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Determination of the domain of attraction and regions of guaranteed cost for robust model predictive controllers based on linear matrix inequalities

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***Abstract*—The robust model-based predictive control (RMPC) formulation originally proposed in [1] ensures convergence of the state trajectory to the origin and satisfaction of operational constraints, provided that a given system of LMIs is feasible at the beginning of the control task. The largest domain of attraction of the origin under the resulting closed-loop control law can be defined as the set of all state values for which the LMIs are feasible. The present paper demonstrates that such a set is convex and symmetric about the origin, which allows the determination of extreme points through the solution of a modified version of the original RMPC optimization problem. An inner approximation of the largest domain of attraction can then be generated as the convex hull of these extreme points. The convexity and symmetry properties are also demonstrated for regions of guaranteed cost, defined as the set of initial states for which the resulting cost is upper-bounded by a given value. Inner approximations of such regions can also be obtained by solving a modified version of the RMPC optimization problem. For illustration, a numerical simulation model of an angular positioning system is employed, as in [1]. In this example, the proposed approximations were found to be in agreement with the feasibility and cost results obtained in a pointwise manner for a grid of initial conditions.**

***Index Terms*—Robust model predictive control, linear matrix inequalities, domain of attraction, convex optimization.**

I. INTRODUCTION

The term Model-based Predictive Control (MPC) refers to a body of techniques that involve the solution of an optimal control problem within a receding horizon [2]. MPC has become widespread in several application areas, mainly due to the possibility of addressing operational constraints in an explicit manner [3]. Constraint satisfaction and closed-loop stability can be guaranteed by a proper formulation of the receding-horizon optimization problem. Usually, the adoption of target sets that are invariant under locally stabilizing control laws is employed for this purpose [4]. However, the design guarantees may be lost if the prediction model does not match the actual plant dynamics. Such a mismatch may arise due to modelling approximations (linearization and order reduction, for example), parameter uncertainties or variations in the plant behaviour due to faults or natural aging. This problem has motivated the development of robust MPC (RMPC) techniques.

Early formulations of RMPC involved the online solution of a min-max optimization problem, where the worst case value of the cost function was evaluated over a set of uncertain plants [5], [6], [7]. However, the computational cost of the resulting problem could become prohibitive for actual implementation. In this context, Kothare and collaborators [1] proposed an RMPC approach based on linear matrix inequalities (LMIs). By using the proposed approach, the optimization problem was cast into a semi-definite programming (SDP) form [8], which allowed the use of efficient numerical solvers to obtain the optimal control in polynomial time. Moreover, operational constraints could be easily introduced by augmenting the problem formulation with additional LMIs. This seminal work was later extended to encompass the use of output feedback [9], [10], [11], as well as the control of nonlinear systems [12] systems with uncertain time delay [13], [14], [15], and systems with asymmetric output constraints [16]. Modifications to the LMI formulation aimed at reducing conservatism have also been proposed [17], [18], [19], [20].

Within the framework developed in [1], the state trajectory is guaranteed to converge to the origin with satisfaction of the operational constraints, provided that the system of LMIs is feasible at the beginning of the control task. Therefore, the largest domain of attraction of the origin under the closed-loop control law can be defined as the set of all state values (i.e. initial conditions for the state trajectory) for which the LMIs are feasible. However, the analytical or numerical characterization of such a domain of attraction was not discussed in [1]. In fact, although the solution of the SDP problem for a given initial condition can be used to establish an asymptotically stable invariant ellipsoid [1], [21], such an ellipsoid is not necessarily the largest domain of attraction for the origin. It is possible to maximize the size of this invariant ellipsoid by using a determinant maximization procedure, as proposed in [12]. Yet, one may still argue that the largest domain of attraction is not necessarily of ellipsoidal shape.

The present paper establishes some properties of the largest domain of attraction $\mathcal{D}$ for the RMPC approach developed in [1]. More specifically, it is demonstrated that $\mathcal{D}$ is convex and symmetric about the origin. In view of such properties, extreme points of $\mathcal{D}$ can be found by solving a modified version of the original RMPC optimization problem. An inner approximation of $\mathcal{D}$ can then be generated as the convex hull of the extreme points thus obtained. The resulting approximation can be useful, for instance, to design schemes for the commutation between different RMPC controllers, as well as to choose an appropriate initial condition during the planning stage of a control manoeuvre.

