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# Singularly Impulsive Dynamical Systems with time delay: Mathematical Model and Stability

Nataša A. Kablar, Vlada Kvrgić,
*Lola Institute, Kneza Viseslava 70a, Belgrade 11000, Serbia*
nkablar.ae01@gtalumni.org, vlada.kvrgic@li.rs

**Abstract** - *In this paper we introduce new class of system, so called singularly impulsive or generalized impulsive dynamical systems with time delay. Dynamics of this system is characterized by the set of differential and difference equations with time delay, and algebraic equations. They represent the class of hybrid systems, where algebraic equations represent constraints that differential and difference equations with time delay need to satisfy. In this paper we present model, assumptions on the model, two classes of singularly impulsive dynamical systems with delay - time dependenet and state dependent. Further, we present Lyapunov and asymptotic stability theorems for nonlinear time-dependent and state-dependent singularly impulsive dynamical systems with time delay.*

## I. Introduction

Modern complex engineering systems as well as biological and physiological systems typically possess a multi-echelon hierarchical hybrid architecture characterized by continuous-time dynamics at the lower levels of hierarchy and discrete-time dynamics at the higher levels of the hierarchy. Hence, it is not surprising that hybrid systems have been the subject of intensive research over the past recent years (see Branicky et al. (1998), Ye et al. (1998 b), Haddad, Chellaboina and Kablar (2001a-b)). Such systems include dynamical switching systems Branicky (1998), Leonessa et al. (2000), nonsmooth impact and constrained mechanical systems, Back et al. (1993), Brogliato (1996), Brogliato et al. (1997), biological systems Lakshmikantham et al. (1989), demographic systems Liu (1994), sampled-data systems Hagiwara and Araki (1988), discrete-event systems Passino et al. (1994), intelligent vehicle/highway systems Lygeros et al. (1998) and flight control systems, etc. The mathematical descriptions of many of these systems can be characterized by impulsive differential equations, Simeonov and Bainov (1985), Liu (1988), Lakshmikantham et al. (1989, 1994), Bainov and Simeonov (1989, 1995), Kulev and Bainov (1989), Lakshmikantham and Liu (1989), Hu et al. (1989), Samoilenko and Perestyuk (1995), Haddad, Chellaboina and Kablar (2001a-b). Impulsive dynamical systems can be viewed as a subclass of hybrid systems.

Motivated by the results on impulsive dynamical systems presented in Haddad, Chellaboina, and Kablar (2001, 2005), the authors previous work on singular or generalized systems, and results on singularly impulsive dynamical systems published in Kablar(2003, 2010) we presented new class of *singularly impulsive* or *generalized impulsive dynamical systems with time delay*. It presents novel class of hybrid systems and generalization of impulsive dynamical systems to incorporate singular nature of the systems and time delays. Extensive applications of this class of systems can be found in contact problems and in hybrid systems.

We present mathematical model of the singularly impulsive dynamical systems with time delay. We show how it can be viewed as general systems from which impulsive dynamical systems with time delay, singular continuous-time systems with time delay and singular dicrete-time systems with time delay, as well as without time delay,follow. Then we present Assumptions needed for the model and the division of this class of systems to time-dependent and state-dependent singularly impulsive dynamical systems with time delay with respect to the resetting set. Finally, we draw some conclusions and define future work.

In this paper for the class of nonlinear singularly impulsive dynamical systems with time delay we develop Lyapunov and asymptotic stability results. Results are further specialized to linear case. Note that for addressing the stability of the zero solution of a singularly impulsive dynamical system the usual stability definitions are valid. Then we draw some conclusions and define future work.

At first, we establish definitions and notations. Let $\mathbb{R}$ denote the set of real numbers, let $\mathbb{R}^n$ denote the set of $n \times 1$ real column vectors, let $\mathcal{N}$ denote the set of nonnegative integers, and let $I_n$ or $I$ denote the $n \times n$ identity matrix. Furthermore, let $\partial\mathcal{S}, \dot{\mathcal{S}}, \bar{\mathcal{S}}$ denote the boundary, the interior, and a closure of the subset $\mathcal{S} \subset \mathbb{R}^n$, respectively. Finally, let $\mathrm{C}^0$ denote the set of continuous functions and $\mathrm{C}^{\mathrm{r}}$ denote the set of functions with $r$ continuous derivatives.

