UKACC International Conference on Control 2012
Cardiff, UK, 3-5 September 2012

Necessary and Sufficient Conditions for Consensus of Multi-Agent Systems with Nonlinear Dynamics and Variable Topology

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Abstract—**This paper studies some necessary and sufficient conditions for consensus of continuous multi-agent systems with nonlinear node dynamics and variable topology. The multi-agent systems are under variable topology. Basic theoretical analysis is carried out for the case where for each agent the nonlinear dynamics are governed by the position terms of the neighbor nodes. A necessary and sufficient condition associated with eigenvalues is given to ensure consensus of the nonlinear multi-agent system. Based on this result, a simulation example is given to verify the theoretical analysis.**

Keywords- Nonlinear Multi-agent system; Consensus; Variable Topology

I. Introduction

Since consensus of multi-agent systems (MAS) is a fundamental problem in the MAS research area, it has attracted increasing attention of researchers from various disciplines of engineering, biology and science. In networks of agents, consensus means to reach an agreement regarding a certain quantity of interest that depends on the state of all agents. A consensus algorithm is an interaction rule that specifies the information exchange between an agent and all of its neighbors on the network. Such problems have been formulated as consensus of leaderless problems or leader-following problems [1-3]. For a cooperative multi-agent system, leaderless consensus means that each agent updates its state based on local information of its neighbors such that all agents eventually reach an agreement on a common value, while leader-following consensus means that there exists a virtual leader which specifies an objective for all agents to follow.

In the past few years, the multi-agent systems with integer dynamics [4-6] or invariant topology [1, 4, 5] have been widely studied by many researchers due to its simple construction and convenience to analyze. Certainly, there are some researchers spend efforts on multi-agent system with nonlinear dynamics [1, 7] or switching topologies [2, 6, 8], and there have been some outcomes. In [1] a pinning control algorithm was proposed to achieve leader-following consensus in a network of agents with nonlinear second-order dynamics. [7] proposed an adaptive distributed controller with a disturbance estimator to solve the consensus problem under fix topology. By using a common Lyapunov function, [2] extended leader-following consensus control for multi-agent systems, which ensured strong mean square consensus the result, to the switching topology case. In [6], the sampled control protocols are induced from continuous-time linear consensus protocol by using periodic sampling technology and zero-order hold circuit. Nevertheless, since consensus problem of multi-agent associated with both nonlinear dynamics and variable topology is extremely difficult and complicated, people rarely discuss it.

However, considering the fact that almost all the physical plants contain nonlinearity and the communication topology may change from time to time for that the velocity of each agent is time-varying and the communication radius is finite, nonlinear dynamics associated with variable topology consensus is of vital necessity. In order to overcome this problem, some knowledge of complex dynamical network is employed. Broadly speaking, the first-order nonlinear consensus problem of multi-agent systems can be treated as a special case of the synchronization problem of complex dynamical networks, which has been extensively studied in the past few decades [9-11]. [9] presents a sufficient and necessary condition of synchronization criteria for a time-varying complex dynamical network, which is cited as Lemma1 in this paper. Based on the author's work in [9], a necessary and sufficient condition associated with eigenvalues, which is briefer and easier to verify, is given to ensure consensus of the nonlinear multi-agent system with variable topology.

The paper is organized as follows. Some preliminaries of graph theory are briefly reviewed in Section 2. Main results are given in Section 3. To illustrate the proposed theoretical results, a numerical simulation is provided in Section 4. And finally, conclusions are drawn in Section 5.