It is worth noting that the initial feasibility of the RMPC optimization problem guarantees that the state trajectory will converge to the origin, but does not ensure that the performance will be acceptable. Therefore, it would also be of value to characterize a set $\mathcal{D}^{\bar{\gamma}}$ of initial conditions for which the resulting cost function value is smaller or equal to a given scalar $\bar{\gamma} > 0$. In this work, such a set will be termed "region of guaranteed cost". As an additional contribution of the present paper, the symmetry and convexity properties of $\mathcal{D}$ are also demonstrated for $\mathcal{D}^{\bar{\gamma}}$. Thus, an inner approximation of $\mathcal{D}^{\bar{\gamma}}$ can also be obtained by solving a modified version of the original RMPC optimization problem.

For illustration, a numerical simulation model of an angular positioning system is employed, as in [1]. The remaining of this paper is organized as follows. Section II describes the LMI-based RMPC formulation adopted in the present work. Section III demonstrates the symmetry and convexity properties of the largest domain of attraction $\mathcal{D}$ and proposes a procedure for obtaining extreme points of $\mathcal{D}$ through the solution of an SDP problem. The corresponding developments for regions of guaranteed cost $\mathcal{D}^{\bar{\gamma}}$ are derived in Section IV. The numerical example is discussed in Section V. Finally, concluding remarks are presented in Section VI.

II. Robust model predictive control employing linear matrix inequalities

The RMPC approach under consideration is concerned with uncertain state-space models of the form:

$$x(k+1) = A(k)x(k) + B(k)u(k),\ [A(k),\, B(k)] \in \Omega \quad (1)$$

where $x(k) \in \mathbb{R}^{n_x}$, $u(k) \in \mathbb{R}^{n_u}$ are the state and input variables, respectively, and Ω is an uncertainty polytope with known vertices $A_i \in \mathbb{R}^{n_x \times n_x}$, $B_i \in \mathbb{R}^{n_x \times n_u}$, $i = 1, 2, \ldots, L$. It is assumed that component-wise amplitude constraints are to be imposed on the inputs $u(k)$, as well as on n_y output variables defined as $y_l(k) = C_l x(k)$ $(l = 1, 2, \ldots, n_y)$, where $C_l \in \mathbb{R}^{1\times n}$ are known matrices.

Let $J_\infty(k)$ denote the following infinite-horizon cost function:

$$J_\infty(k) = \sum_{j=0}^{\infty} \left[||x(k+j|k)||_S^2 + ||u(k+j|k)||_R^2 \right] \quad (2)$$

where $S \in \mathbb{R}^{n_x \times n_x}$ and $R \in \mathbb{R}^{n_u \times n_u}$ are positive-definite weight matrices and $(\bullet|k)$ denotes a predicted value, which is computed on the basis of the information available at time k. It is assumed that the system state is directly measured, so that $x(k|k) = x(k)$.

The optimization problem to be solved at time k can be formulated as

$$\min_{u(k+j|k),\ i \geq 0} \quad \max_{[A(k),\, B(k)] \in \Omega} J_\infty(k) \quad (3)$$

subject to

$$|u_r(k+j|k)| \leq u_{r,max},\ r = 1, 2, \ldots, n_u,\ j \geq 0 \quad (4)$$

$$|y_l(k+j|k)| \leq y_{l,max},\ l = 1, 2, \ldots, n_y,\ j \geq 1 \quad (5)$$

where $u_{r,max}$, $y_{l,max}$ denote the bounds on the magnitude of the rth input and lth output variables, respectively.

As demonstrated in [1], an upper bound γ on the cost $J_\infty(k)$ is minimized by solving the following semidefinite programming (SDP) problem:

$$\min_{\gamma, Q>0, Y, X, Z} \gamma \quad (6)$$

subject to[1]

$$\begin{bmatrix} Q & x(k) \\ * & 1 \end{bmatrix} \geq 0 \quad (7)$$

$$\begin{bmatrix} Q & 0 & 0 & A_iQ + B_iY \\ * & \gamma I & 0 & S^{1/2}Q \\ * & * & \gamma I & R^{1/2}Y \\ * & * & * & Q \end{bmatrix} \geq 0,\ i = 1, 2, \ldots, L \quad (8)$$

$$\begin{bmatrix} X & Y \\ * & Q \end{bmatrix} \geq 0 \quad (9)$$

$$X_{rr} \leq u_{r,max}^2,\ r = 1, 2, \ldots, n_u \quad (10)$$

$$\begin{bmatrix} Z & C_l(A_iQ + B_iY) \\ * & Q \end{bmatrix} \geq 0, \quad \begin{matrix} l = 1, 2, \ldots, n_y, \\ i = 1, 2, \ldots, L \end{matrix} \quad (11)$$

$$Z_{ll} \leq y_{l,max}^2,\ l = 1, 2, \ldots, n_y, \quad i = 1, 2, \ldots, L \quad (12)$$

and then using a state feedback control law $u(k+j|k) = Fx(k+j|k)$ over the prediction horizon, with $F = YQ^{-1}$. It is worth noting that the solution of this optimization problem depends on the present state $x(k)$, which appears in the first LMI (7). Therefore, in what follows the SDP given by (6) - (12) will be termed $\mathbb{P}(x(k))$.