## II. Mathematical Model of Singularly Impulsive Dynamical Systems with Time Delay

A singularly impulsive dynamical system with delay consists of three elements:

1. A possibly singular continuous-time dynamical equation with time delay, which governs the motion of the system between resetting events;
2. A possibly singular difference equation with time delay, which governs the way the states are instantaneously changed when a resetting occurs; and

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3. A criterion for determining when the states of the system are to be reset.

Mathematical model of these systems is described with

$$E_{\mathrm{c}}\dot{x}(t)=f_{\mathrm{c}}(x(t,\tau))+G_{\mathrm{c}}(x(t,\tau))u_{\mathrm{c}}(t),\quad (t,x(t,\tau),u_{\mathrm{c}}(t))\notin\mathcal{S}, \tag{II.1}$$

$$E_{\mathrm{d}}\triangle x(t)=f_{\mathrm{d}}(x(t,\tau))+G_{\mathrm{d}}(x(t,\tau))u_{\mathrm{d}}(t),\quad (t,x(t,\tau),u_{\mathrm{c}}(t))\in\mathcal{S}, \tag{II.2}$$

$$y_{\mathrm{c}}(t)=h_{\mathrm{c}}(x(t,\tau))+J_{\mathrm{c}}(x(t,\tau))u_{\mathrm{c}}(t),\quad (t,x(t,\tau),u_{\mathrm{c}}(t))\notin\mathcal{S}, \tag{II.3}$$

$$y_{d}(t)=h_{\mathrm{d}}(x(t,\tau)+J_{\mathrm{d}}(x(t,\tau))u_{\mathrm{d}}(t),\quad (t,x(t,\tau),u_{\mathrm{c}}(t))\in\mathcal{S}, \tag{II.4}$$

where $t\geq 0$, $\tau>0$, $x(0)=x_0$, $x(t,\tau)\in\mathcal{D}\subset\mathbb{R}^{\ltimes}\times\mathbb{N}$, $\mathcal{D}$ is an open set with $0\in\mathcal{D}$, $u_{\mathrm{c}}\in\mathcal{U}_{\mathrm{c}}\subset\mathbb{R}^{\gtrdot_{\mathrm{c}}}$, $u_{\mathrm{d}}(t_k)\in\mathcal{U}_{\mathrm{d}}\subset\mathbb{R}^{\gtrdot_{\mathrm{d}}}$, $t_k$ denotes $k^{\mathrm{th}}$ instant of time at which $(t,x(t,\tau),u_{\mathrm{c}}(t))$ intersects $\mathcal{S}$ for a particular trajectory $x(t,\tau)$ and input $u_{\mathrm{c}}(t),y_{\mathrm{c}}(t)\in\mathbb{R}^{\lessdot_{\mathrm{c}}}$, $y_{\mathrm{d}}(t_k)\in\mathbb{R}^{\lessdot_{\mathrm{d}}}$, $f_{\mathrm{c}}:\mathcal{D}\to\mathbb{R}^{\ltimes}$ is Lipschitz continuous and satisfies $f_{\mathrm{c}}(0)=0$, $G_{\mathrm{c}}:\mathcal{D}\to\ltimes\times\gtrdot_{\mathrm{c}}$, $f_{\mathrm{d}}:\mathcal{D}\to\mathbb{R}^{\ltimes}$ is continuous and satisfies $f_{\mathrm{d}}(0)=0$, $G_{\mathrm{d}}:\mathcal{D}\to\mathbb{R}^{n\times m_{\mathrm{d}}}$, $h_{\mathrm{c}}:\mathcal{D}\to\mathbb{R}^{l_{\mathrm{c}}}$ and satisfies $h_{\mathrm{c}}(0)=0$, $J_{\mathrm{c}}:\mathcal{D}\to\mathbb{R}^{l_{\mathrm{c}}\times m_{\mathrm{c}}}$, $h_{\mathrm{d}}:\mathcal{D}\to\mathbb{R}^{l_{\mathrm{d}}}$ and satisfies $h_{\mathrm{d}}(0)=0$, $J_{\mathrm{d}}:\mathcal{D}\to\mathbb{R}^{l_{\mathrm{d}}\times m_{\mathrm{d}}}$, and $\mathcal{S}\subset[0,\infty)\times\mathbb{R}^n\times\mathcal{U}_{\mathrm{c}}$ is the *resetting set*. Here, as in Haddad, Chellaboina, and Kablar (2001a) we assume that $u_{\mathrm{c}}(\cdot)$ and $u_{\mathrm{d}}(\cdot)$ are restricted to the class of *admissible* inputs consisting of measurable functions $(u_{\mathrm{c}}(t),u_{\mathrm{d}}(t))\in\mathcal{U}_{\mathrm{c}}\times\mathcal{U}_{\mathrm{d}}$ for all $t\geq 0$ and $k\in\mathcal{N}_{[0,t)}\equiv k:0\leq t_k<t$, where the constraint set $\mathcal{U}_{\mathrm{c}}\times\mathcal{U}_{\mathrm{d}}$ is given with $(0,0)\in\mathcal{U}_{\mathrm{c}}\times\mathcal{U}_{\mathrm{d}}$. We refer to the differential equation (II.1) as the *continuous-time dynamics with time delay*, and we refer to the difference equation (II.2) as the *resetting law*.