II. Problem Statement

A. Notations

This work was supported by National Natural Science Foundation of Youth Science Fund(No. 61104086) the Key Technology Support Program of Scientific Research Base in Beijing Institute of Technology (No.2011CX02010), Excellent Young Scholars Research Fund of Beijing Institute of Technology(No. 2011CX04031) and Research Foundation of Beijing Institute of Technology(No. 20100642007)

Some mathematical notations are used throughout this paper. Denote $I_N \in \mathbb{R}^{N\times N}$ as an N-dimensional identity matrix, $1_N = [1,1,\cdots,1]^T \in \mathbb{R}^N$ as a vector of all ones. Let A^T and A^{-1} be the transpose and the inverse of matrix A, respectively. $\lambda_{\max}(A)$ denotes the maximal eigenvalue of matrix A. $\|\bullet\|$ denotes Euclidean norm.

B. Preliminaries in graph theory

A directed graph, denoted by $G=\{v,\varepsilon,A\}$ be a weighted digraph with a node set $v=\{1,2,\ldots,N\}$, an edge set $\varepsilon \subseteq v\times v$ and a weighted adjacency matrix $A=(a_{ij})_{N\times N}$ with nonnegative elements [12].We consider that $(i,j)\in\varepsilon$ if and only if vertex (node) i can send its information to vertex j. If $(i,i)\in e$, we say that vertex i has self-loop. In this paper, it is assumed that no self-loop exists. The set of neighbors of vertex i is denoted by $N_i=\{j \mid j\in v,(j,i)\in e\}$, where $j\notin N_i$, which means there is no information flow from vertex j to vertex i, then $a_{ij}=0$, otherwise $a_{ij}>0$. The in-degree and out-degree of node i are, respectively, defined as [13].

$$\deg_{in}(i)=\sum_{j=1,j\neq i}^{N} a_{ij} \quad , \quad \deg_{out}(i)=\sum_{j=1,j\neq i}^{N} a_{ji}$$

A digraph is called balanced if $\deg_{in}(i)=\deg_{out}(i)$, $\forall i\in v$ [12].

Denote $D \triangleq diag\{\deg_{in}(1),\deg_{in}(2),\ldots,\deg_{in}(N)\}$. Then the Laplacian matrix L of the graph G is defined as $L\triangleq D-A$.

C. Equations

The system to be considered in this paper is a multi-agent system composed of N nonlinear coupled agents, labeled from 1to N, which means $v=\{1,2,\ldots,N\}$. The multi-agent system with variable topology is assumed to have the following dynamics:

$$\dot{x}_i(t)=f(x_i(t))+u_i(t), i\in v \tag{1}$$

With control protocol

$$\begin{aligned} u_i(t)=\sum_{j\in N_i(t)}^{N} a_{ij}(t)\big(x_i(t)-x_j(t)\big)+ \\ b_i(t)\big(x_i(t)-s(t)\big), i\in v \end{aligned} \tag{2}$$

where $x_i\in\mathbb{R}$ and $u_i\in\mathbb{R}$ are the state and control input of agent i, respectively, $f(x_i)\in\mathbb{R}$ is a nonlinear continuous function to describe the self-dynamics of agent i. $a_{ij}(t)$ is the (i,j)-th entry of the adjacency matrix $A(t)\in\mathbb{R}^{N\times N}$ at time t, where $s\in\mathbb{R}$ is the state of the virtual leader for multi-agent system (1), which is an isolated agent described by $\dot{s}(t)=f(s(t))$. $b_i(t)$ indicates the accessibility of $s(t)$ by agent i at time t. $b_i(t)>0$ indicates the case that $s(t)$ is accessible by agent i, and $b_i(t)=0$ indicates the case that $s(t)$ is not accessible by agent i.

Let

$$B(t)=diag\{b_1(t),b_2(t),\cdots,b_N(t)\}$$
$$b(t)=col\{b_1(t),b_2(t),\cdots,b_N(t)\}.$$

Definition 1: The multi-agent system (1) is said to achieve synchronization if $\lim_{t\to\infty}\|x_i(t)-s(t)\|=0, i\in v$ for any initial condition.

Denote $\bar{G}=\{\bar{v},\bar{\varepsilon},\bar{A}\}$ as the graph considering $s(t)$ into G. Denote $\bar{L}(t)=L(t)+B(t)$ as the Laplacian matrices of $\bar{G}$ at time t.