By applying the control law in a receding horizon manner, i.e. by solving $\mathbb{P}(x(k))$ in order to obtain a new gain matrix F at each sampling time k, the closed-loop system can be shown to be robustly asymptotically stable, provided that $\mathbb{P}(x(k))$ is feasible at the initial time $k = 0$ [1]. Henceforth, with a slight abuse of language, an initial condition $x(0) = \xi \in \mathbb{R}^{n_x}$ will be termed "feasible" if the optimization problem $\mathbb{P}(\xi)$ is feasible, i.e. if there exists a feasible solution $(\gamma, Q > 0, Y, X, Z)$ to the system of LMIs (7) - (12) with $x(k)$ replaced with ξ.

III. Domain of attraction

The RMPC controller described in the previous section is a regulator that steers the system state $x(k)$ to the origin, starting from a given initial condition $x(0)$. Asymptotic stability and constraint satisfaction are guaranteed, provided that the optimization problem is feasible at the initial time $k = 0$ [1]. Therefore, the largest domain of attraction $\mathcal{D}$ of the origin for the closed-loop system can be defined as the set of all feasible initial conditions $x(0) \in \mathbb{R}^{n_x}$. Henceforth, the term "largest"

[1]Symbol $*$ is used to represent the elements below the main diagonal of a symmetric matrix.

will be omitted for brevity. In what follows, some properties of $\mathcal{D}$ will be established.

Proposition 1 (Symmetry of $\mathcal{D}$)**.** The domain of attraction $\mathcal{D}$ is symmetric about the origin, i.e. if $\xi \in \mathcal{D}$, then $-\xi \in \mathcal{D}$.

Proof: Initially, it should be noted that $x(k)$ only appears in the first LMI (7) in the definition of $\mathbb{P}(x(k))$. By using the Schur complement [8], [2], the LMI (7) with $Q > 0$ is seen to be equivalent to $1 - x^T(k)Q^{-1}x(k) \geq 0$. If this inequality is satisfied with $x(k) = \xi$, it is also satisfied with $x(k) = -\xi$. Now, assume that ξ is an element of $\mathcal{D}$. By definition, there exists a feasible solution $(\gamma_\xi, Q_\xi, Y_\xi, X_\xi, Z_\xi)$ to $\mathbb{P}(\xi)$. It can then be seen that $(\gamma_\xi, Q_\xi, Y_\xi, X_\xi, Z_\xi)$ is also a feasible solution to $\mathbb{P}(-\xi)$, which shows that $-\xi$ is an element of $\mathcal{D}$. ■

Proposition 2 (Convexity of $\mathcal{D}$)**.** The domain of attraction $\mathcal{D}$ is a convex set, i.e. if ξ_1, $\xi_2 \in \mathcal{D}$, then $\lambda\xi_1 + (1-\lambda)\xi_2 \in \mathcal{D}$ for any $\lambda \in [0, 1]$.

Proof: Let ξ_1, ξ_2 be two elements of $\mathcal{D}$. Then, by definition, there exist feasible solutions $(\gamma_1, Q_1 > 0, Y_1, X_1, Z_1)$ and $(\gamma_2, Q_2 > 0, Y_2, X_2, Z_2)$ to the system of LMIs (7) - (12) with $x(k)$ replaced with ξ_1 and ξ_2, respectively. Now, let $\xi_3 = \lambda\xi_1 + (1-\lambda)\xi_2$ and $(\gamma_3, Q_3, Y_3, X_3, Z_3) = \lambda(\gamma_1, Q_1, Y_1, X_1, Z_1) + (1-\lambda)(\gamma_2, Q_2, Y_2, X_2, Z_2)$, with $\lambda \in [0, 1]$. It follows that $Q_3 = \lambda Q_1 + (1-\lambda)Q_2 > 0$. Moreover:

$$\begin{bmatrix} Q_3 & \xi_3 \\ * & 1 \end{bmatrix} = \begin{bmatrix} \lambda Q_1 + (1-\lambda)Q_2 & \lambda\xi_1 + (1-\lambda)\xi_2 \\ * & 1 \end{bmatrix} = \lambda \underbrace{\begin{bmatrix} Q_1 & \xi_1 \\ * & 1 \end{bmatrix}}_{\geq 0} + (1-\lambda) \underbrace{\begin{bmatrix} Q_2 & \xi_2 \\ * & 1 \end{bmatrix}}_{\geq 0} \geq 0 \tag{13}$$

