solution of

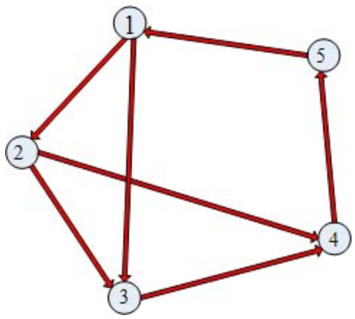

Fig. 1. Network topology in Example 1.

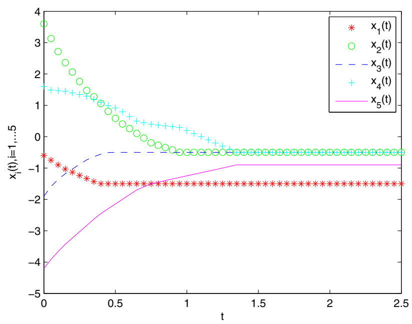

Fig. 2. The states of the multi-agent network in Example 1.

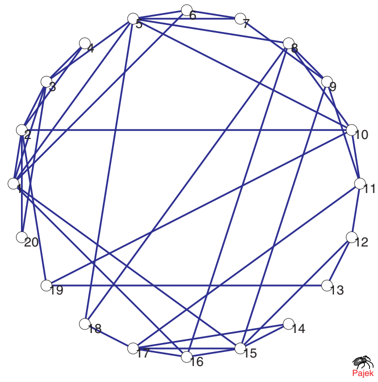

Fig. 3. Network topology in Example 2.

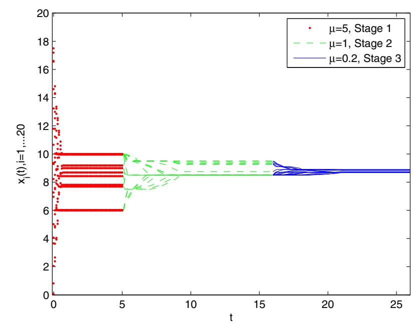

Fig. 4. The states trajectories of the multi-agent networks at different stages with respect to different μ.

system (13) is defined in (17) and $\psi_j(s)$ $(s \in [-\tau, 0])$ is the initial condition of measurable selection of $\gamma_j(s)$. Let $\eta(0) = \frac{1}{N}\sum_{i=1}^{N} x_i(0) + \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1,\, j\neq i}^{N} a_{ij}\int_{-\tau}^{0}\psi_j(s)ds$ and $A = \mu(\Delta - \frac{\tau}{N}\sum_{i=1}^{N} a_{ii})$, where μ and Δ are quantization parameters and error bound of the quantizer.

Theorem 3: Consider the undirected multi-agent network (13) with connected graph G. Then, for any finite communication delay τ, each agent in the network will converge to the set of $\Omega_1 = [(k-\frac{1}{2})\mu\Delta,\ (k+\frac{1}{2})\mu\Delta]$ asymptotically, where $k = \lfloor \frac{\eta(0)}{A} \rfloor$ or $\lfloor \frac{\eta(0)}{A} \rfloor + 1$.

V. Numerical Examples

In this section, two examples are given to illustrate the correctness of the theoretical results.

Example 1: Consider the multi-agent system (5) with five agents, where $\mu = 1$ and $\Delta = 1$. The directed network topology is displayed in Figure 1, and the weight of each edge is set as 1.

Figure 2 shows the state trajectories of (5) with the initial condition randomly chosen from (-5,5). It can be observed from Figure 1 that the state of each agent converges to a practical consensus set in a finite time, which illustrates Theorem 1 very well.

In many real multi-agent consensus problems, the size of this consensus set is required to be very small. In the following example, we will show that the size of the practical consensus set can be reduced by decreasing the quantization parameter μ, which also illustrates Corollary 1 well.

Example 2: In order to illustrate how quantization parameter μ affects the practical consensus set and how to reduce the size of the practical consensus set, a multi-agent network (5) with 20 agents is considered. The graph (Figure 3) is assumed to be a small-world network (undirected), in which each node has 2 nearest neighbors and the rewiring probability of the edges is 0.5 (see [26]). Let $\Delta = 1$ and initial conditions are randomly chosen from (0,20). Let the quantization parameter μ be updated at each stage of practical consensus:

$$\mu = \begin{cases} 5, & 5 > t \geq 0\ (Stage\,1), \\ 1, & 16 > t \geq 5,\ (Stage\,2) \\ 0.2, & 42 > t \geq 16\ (Stage\,3). \end{cases} \quad (18)$$

Note that the value of parameter μ is reduced by 80% after each stage of practical consensus in order to reduce the size of the consensus set.

Figure 4 shows the states of the 20 agents at different stages with respect to $\mu = 5$, 1 and 0.2, respectively. At each stage, all the agents converge to a practical consensus set in a finite time. After a smaller value of μ is chosen, the agents converge to a new set with a smaller range. The final consensus states of the 20 agents at different stages are respectively shown in Figure 5. From Figures 4 and 5, we can observe that the size of the consensus set is greatly reduced as μ decreases at different stages. These two figures also verify Corollary 1, that is, the size of consensus set is controlled by μ.

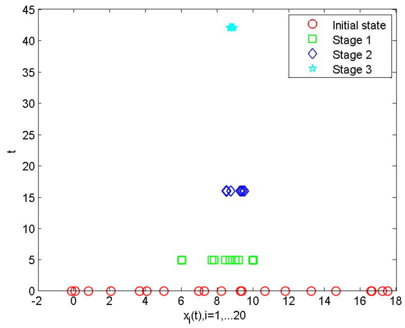

Fig. 5. The final states of the 20 agents at different stages with respect to different μ.

VI. Conclusion

This paper investigates the continuous-time consensus problem of multi-agent network where each agent can only obtain the quantized and delayed measurements of the states of its neighbors. Filippov solutions of the resulting system exist for any initial condition. We have proved that under certain network topology, the states of the multi-agent network which only considers quantization effect will converge to a practical consensus set in a finite time. For the multi-agent network model considering quantization and time delay simultaneously, it is shown that Filippov solutions converge to a practical consensus set asymptotically. Moreover, we also present how initial state of the agents, time delay and quantization parameter affect the final practical consensus set. The theoretical results have been well illustrated by two numerical examples.

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