III. MAIN RESULTS

This section presents necessary and sufficient conditions for nonlinear multi-agent system with variable topology.

Define

$$x(t)=col\{x_1(t),x_2(t),\cdots,x_N(t)\}$$

$$S(t)=col\{s(t),s(t),\cdots,s(t)\}$$

With the fact that $L1_N=0$ and $B1_N=b$, we can rewrite the multi-agent system (1) as

$$\begin{aligned}\dot{x}(t)&=f(x(t))+(L(t)+B(t))x(t)-B(t)1_N s(t)\\ &=f(x(t))+(L(t)+B(t))(x(t)-S(t))\end{aligned} \tag{3}$$

Define

$\xi(t)=col\{\xi_1(t),\xi_2(t),\cdots,\xi_N(t)\}$ with $\xi_i(t)=x_i(t)-s(t)$.

Then the error closed-loop system can be deduced,

$$\dot{\xi}(t)=f(\xi(t)+s(t))-f(s(t))+(L(t)+B(t))\xi(t) \tag{4}$$

Lemma1*[9]:* Suppose that $Df(s(t))$, which is the Jacobian of f evaluated at $s(t)$, is bounded. There exists a real matrix $\phi(t)$, nonsingular for all t, such that $\phi^{-1}(t)(L(t)+B(t))^T\phi(t)=diag\{\lambda_1(t),\lambda_2(t),\cdots,\lambda_N(t)\}$ and $\dot{\phi}^{-1}(t)\phi(t)=diag\{\beta_1(t),\beta_2(t),\cdots,\beta_N(t)\}$. The control protocol (2) solves the consensus problem of nonlinear multi-agent system with variable topology is stable if and only if the linear time-varying systems

$$\dot{w}_k(t) = \left[Df(s(t)) + \lambda_k(t) - \beta_k(t)\right] w_k(t) \quad k = 1,2,\cdots,N \tag{5}$$

are stable.

For linear continuous time-varying system given by

$$\dot{y}(t) = A(t)\,y(t), y(t_0) = y_0 \tag{6}$$

with $y(t) \in \mathbb{R}^n$, it is assumed that $A(t)$ is continuous in t. The following classical notation of stability, known as stability in sense of Lyapunov is recalled.

Definition 2: Let t_0 be any real number. If for all $\delta > 0$, there exists $r(t_0,\delta) > 0$ such that $\|y(t_0)\| < r(t_0,\delta)$ implies $\|y(t)\| < \delta$ for all $t \geq t_0$, then the system (6) is stable in the sense of Lyapunov.

Define

$$w(t) = col\{w_1(t), w_2(t), \cdots, w_N(t)\}$$

System (5) can be rewritten as

$$\dot{w}(t) = \Lambda(t)\,w(t), w(t_0) = w_0 \tag{7}$$

with

$$\Lambda(t) = diag\{Df(s(t)) + \lambda_1(t) - \beta_1(t), Df(s(t)) + \lambda_2(t) - \beta_2(t), \cdots, Df(s(t)) + \lambda_N(t) - \beta_N(t)\}$$

Consider the following sets

$$\Omega_0 = \left\{w_0 \in \mathbb{R}^n : \|w_0\|^2 \leq \rho_0^2, \rho_0 > 0\right\} \tag{8}$$

and

$$E_t = \left\{w \in \mathbb{R}^n : \|w(t)\|^2 \leq \rho(t)^2, \rho(t) > 0, \forall t \geq t_0\right\} \tag{9}$$

with $\Omega_0 \subseteq E_{t0}$, then the following results can be deduced.

Theorem 1: Consider the linear time-varying system (7) and suppose that $w(t_0) = w_0 \in \Omega_0$. Then $w(t) \in E_t$ for all $t \geq t_0$ if and only if

$$\rho(t) \geq \hat{\rho}(t) = \rho_0 \lambda_{\max}(\Phi(t,t_0)) \tag{10}$$

where $\Phi(t,t_0)$ is the state transition matrix of linear time-varying system (7).