$$\begin{bmatrix} Q_3 & 0 & 0 & A_iQ_3 + B_iY_3 \\ * & \gamma I & 0 & S^{1/2}Q_3 \\ * & * & \gamma I & R^{1/2}Y_3 \\ * & * & * & Q_3 \end{bmatrix} = \begin{bmatrix} \lambda Q_1 + (1-\lambda)Q_2 & 0 & 0 & A_i[\lambda Q_1 + (1-\lambda)Q_2] + B_i[\lambda Y_1 + (1-\lambda)Y_2] \\ * & \gamma I & 0 & S^{1/2}[\lambda Q_1 + (1-\lambda)Q_2] \\ * & * & \gamma I & R^{1/2}[\lambda Y_1 + (1-\lambda)Y_2] \\ * & * & * & \lambda Q_1 + (1-\lambda)Q_2 \end{bmatrix}$$

$$= \lambda \underbrace{\begin{bmatrix} Q_1 & 0 & 0 & A_iQ_1 + B_iY_1 \\ * & \gamma I & 0 & S^{1/2}Q_1 \\ * & * & \gamma I & R^{1/2}Y_1 \\ * & * & * & Q_1 \end{bmatrix}}_{\geq 0} + (1-\lambda) \underbrace{\begin{bmatrix} Q_2 & 0 & 0 & A_iQ_2 + B_iY_2 \\ * & \gamma I & 0 & S^{1/2}Q_2 \\ * & * & \gamma I & R^{1/2}Y_2 \\ * & * & * & Q_2 \end{bmatrix}}_{\geq 0} \geq 0, \quad i = 1, 2, \ldots, L \tag{14}$$

$$\begin{bmatrix} X_3 & Y_3 \\ * & Q_3 \end{bmatrix} = \begin{bmatrix} \lambda X_1 + (1-\lambda)X_2 & \lambda Y_1 + (1-\lambda)Y_2 \\ * & \lambda Q_1 + (1-\lambda)Q_2 \end{bmatrix} = \lambda \underbrace{\begin{bmatrix} X_1 & Y_1 \\ * & Q_1 \end{bmatrix}}_{\geq 0} + (1-\lambda) \underbrace{\begin{bmatrix} X_1 & Y_1 \\ * & Q_1 \end{bmatrix}}_{\geq 0} \geq 0 \tag{15}$$

$$X_{3,rr} = \lambda \underbrace{X_{1,rr}}_{\leq u^2_{r,max}} + (1-\lambda) \underbrace{X_{2,rr}}_{\leq u^2_{r,max}} \leq u^2_{r,max}, \quad r = 1, 2, \ldots, n_u \tag{16}$$

$$\begin{bmatrix} Z_3 & C_l(A_iQ_3 + B_iY_3) \\ * & Q_3 \end{bmatrix} = \lambda \underbrace{\begin{bmatrix} Z_1 & C_l(A_iQ_1 + B_iY_1) \\ * & Q_1 \end{bmatrix}}_{\geq 0} + (1-\lambda) \underbrace{\begin{bmatrix} Z_2 & C_l(A_iQ_2 + B_iY_2) \\ * & Q_2 \end{bmatrix}}_{\geq 0} \geq 0 \tag{17}$$

$$Z_{3,ll} = \lambda \underbrace{Z_{1,ll}}_{\leq y^2_{l,max}} + (1-\lambda) \underbrace{Z_{2,ll}}_{\leq y^2_{l,max}} \leq y^2_{l,max}, \quad l = 1, 2, \ldots, n_y, \quad i = 1, 2, \ldots, L \tag{18}$$

Therefore, $(\gamma_3, Q_3 > 0, Y_3, X_3, Z_3)$ is a feasible solution to the system of LMIs (7) - (12) with $x(k)$ replaced with ξ_3, which shows that $\xi_3 \in \mathcal{D}$. ■

Given the properties of convexity and symmetry about the origin, extreme points of $\mathcal{D}$ can be obtained by solving an SDP problem of the form

$$\min_{\beta, \gamma, Q>0, Y, X, Z} \beta \tag{19}$$

subject to

$$\begin{bmatrix} Q & \beta\xi \\ * & 1 \end{bmatrix} \geq 0 \tag{20}$$

and the remaining LMIs (8) – (12) of the original RMPC optimization problem. In (20), $\xi \in \mathbb{R}^{n_x}$ is a constant vector that defines the direction along which the extreme point is to be found. The extreme point will be given by $\beta^*\xi$, where β^* is the minimal value of β resulting from the optimization process, provided that $\mathcal{D}$ is bounded along the direction of ξ. It is worth noting that $-\beta^*\xi$ will also be an extreme point, due to the symmetry property. Fig. 1a illustrates the process of obtaining an extreme point of $\mathcal{D}$ in a two-dimensional case ($n_x = 2$).