Matrices $E_{\mathrm{c}}$, $E_{\mathrm{d}}$ may be singular matrices. In case $E_{\mathrm{c}}=I$, $E_{\mathrm{d}}=I$, and $\tau=0$ (II.1)–(II.4) represent standard impulsive dynamical systems described in Haddad, Chellaboina, and Kablar (2001a), and Haddad, Kablar, and Chellaboina (2000, 2005), where stability, dissipativity, feedback interconnections, optimality, robustness, and disturbance rejection has been analyzed. In absence of discrete dynamics they specialize to singular continuous-time systems, with further specialization $E_{\mathrm{c}}=I$ to standard continuous-time systems. If only discrete dynamics is present they specialize to singular discrete-time systems, with further specialization $E_{\mathrm{d}}=I$ to standard discrete-time systems.

In case $E_{\mathrm{c}}=I$, $E_{\mathrm{d}}=I$, and $\tau\neq 0$, (II.1)–(II.4) represent standard impulsive dynamical systems with time delay. In absence of discrete dynamics they specialize to singular continuous-time systems with time delay, with further specialization $E_{\mathrm{c}}=I$ to standard continuous-time systems with time delay. If only discrete dynamics is present they specialize to singular discrete-time systems with time delay, with further specialization $E_{\mathrm{d}}=I$ to standard discrete-time systems with time delay.

Therefore, theory of the singularly impulsive or generalized impulsive dynamical systems with time delay once developed, can be viewed as a generalization of the singular and impulsive dynamical system with time delay theory, unifying them into more general new system theory.

In what follows is given basic setting and division of this class of systems with respect to the definition of the resetting sets, accompanied with adequate assumptions needed for the model.

We make the following additional assumptions:

A1. $(0,x_0,u_{\mathrm{c}0})\notin\mathcal{S}$, where $x(0)=x_0$ and $u_{\mathrm{c}}(0)=u_{\mathrm{c}0}$, that is, the initial condition is not in $\mathcal{S}$.

A2. If $(t,x(t,\tau),u_{\mathrm{c}}(t))\in\bar{\mathcal{S}}\backslash\mathcal{S}$ then there exists $\epsilon>0$ such that, for all $0<\delta<\epsilon$, $s(t+\delta;t,x(t,\tau),u_{\mathrm{c}}(t+\delta))\notin\mathcal{S}$.

A3. If $(t_k,x(t_k),u_{\mathrm{c}}(t_k))\in\partial\mathcal{S}\cap\mathcal{S}$ then there exists $\epsilon>0$ such that, for all $0<\delta<\epsilon$ and $u_{\mathrm{d}}(t_k)\in\mathcal{U}_{\mathrm{d}}$, $s(t_k+\delta;t_k,E_{\mathrm{d}}x(t_k)+f_{\mathrm{d}}(x(t_k))+G_{\mathrm{d}}(x(t_k))u_{\mathrm{d}}(t_k),u_{\mathrm{c}}(t_k+\delta))\notin\mathcal{S}$.

A4. We assume consistent initial conditions (and prior and after every resetting).