Proof

Sufficiency: Consider the following optimization problem for a given instant of time t

$$\begin{cases} \max\limits_{w_0} \rho(t)^2 = w_0^T \Phi(t,t_0)^T \Phi(t,t_0)\, w_0 \\ w_0^T w_0 \leq \rho_0^2 \end{cases} \tag{11}$$

The Lagrangian of this optimization problem is

$$L(w_0,\gamma) = w_0^T \Phi(t,t_0)^T \Phi(t,t_0)\, w_0 + \gamma\left(w_0^T w_0 - \rho_0^2\right) \tag{12}$$

with $\gamma \leq 0$. The optimal condition $\frac{\partial L}{\partial w_0} = 0$ yields

$$2\Phi(t,t_0)^T \Phi(t,t_0)\, w_0 + 2\gamma w_0 = 0 \tag{13}$$

Considering the reversibility of state transition matrix $\Phi(t,t_0)$,(13) can be rewritten as

$$2\gamma\Phi(t,t_0)^T \left(\Phi(t_0,t)^T \Phi(t_0,t) + \gamma^{-1} I\right)\Phi(t,t_0)\, w_0 = 0 \tag{14}$$

The optimal condition $\frac{\partial L}{\partial \gamma} = 0$ yields

$$w_0^T w_0 - \rho_0^2 = 0 \tag{15}$$

Multiplying (13) on the left by w_0^T, (16) is obtained,

$$w_0^T \Phi(t,t_0)^T \Phi(t,t_0)\, w_0 + \gamma w_0^T w_0 = 0 \tag{16}$$

and considering (11) and (15), it yields that

$$\rho(t)^2 = -\rho_0^2 \gamma, \forall t \geq t_0 \tag{17}$$

To obtain a solution $w_0 \neq 0$ satisfying condition (14)and (15), it must have

$$\det\left(\Phi(t_0,t)^T \Phi(t_0,t) + \gamma^{-1} I\right) = 0 \tag{18}$$

which implies that $-\gamma^{-1}$ is an eigenvalue of matrix $\Phi(t_0,t)^T \Phi(t_0,t)$. Then, from (17) and the reversibility of state transition matrix $\Phi(t,t_0)$, it follows that

$$\begin{aligned} \hat{\rho}(t)^2 &= \max \rho(t)^2 \\ &= \rho_0^2 \lambda_{\max}\left(\Phi(t,t_0)^T \Phi(t,t_0)\right) \end{aligned} \tag{19}$$

In this paper, the state matrix of system (7) , $\Lambda(t)$, is diagonal, then the state transition matrix $\Phi(t,t_0)$ is also diagonal, which results that

$$\lambda_{\max}\left(\Phi(t,t_0)^T \Phi(t,t_0)\right) = \lambda_{\max}^2(\Phi(t,t_0))$$

Namely that

$$\hat{\rho}(t) = \lambda_{\max}(\Phi(t,t_0)) \tag{20}$$

If $\rho(t) \geq \hat{\rho}(t)$ for all $t \geq t_0$, the state of the system is confined in the family of sets E_t.

Necessity: Suppose $w(t) \in E_t$ and that for $t > t_0$, $\rho(t) < \hat{\rho}(t)$. This implies that there exist both $w_0 \in \Omega_0$ and

$t > t_0$ such that $w(t)$ does not belong to the ellipsoid E_t, which leads to a contradiction. The proof is completed.

For the multi-agent system (1) with nonlinear dynamics and variable topology $\bar{G} = \{\bar{v}, \bar{\varepsilon}, \bar{A}\}$, the necessary and sufficient conditions of the protocol (2) solving the consensus problem will be presented as follows.