By solving this SDP problem with different ξ vectors, a number of extreme points of $\mathcal{D}$ can be obtained. An inner approximation of $\mathcal{D}$ can then be generated as the convex hull of those points, as illustrated in Fig. 1b.

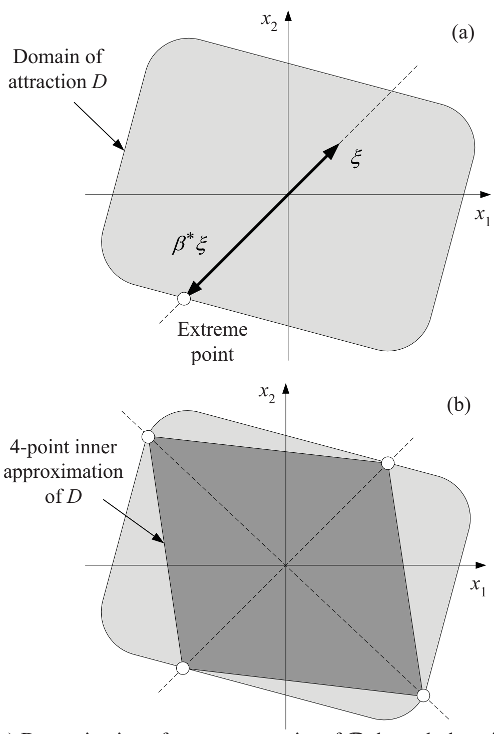

Fig. 1. (a) Determination of an extreme point of $\mathcal{D}$ through the minimization of β. (b) Inner approximation of $\mathcal{D}$ obtained as the convex hull of four extreme points.

IV. Regions of Guaranteed Cost

The domain of attraction $\mathcal{D}$ involves only the feasibility of the RMPC optimization problem (6) – (12), regardless of the achievable value for the cost function γ. By including a constraint on γ, it is possible to characterize a region of initial conditions for which the cost is guaranteed to be smaller than a certain bound, as defined below.

Definition 1 (Region of guaranteed cost)**.** Given a value of $\bar{\gamma} > 0$, the region $\mathcal{D}^{\bar{\gamma}}$ is defined as the set of initial conditions $x(0) \in \mathcal{D}$ for which the optimal solution γ^* obtained by solving $\mathbb{P}(x(0))$ is smaller or equal to $\bar{\gamma}$.

The symmetry and convexity of $\mathcal{D}^{\bar{\gamma}}$ are established in the two propositions below.

Proposition 3 (Symmetry of $\mathcal{D}^{\bar{\gamma}}$)**.** The region of guaranteed cost $\mathcal{D}^{\bar{\gamma}}$ is symmetric about the origin, for any given $\bar{\gamma} > 0$.

Proof: Symmetry can be demonstrated by applying the Schur complement to LMI (7), as in the proof of Proposition 1. ■

Proposition 4 (Convexity of $\mathcal{D}^{\bar{\gamma}}$)**.** The region of guaranteed cost $\mathcal{D}^{\bar{\gamma}}$ is a convex set, for any given $\bar{\gamma} > 0$.

Proof: Let $\xi_1 \in \mathbb{R}^{n_x}$ and $\xi_2 \in \mathbb{R}^{n_x}$ be two elements of $\mathcal{D}^{\bar{\gamma}}$ for a given $\bar{\gamma} > 0$, and let $(\gamma_1^*, Q_1^*, Y_1^*, X_1^*, Z_1^*)$ and $(\gamma_2^*, Q_2^*, Y_2^*, X_2^*, Z_2^*)$ be the optimal solutions of $\mathbb{P}(\xi_1)$ and $\mathbb{P}(\xi_2)$. From the definition of $\mathcal{D}^{\bar{\gamma}}$, it follows that