Assumption A1 ensures that the initial condition for the resetting differential equation (II.1), (II.2) is not a point of discontinuity, and this assumption is made for convenience. If $(0,x_0,u_{\mathrm{c}0})\in\mathcal{S}$, then the system initially resets to $E_{\mathrm{d}}x_0^+=E_{\mathrm{d}}x_0+f_{\mathrm{d}}(x_0)+G_{\mathrm{d}}(x_0)u_{\mathrm{d}}(0)$ which serves as the initial condition for the continuous dynamics (II.1). It follows from A3 that the trajectory then leaves $\mathcal{S}$. We assume in A2 that if a trajectory reaches the closure of $\mathcal{S}$ at a point that does not belong to $\mathcal{S}$, then the trajectory must be directed away from $\mathcal{S}$, that is, a trajectory cannot enter $\mathcal{S}$ through a point that belongs to the closure of $\mathcal{S}$ but not to $\mathcal{S}$. Finally, A3 ensures that when a trajectory intersects the resetting set $\mathcal{S}$, it instantaneously exits $\mathcal{S}$, see Figure 1. We make the following remarks.

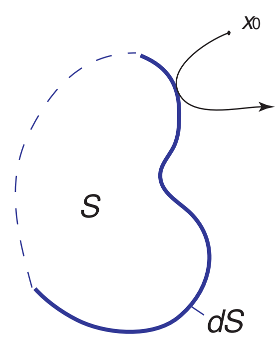


**Figure 1.** Resetting Set.

**Remark II.1.** *It follows from A3 that resetting removes the pair $(t_k,x_k,u_{\mathrm{c}}(t_k))$ from the resetting set $\mathcal{S}$. Thus, immediately after resetting occurs, the continuous-time dynamics (II.1), and not the resetting law (II.2), becomes the active element of the singularly impulsive dynamical system.*

**Remark II.2.** *It follows from A1-A3 that no trajectory can intersect the interior of $\mathcal{S}$. According to A1, the trajectory $x(t)$ begins outside the set $\mathcal{S}$. Furthermore, it follows from A2 that a trajectory can only reach $\mathcal{S}$ through a point belonging to both $\mathcal{S}$ and its boundary. Finally, from A3, it follows that if a trajectory reaches a point $\mathcal{S}$ that is on the boundary of $\mathcal{S}$, then the trajectory is instantaneously removed from $\mathcal{S}$. Since* *a continuous trajectory starting outside of $\mathcal{S}$ and intersecting the interior of $\mathcal{S}$ must first intersect the boundary of $\mathcal{S}$, it follows that no trajectory can reach the interior of $\mathcal{S}$.*

**Remark II.3.** *It follows from A1-A3 and Remark 1.2 that $\partial\mathcal{S} \cup \mathcal{S}$ is closed and hence the resetting times $t_k$ are well defined and distinct.*

**Remark II.4.** *Since the resetting times are well defined and distinct, and since the solutions to (II.1) exist and are unique, it follows that the solutions of the singularly impulsive dynamical system (II.1), (II.2) also exist and are unique over a forward time interval.*

In Haddad, Chellaboina and Kablar (2001a), the resetting set $\mathcal{S}$ is defined in terms of a countable number of functions $n_k : \mathbb{R}^n \to (0, \infty)$, and is given by

$$\mathcal{S} = \cup_k \{(n_k(x), x, u_{\mathrm{c}}(n_k(x)) : x \in \mathbb{R}^n\}. \tag{II.5}$$

The analysis of singularly impulsive dynamical systems with time delay and with a resetting set of the form (II.5) can be quite involved. In particular, such systems exhibit Zenoness, beating, as well as confluence phenomena wherein solutions exhibit infinitely many transitions in a finite times, and coincide after a given point of time, Haddad, Chellaboina and Kablar (2001a). In this paper we assume that existence and uniqueness properties of a given singularly impulsive dynamical system with time delay are satisfied in forward time. Furthermore, since singularly impulsive dynamical systems of the form (II.1)-(II.4) involve impulses at variable times they are time-varying systems.

Here we will consider singularly impulsive dynamical systems involving two distinct forms of the resetting set $\mathcal{S}$. In the first case, the resetting set is defined by a prescribed sequence of times which are independent of state $x$. These equations are thus called *time-dependent singularly impulsive dynamical systems with time delay*. In the second case, the resetting set is defined by a region in the state space that is independent of time. These equations are called *state-dependent singularly impulsive dynamical systems with time delay*.