Theorem 2: Consider the nonlinear multi-agent system (1) with variable topology $\bar{G} = \{\bar{v}, \bar{\varepsilon}, \bar{A}\}$. Then the control protocol (2) solves the consensus problem if and only if

$$\rho_M = \max_{t \ge t_0} \{\rho_0 \lambda_{\max}(\Phi(t, t_0))\} < \infty$$

where $\Phi(t, t_0)$ is the state transition matrix of linear time-varying system (7).

Proof. According to Lemma1, it is apparent that we just need to proof the stability of system (7) if we want to proof the stability of system (1).

Sufficiency: According to Theorem1, we know that for initial condition $\|w_0\|^2 \le \rho_0^2$, it yields $\|w(t)\|^2 \le \rho(t)^2$.

Associated with equation (17) $\rho(t)^2 = -\rho_0^2 \gamma, \forall t \ge t_0$, choose $-\gamma = \left(\dfrac{\delta}{\lambda_{\max}(\Phi(t, t_0))} \right)^2$, where $\delta > 0$ is an arbitrary scalar.

Consider the set

$$\{w_0 \in \mathbb{R}^n : \|w_0\|^2 \le r_0^2\} \tag{21}$$

with $r_0 = \min_{t \ge t_0} \delta \lambda_{\max}^{-1}(\Phi(t, t_0)) \le \dfrac{\delta \rho_0}{\rho_M}$

For all time $t \ge t_0$, (21) presents the set of initial conditions such that $\|w(t)\|^2 \le \delta^2$.

Then, for all $\delta > 0$, there exists $r = \dfrac{\delta \rho_0}{\rho_M}$ such that $\|w_0\| < r$ which results in $\|w(t)\| < \delta, \quad \forall t \ge t_0$ provided that $\rho_M < \infty$. Then, according to Definition2, the system (7) is stable in sense of Lyapunov. This completes the proof of sufficiency.

Necessity: Theorem1 defines in the state space a tube containing all the trajectories of the system (7) under $\forall t \ge t_0$. Then, there exists $t = \hat{t}$ corresponding to the $\hat{\rho}(t)$ such that $\|w(t)\|^2 = \rho_M^2$. If the system (7) is stable, ρ_M must be finite, which yields the fact that r is independent of the initial time $\hat{t}_0 \ge t_0$. This completes the proof of necessity.

According to the above, we can obtain that the condition $\rho_M = \max_{t \ge t_0} \{\rho_0 \lambda_{\max}(\Phi(t, t_0))\} < \infty$ ensures the stability of system (7). From Lemma1, the protocol (2) solves the consensus problem of nonlinear multi-agent system (1) with variable topology if and only if $\rho_M < \infty$. This completes the prove.

IV. Numerical Example

In this section, a numerical example is given to illustrate the Theorem2 with network (1) under control protocol (2). Consider the nonlinear multi-agent system with variable topology:

$$\dot{x}(t) = -\arctan(x(t)) + (L(t) + B(t))(x(t) - s(t)1_N) \tag{22}$$

with $\dot{s}(t) = -\arctan(s(t))$, $x(t) = [x_1(t), x_2(t), x_3(t)]$ and $N = 3$ .Then the Jacobian matrix is $Df(s(t)) = -\dfrac{1}{1 + s(t)^2}$. Assume that the coupling matrix of system (22) is

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$$L(t) + B(t) = \frac{1}{2e^2 - e - 1} \begin{pmatrix} (e^2 - 1)th(t) + e\arctan(t) & (1 - e)th(t) - 2e\arctan(t) & (e - e^2)th(t) + e\arctan(t) \\ 2(e^2 - 1)th(t) + e^2\arctan(t) & 2(1 - e)th(t) - 2e^2\arctan(t) & 2(e - e^2)th(t) + e^2\arctan(t) \\ 3(e^2 - 1)th(t) + \arctan(t) & 3(1 - e)th(t) - 2\arctan(t) & 3(e - e^2)th(t) + \arctan(t) \end{pmatrix}$$