$$\gamma_1^* \leq \bar{\gamma}, \ \gamma_2^* \leq \bar{\gamma}. \tag{21}$$

Now, let $\xi_3 = \lambda\xi_1 + (1-\lambda)\xi_2$ and $(\gamma_3, Q_3, Y_3, X_3, Z_3) = \lambda(\gamma_1^*, Q_1^*, Y_1^*, X_1^*, Z_1^*) + (1-\lambda)(\gamma_2^*, Q_2^*, Y_2^*, X_2^*, Z_2^*)$, with $\lambda \in [0,1]$. In view of (21), one has $\gamma_3 = \lambda\gamma_1^* + (1-\lambda)\gamma_2^* \leq \bar{\gamma}$. Moreover, a demonstration similar to that of Proposition 2 can be used to prove that $(\gamma_3, Q_3, Y_3, X_3, Z_3)$ is a feasible solution to $\mathbb{P}(\xi_3)$. Finally, let γ_3^* be the optimal value of the cost γ for $\mathbb{P}(\xi_3)$. Given that the minimal value of the cost must be smaller or equal to the cost of any feasible solution, it follows that $\gamma_3^* \leq \gamma_3 \leq \bar{\gamma}$. Therefore, $\xi_3 \in \mathcal{D}^{\bar{\gamma}}$, which shows that $\mathcal{D}^{\bar{\gamma}}$ is a convex set. ■

Extreme points of $\mathcal{D}^{\bar{\gamma}}$ can be obtained by solving an SDP problem of the form

$$\min_{\beta,\gamma,Q>0,Y,X,Z} \beta \tag{22}$$

subject to

$$\gamma \leq \bar{\gamma} \tag{23}$$

$$\begin{bmatrix} Q & \beta\xi \\ * & 1 \end{bmatrix} \geq 0 \tag{24}$$

and the remaining LMIs (8) – (12) of the original RMPC optimization problem. As in the previous section, the extreme point will be given by $\beta^*\xi$, where $\xi \in \mathbb{R}^{n_x}$ is a constant vector that defines the direction along which the extreme point is to be found. An inner approximation of $\mathcal{D}^{\bar{\gamma}}$ can be generated as the convex hull of extreme points obtained with different ξ vectors.

V. Numerical Example

The angular positioning system described in [1] will be adopted to illustrate the proposed method. The problem under consideration involves the control of a rotating antenna driven by an electric motor. The plant dynamics are described by the following discrete-time state equation:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1.0 & 0.1 \\ 0 & 1-0.1\alpha(k) \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0.0787 \end{bmatrix} u(k) \tag{25}$$

where the state variables x_1 and x_2 denote the angular position (rad) and velocity (rad/s) of the antenna, respectively, and the control variable u corresponds to the input voltage (V) of the electric motor. The viscous friction in the moving parts of the antenna is associated to the uncertain parameter $\alpha(k)$, which is assumed to be in a given range $[\alpha_{min}, \alpha_{max}]$. Therefore, the model is of the form (1), with $A(k) \in Co\{A_1, A_2\}$ where

$$A_1 = \begin{bmatrix} 1.0 & 0.1 \\ 0 & 1-0.1\alpha_{min} \end{bmatrix}, \ A_2 = \begin{bmatrix} 1.0 & 0.1 \\ 0 & 1-0.1\alpha_{max} \end{bmatrix} \tag{26}$$

It is worth noting that the actual dependence of $\alpha(k)$ on the time k does not affect the RMPC formulation, because the LMIs only involve the vertices A_1, A_2 of the uncertainty polytope. In what follows, the uncertain parameter will be denoted simply by α for brevity.

The cost function weights S and R were set to $I_{2\times 2}$ and 1, respectively. Moreover, the control variable u was constrained to the range $[-2V, +2V]$, as in [1], and the position x_1 was constrained to the range $[-1rad, +1rad]$. Such a constraint can be cast into the form (5) by defining an output variable $y = Cx$, with $C = [1\ 0]$. All numerical results were obtained by using the LMI Lab package for Matlab.

A. Results

Figure 2a presents the inner approximations of the attraction domain $\mathcal{D}$ obtained by using 4 and 16 extreme points. In this case, the bounds on the uncertain parameter α were set to $\alpha_{min} = 0.1$ and $\alpha_{max} = 10$. Moreover, Fig. 2a shows a grid of states that were employed to test the feasibility of the original SDP problem (6) - (12). As can be seen, all grid points inside the obtained polygons correspond to feasible initial conditions. It is worth noting that points $[-1\ 0]^T$ and $[+1\ 0]^T$, which correspond to unfeasible initial conditions, are outside the polygons, as shown in Fig. 2b. The use of 16 extreme points provides a better approximation of the attraction domain, in that the resulting polygon encompasses a larger region of feasible initial conditions, as compared to the 4-point approximation.