### *A. Time-Dependent Singularly Impulsive Dynamical Systems with Time Delay*

Time-dependent singularly impulsive dynamical systems with time delay can be written as (II.1)–(II.4) with $\mathcal{S}$ defined as

$$\mathcal{S} = n \times \mathbb{R}^n \times \mathcal{U}_{\mathrm{c}}, \tag{II.6}$$

where

$$n = t_1, t_2, \ldots \tag{II.7}$$

and $0 < t_1 < t_2 < \ldots$ are prescribed resetting times. When an infinite number of resetting times are used and $t_k \to \infty$ as $k \to \infty$, then $\mathcal{S}$ is closed. Now (II.1)–(II.4) can be rewritten in the form of the *time-dependent singularly impulsive dynamical system with time delay*

$$E_{\mathrm{c}}\dot{x}(t) = f_{\mathrm{c}}(x(t,\tau)) + G_{\mathrm{c}}(x(t,\tau))u_{\mathrm{c}}(t),\ t \neq t_k, \tag{II.8}$$

$$E_{\mathrm{d}}\triangle x(t) = f_{\mathrm{d}}(x(t,\tau)_+ G_{\mathrm{d}}(x(t,\tau))u_{\mathrm{d}}(t),\ t = t_k, \tag{II.9}$$

$$y_c(t) = h_{\mathrm{c}}(x(t,\tau)) + J_{\mathrm{c}}(x(t,\tau))u_{\mathrm{c}}(t),\ t \neq t_k, \tag{II.10}$$

$$y_d(t) = h_{\mathrm{d}}(x(t,\tau)) + J_{\mathrm{d}}(x(t,\tau))u_{\mathrm{d}}(t),\ t = t_k \tag{II.11}$$

Since $0 \notin \tau$ and $t_k < t_{k+1}$, $\tau > 0$, it follows that the assumptions A1–A3 are satisfied. Since time-dependent singularly impulsive dynamical systems with time delay involve impulses at a fixed sequence of times, they are time-varying systems.

**Remark II.5.** *The time-dependent singularly impulsive dynamical system with time delay (II.8)–(II.11), with $E_{\mathrm{c}} = I$ and $E_{\mathrm{d}} = I$ includes as a special case the impulsive control problem addressed in the literature wherein at least one of the state variables of the continuous-time plant can be changed instantaneously to any given value given by an impulsive control at a set of control instants $\tau$, Haddad, Chellaboina and Kablar (2001a).*

### *B. State-Dependent Singularly Impulsive Dynamical Systems with Time Delay*

State-dependent singularly impulsive dynamical systems with time delay can be written as (II.1)–(II.4) with $\mathcal{S}$ defined as

$$\mathcal{S} = [0, \infty) \times \mathcal{Z}, \tag{II.12}$$

where $\mathcal{Z} = \mathcal{Z}_x \times \mathcal{U}_{\mathrm{c}}$ and $\mathcal{Z}_x \subset \mathbb{R}^n$. Therefore, (II.1)–(II.4) can be rewritten in the form of the *state-dependent singularly impulsive dynamical system with time delay*

$$E_{\mathrm{c}}\dot{x}(t) = f_{\mathrm{c}}(x(t,\tau)) + G_{\mathrm{c}}(x(t,\tau))u_{\mathrm{c}}(t), \quad (x(t,\tau), u_{\mathrm{c}}(t)) \notin \mathcal{Z}, \tag{II.13}$$

$$E_{\mathrm{d}}\triangle x(t) = f_{\mathrm{d}}(x(t,\tau)) + G_{\mathrm{d}}(x(t,\tau))u_{\mathrm{d}}(t), \quad (x(t,\tau), u_{\mathrm{c}}(t)) \in \mathcal{Z}, \tag{II.14}$$

$$y_c(t) = h_{\mathrm{c}}(x(t,\tau)) + J_{\mathrm{c}}(x(t,\tau))u_{\mathrm{c}}(t), \quad (x(t,\tau), u_{\mathrm{c}}(t)) \notin \mathcal{Z}, \tag{II.15}$$

$$y_d(t) = h_{\mathrm{d}}(x(t,\tau)) + J_{\mathrm{d}}(x(t,\tau))u_{\mathrm{d}}(t), \quad (x(t,\tau), u_{\mathrm{c}}(t)) \in \mathcal{Z}. \tag{II.16}$$