It is easy to verify that there exists a nonlinear real matrix

$$\phi(t) = \frac{1}{2e^2 - e - 1} \begin{pmatrix} 3e^2 - 2 & (1 - e^2)e^t & -e^{1+\sin(t)} \\ 1 - 3e & (e - 1)e^t & 2e^{1+\sin(t)} \\ 2e - e^2 & (e^2 - e)e^t & -e^{1+\sin(t)} \end{pmatrix}$$

Such that

$$f^{-1}(t)(L(t) + B(t))^T f(t) = diag\{0, -th(t), -\arctan(t)\}$$

and $\dot{f}^{-1}(t) f(t) = diag\{0, -1, -\cos(t)\}$

Thus, system (7) is developed as (23) for instantiation.

$$\dot{w}(t) = \breve{L}(t) w(t) \tag{23}$$

with

$$\tilde{L}(t) = diag\left\{ -\frac{1}{1+s(t)^2}, -\frac{1}{1+s(t)^2} - th(t) + 1, -\frac{1}{1+s(t)^2} + \cos(t) - \arctan(t) \right\}$$

The state transition matrix is

$$\tilde{\Phi}(t) = diag\left\{ e^{\int_{t_0}^{t} -\frac{1}{1+s(t)^2} dt}, e^{\int_{t_0}^{t} \left(\frac{s(t)^2}{1+s(t)^2} - th(t)\right) dt}, e^{\int_{t_0}^{t} \left(-\frac{1}{1+s(t)^2} + \cos(t) - \arctan(t)\right) dt} \right\}$$

It is obvious that $\max_{t \geq t_0} \{\rho_0 \lambda_{\max}(\Phi(t,t_0))\} < \infty$. Therefore, from Theorem2, we know that the consensus of nonlinear multi-agent system (22) with variable topology and control protocol (2) is achieved.

Fig. 1 describes the errors of three followers with the leader in the multi-agent system under the initial condition $s(0)=1$, $x(0)=[-0.2\ \ 0.5\ \ 2]$.

From Fig. 1, we can see the errors of the three following agents with the leader agent tend to zero as time goes on, which means the multi-agent system (22) reaches an absolute consensus on the state x. This verified that the necessary and sufficient conditions can solve the consensus problem of multi-agent system with both nonlinear dynamics and variable topology effectively.

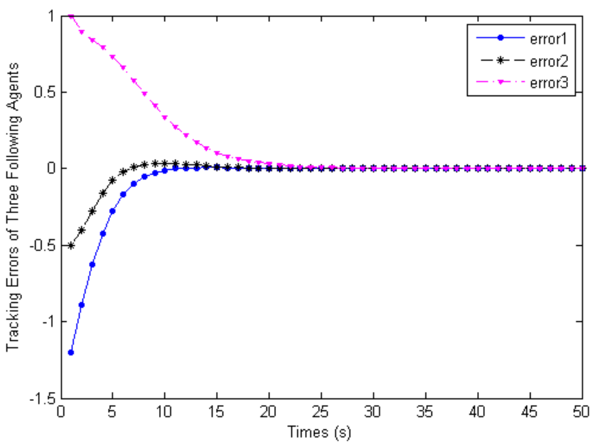

Figure 1. Errors of agents with the leader

V. CONCLUSIONS

A necessary and sufficient condition associated with eigenvalues has been given to ensure consensus of the nonlinear multi-agent system with variable topology. The condition merges the work of [9], which studied controlled synchronization criteria of complex dynamical networks, and some stability theory on linear time-varying system. The necessary and sufficient condition is expressed in a brief way and it is easy to verify.

In the future, second-order consensus problem will be studied based on the work in this paper. Second-order multi-agent system better confirms to the actual situation than the first-order. Therefore, the extension of consensus algorithms from first-order to second-order is non-trivial. The second-order consensus problem is more complicated and challenging.

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