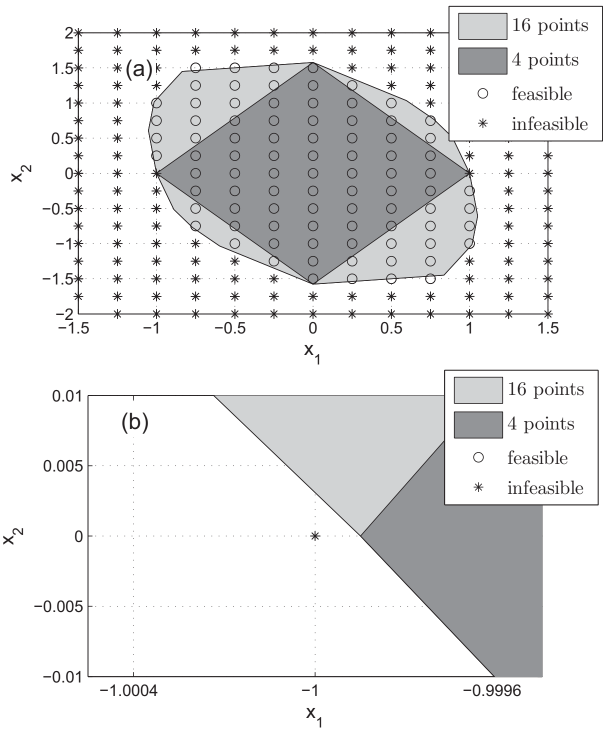

Fig. 2. (a) Inner approximations of the attraction domain $\mathcal{D}$. (b) Detail of the region around point $[-1\ 0]^T$.

From a physical point of view, unfeasibility arises if the position x_1 is close to the ± 1 bounds and the velocity is such that the position is changing towards the bound. It is interesting to notice that part of the 16-point polygon is located outside the $[-1, +1]$ range of admissible values for $y = x_1$. This result can be explained by noting that the output constraints in (5) are only enforced after one time step ahead of the present time. Therefore, if the initial condition is such that the output y can be steered to the admissible range in a single time step, the output constraint in (5) will be satisfied.

An interesting investigation that could be performed at this point concerns the relation between the domain of attraction $\mathcal{D}$ and the range of values for the uncertain model parameter α. It is expected that $\mathcal{D}$ will be larger if the characterization of α is more precise, i.e., if the range of uncertainty is smaller. To investigate this issue, the proposed method was employed to obtain new extreme points of $\mathcal{D}$, with bounds on α set to $\alpha_{min} = 0.5$ and $\alpha_{max} = 2$. The results are presented in Fig. 3. As expected, the polygon obtained for $0.5 \leq \alpha \leq 2$ contains the polygon obtained for the wider range $0.1 \leq \alpha \leq 10$.

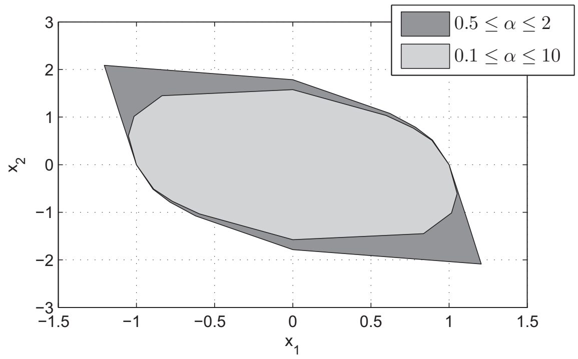

Fig. 3. Inner approximations of the attraction domain $\mathcal{D}$ for two different uncertainty ranges.

Figure 4 presents the inner approximations of the region of guaranteed cost $\mathcal{D}^{\bar{\gamma}}$ obtained by using 4 and 16 extreme points with $\bar{\gamma} = 100$. In this case, the bounds on the uncertain parameter α were again set to $\alpha_{min} = 0.1$ and $\alpha_{max} = 10$. The grid of states shown in this figure was employed to solve the SDP problem (6) - (12) in order to obtain the minimal value of γ, which is denoted by γ^*. As expected, all grid points inside the obtained polygons correspond to initial conditions for which $\gamma^* \leq \bar{\gamma} = 100$. As in Fig. 2a, the use of 16 extreme points provides a better approximation of the region under consideration.