We assume that $(x_0, u_{\mathrm{c}0}) \notin \mathcal{Z}$, $\tau > 0$, $(0,0) \notin \mathcal{Z}$, and that the resetting action removes the pair $(x, u_{\mathrm{c}})$ from the set $\mathcal{Z}$; that is, if $(x, u_{\mathrm{c}}) \in \mathcal{Z}$ then $(E_{\mathrm{d}}x + f_{\mathrm{d}}(x) + G_{\mathrm{d}}(x)u_{\mathrm{d}}, u_{\mathrm{c}}) \notin \mathcal{Z}$, $u_{\mathrm{d}} \in \mathcal{U}_{\mathrm{d}}$. In addition, we assume that if at time $t$ the trajectory $(x(t,\tau), u_{\mathrm{c}}(t)) \in \bar{\mathcal{Z}} \backslash \mathcal{Z}$, then there exists $\epsilon > 0$ such that for $0 < \delta < \epsilon$, $(x(t + \tau + \delta), u_{\mathrm{c}}(t + \delta)) \notin \mathcal{Z}$.

These assumptions represent the specialization of A1–A3 for the particular resetting set (II.12). It follows from these assumptions that for a particular initial condition, the resetting times $\tau_k(x_0)$ are distinct and well defined. Since the resetting set $\mathcal{Z}$ is a subset of the state space and is independent of time, state-dependent singularly impulsive dynamical systems with time delay are time-invariant systems. Finally, in the case where $\mathcal{S} \equiv [0, \infty) \times \mathbb{R}^n \times \mathcal{Z}_{u_{\mathrm{c}}}$, where $\mathcal{Z}_{u_{\mathrm{c}}} \subset \mathcal{U}_{\mathrm{c}}$ we refer to (II.13)–(II.16) as an input-dependent singularly impulsive dynamical system with time delay. Both these cases represent a generalization to the impulsive control problem considered in the literature.

## III. Lyapunov and Asymptotic Stability of Singularly Impulsive Dynamical Systems with Time Delay

In this section we present Lyapunov and asymptotic stability results of singularly impulsive dynamical systems with time delay.

**Theorem III.1.** *Suppose there exists a continuously differentiable function $V : \mathcal{D} \to [0, \infty)$ satisfying $V(0) = 0$, $V(E_{\mathrm{c/d}}x) \geq 0$, $x \neq 0$, and*

$$V'(E_{\mathrm{c}}x) f_{\mathrm{c}}(x) \leq 0, \qquad x \in \mathcal{D}, \tag{III.17}$$

$$V(E_{\mathrm{d}}x + f_{\mathrm{d}}(x) \leq V(x), \qquad x \in \mathcal{D}. \tag{III.18}$$

*Then the zero solution $x(t,\tau) \equiv 0$ of the undisturbed $((u_{\mathrm{c}}(t), u_{\mathrm{d}}(t_k)) \equiv (0,0))$ time-dependent singularly impulsive dynamical system with time delay (II.8),(II.9) is Lyapunov stable. Furthermore, if the inequality (III.17) is strict for all $x \neq 0$, then the zero solution $x(t,\tau) \equiv 0$ of the undisturbed $((u_{\mathrm{c}}(t), u_{\mathrm{d}}(t_k)) \equiv (0,0))$ time-dependent singularly impulsive dynamical system with time delay (II.8), (II.9) is asymptotically stable. If, in addition, $\mathcal{D} = \mathbb{R}^n$ and*

$$V(E_{\mathrm{c/d}}x) \to \infty \text{ as } \|x\| \to \infty, \tag{III.19}$$

*then the zero solution $x(t,\tau) \equiv 0$ of the undisturbed $((u_{\mathrm{c}}(t), u_{\mathrm{d}}(t_k)) \equiv (0,0))$ time-dependent singularly impulsive dynamical system with time delay (II.8), (II.9) is globally asymptotically stable, Haddad, Chellaboina, and Kablar (2001), Kablar (2003b).*

*Proof:* Prior to the first resetting time, we can determine the value of $V(x(t,\tau))$ as

$$V(E_{\mathrm{c}}x(t,\tau)) = V(E_{\mathrm{c}}x(0)) + \int_0^t V'(x(E_{\mathrm{c}})) f_{\mathrm{c}}(x(s,\tau)\mathrm{d}s, \quad t \in [0, t_1]. \tag{III.20}$$

Between consecutive resetting times $t_k$ and $t_{k+1}$, we can determine the value of $V(x(t,\tau))$ as its initial value plus the integral of its rate of change along the trajectory $x(t,\tau)$, that is,

$$V(E_{\mathrm{c/d}}x(t,\tau) = V(E_{\mathrm{d}}x(t_k) + f_{\mathrm{d}}(x(t_k)) + \int_{t_k}^t V'(x(E_{\mathrm{c}})) f_{\mathrm{c}}(x(s,\tau))\mathrm{d}s, \quad t \in (t_k, t_{k+1}], \tag{III.21}$$

for $k = 1, 2, \ldots$. Adding and subtracting $V(x(E_{\mathrm{d}}t_k))$ to and from the right hand side of the (III.21) yields

$$V(E_{\mathrm{c/d}}x(t,\tau)) = V(E_{\mathrm{c}}x(t_k)) + [V(E_{\mathrm{d}}x(t_k) + f_{\mathrm{d}}(x(t_k)) - V(E_{\mathrm{d}}x(t_k))] + \int_{t_k}^t V'(E_{\mathrm{c}}x(s,\tau)) f_{\mathrm{c}}(x(s,\tau))\mathrm{d}s, \quad t \in (t_k, t_{k+1}], \tag{III.22}$$

and in particular at time $t_{k+1}$,

$$V(E_{\mathrm{d}}x(t_{k+1})) = V(E_{\mathrm{d}}x(t_k)) + [V(E_{\mathrm{d}}x(t_k) + f_{\mathrm{d}}(x(t_k))) - V(E_{\mathrm{d}}x(t_k))] + \int_{t_k}^{t_{k+1}} V'(x(s,\tau)) f_{\mathrm{c}}(x(s,\tau))\mathrm{d}s. \tag{III.23}$$

By recursively substituting (III.23) into (III.22) and ultimately into (III.20), we obtain

$$V(E_{\mathrm{c}}x(t,\tau)) = V(E_{\mathrm{c}}x(0)) + \int_0^t V'(E_{\mathrm{c}}x(s,\tau)) f_{\mathrm{c}}(x(s,\tau))\mathrm{d}s + \sum_{i=1}^k [V(E_{\mathrm{d}}x(t_i) + f_{\mathrm{d}}(x(t_i))) - V(E_{\mathrm{d}}x(t_i))]]. \tag{III.24}$$

If we allow $t_0 = 0$, and $\sum_{i=1}^0 = 0$, then (III.24) is valid for $k \in \mathcal{N}$. From (III.24) and (III.18) we obtain

$$V(E_{\mathrm{c}}x(t,\tau)) \leq V(E_{\mathrm{c}}x(0)) + \int_0^t V'(E_{\mathrm{c}}x(s,\tau)) f_{\mathrm{c}}(x(s,\tau))\mathrm{d}s, \quad t \geq 0. \tag{III.25}$$

Furthermore, it follows from (III.17) that

$$V(E_{\mathrm{c}}x(t,\tau)) \leq V(E_{\mathrm{c}}x(0)), \qquad t \geq 0, \tag{III.26}$$

so that Lyapunov stability follows from standard arguments.

Next, it follows from (III.18) and (III.24) that

$$V(E_{\mathrm{c}}x(t,\tau)) - V(E_{\mathrm{c}}x(s,\tau)) \leq \int_s^t V'(x(E_{\mathrm{c}}s,\tau)) f_{\mathrm{c}}(x(s,\tau))\mathrm{d}s, \quad t > s, \tag{III.27}$$

and, assuming strict inequality in (III.17), we obtain

$$V(E_{\mathrm{c}}x(t,\tau)) < V(E_{\mathrm{c}}x(s,\tau)), \qquad t > s, \tag{III.28}$$

provided $x(s,\tau) \neq 0$. Asymptotic stability, and, with (III.19), global asymptotic stability, then follow from standard arguments. ■