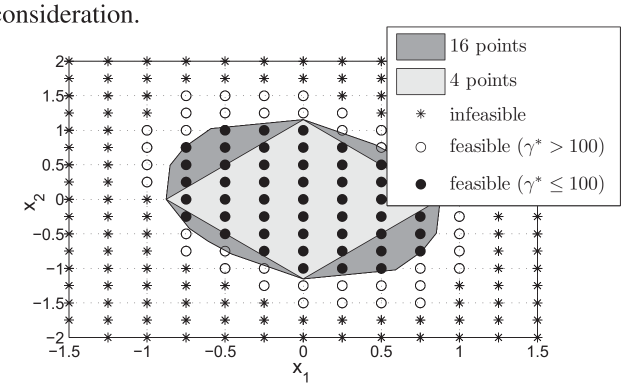

Fig. 4. Inner approximations of the region of guaranteed cost $\mathcal{D}^{\bar{\gamma}}$ for $\bar{\gamma} = 100$.

Figure 5 shows the 16-point polygons obtained for different values of $\bar{\gamma}$. As can be seen from these inner approximations, the region of guaranteed cost $\mathcal{D}^{\bar{\gamma}}$ tends to increase with $\bar{\gamma}$. It can also be seen that, as $\bar{\gamma}$ is increased, $\mathcal{D}^{\bar{\gamma}}$ converges to the domain of attraction $\mathcal{D}$. Indeed, if $\bar{\gamma}$ is made arbitrarily large, the SDP problem becomes equivalent to that involved in the determination of extreme points for $\mathcal{D}$ (i.e. without the γ constraint in (23)).

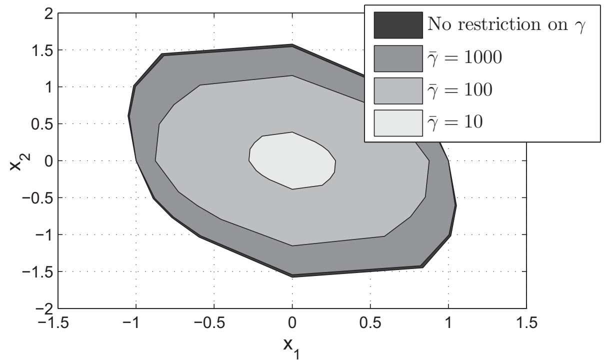

Fig. 5. Inner approximations of the region of guaranteed cost $\mathcal{D}^{\bar{\gamma}}$ for different values of $\bar{\gamma}$. The region obtained with no restriction on γ is an inner approximation of $\mathcal{D}$.

Finally, Fig. 6 presents the 16-point polygonal approximations to the region $\mathcal{D}^{\bar{\gamma}}$ for $\bar{\gamma} = 100$ and two different ranges for the uncertain parameter α, namely $0.1 \leq \alpha \leq 10$ and $0.5 \leq \alpha \leq 2$. As in the case of the attraction domain, the region of guaranteed cost increases as the uncertainty range is reduced.

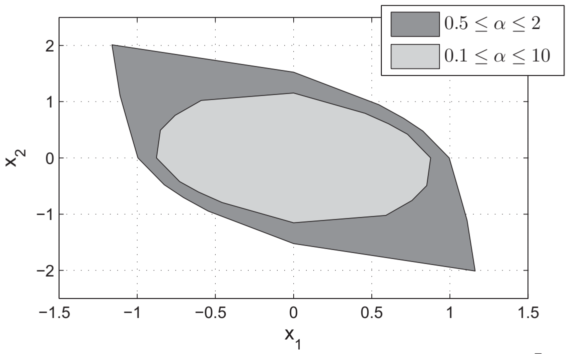

Fig. 6. Inner approximations of the region of guaranteed cost $\mathcal{D}^{\bar{\gamma}}$ for $\bar{\gamma} = 100$ and two different uncertainty ranges.

VI. Concluding Remarks

This paper demonstrated some properties of the domain of attraction $\mathcal{D}$ and regions of guaranteed cost $\mathcal{D}^{\bar{\gamma}}$ for the LMI-based RMPC formulation originally proposed in [1]. More specifically, $\mathcal{D}$ and $\mathcal{D}^{\bar{\gamma}}$ were shown to be convex and symmetric about the origin, which allowed the determination of extreme points through the solution of modified versions of the original SDP problem involved in the RMPC formulation. Inner approximations of these sets could then be generated as the convex hull of the extreme points. In the numerical example presented for illustration, such approximations were found to be in agreement with the feasibility and cost results obtained in a pointwise manner for a grid of initial conditions.

Future investigations could be concerned with the extension of the present work to other LMI-based RMPC formulations with less conservatism, such as those proposed in [17], [18], [19], [20]. It is expected that a reduction in conservatism should lead to an enlargement in the domain of attraction, as well as the regions of guaranteed cost, which would be an additional advantage of those formulations with respect to [1].

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