**Remark III.1.** *If in Theorem III.1 the inequality (III.18) is strict for all $x \neq 0$ as opposed to the inequality (III.17), and an infinite number of resetting times are used, that is, the set $\tau = \{t_1, t_2, \ldots\}$ is infinitely countable, then the zero solution $x(t,\tau) \equiv 0$ of the undisturbed time-dependent singularly impulsive dynamical system with time delay (II.8), (II.9) is also asymptotically stable. A similar remark holds for Theorem 2.2.2.*

**Remark III.2.** *In the proof of Theorem III.1, we note that assuming strict inequality in (III.17), the inequality (III.28) is obtained* provided $x(s,\tau) \neq 0$. *This proviso is necessary since it may be possible to reset the states to the origin, in which case $x(s,\tau) = 0$ for a finite value of $s$. In this case, for $t > s$, we have $V(E_{\mathrm{c}}x(t,\tau)) = V(E_{\mathrm{c}}x(s,\tau)) = V(0) = 0$. This situation does not present a problem, however, since reaching the origin in finite time is a stronger condition than reaching the origin as $t \to \infty$.*

**Remark III.3.** *If, additionally, in Theorem III.1 there exist scalars* $\alpha, \beta, \epsilon > 0$*, and* $p \geq 1$*, such that* $\alpha\|x\|^p \leq V(E_c x) \leq \beta\|x\|^p$*,* $x \in \mathcal{D}$*, and (III.17) is replaced by* $V'(E_c x) f_c(x) \leq -\epsilon V(E_c x)$*,* $x \in \mathcal{D}$*, then the zero solution* $x(t,\tau) \equiv 0$ *of the undisturbed* $((u_c(t), u_d(t_k)) \equiv (0,0))$ *time-dependent singularly impulsive dynamical system with time delay (II.8), (II.9) is exponentially stable. A similar remark holds for Theorem 2.2.2.*

**Remark III.4.** *Theorem III.1 presents sufficient conditions for time-dependent singularly impulsive dynamical systems with time delay in terms of Lyapunov functions that do not depend explicitly on time. Since time-dependent singularly impulsive dynamical systems are time-varying, Lyapunov functions that explicitly depend on time can also be considered. However, in this case the conditions on the Lyapunov functions required to guarantee stability are significantly harder to verify. For further details see Bainov and Simeonov (1989), Samoilenko and Perestyuk (1995), Ye, Michael, and Hou (1998).*

Next, we state a stability theorem for nonlinear state-dependent singularly impulsive dynamical systems with time delay.

**Theorem III.2.** *Suppose there exists a continuously differentiable function* $V : \mathcal{D} \to [0, \infty)$ *satisfying* $V(0) = 0$*,* $V(E_c x) \geq 0$*,* $x \neq 0$*, and*

$$V'(E_c x) f_c(x) \leq 0, \qquad x \notin \mathcal{Z}_x, \tag{III.29}$$

$$V(E_d x + f_d(x)) \leq V(E_c x), \qquad x \in \mathcal{Z}_x. \tag{III.30}$$

*Then the zero solution* $x(t,\tau) \equiv 0$ *of the undisturbed* $((u_c(t), u_d(t_k)) \equiv (0,0))$ *state-dependent singularly impulsive dynamical system with time delay (II.13), (II.14) is Lyapunov stable. Furthermore, if the inequality (III.29) is strict for all* $x \neq 0$*, then the zero solution* $x(t,\tau) \equiv 0$ *of the undisturbed* $((u_c(t), u_d(t_k)) \equiv (0,0)$ *state-dependent singularly impulsive dynamical system with time delay (II.13), (II.14) is asymptotically stable. If, in addition,* $\mathcal{D} = \mathbb{R}^n$ *and (III.19) is satisfied, then the zero solution* $x(t,\tau) \equiv 0$ *of the undisturbed* $((u_c(t), u_d(t)k)) \equiv (0,0))$ *state-dependent singularly impulsive dynamical system with time delay (II.13), (II.14) is globally asymptotically stable, Haddad, Chellaboina, and Kablar (2001), Kablar (2003b).*

*Proof:* For $\mathcal{S} = [0, \infty) \times \mathcal{Z}_x$ it follows from Assumptions A1–A3 that the resetting times $n_k(x_0)$ are well defined and distinct for every trajectory of (II.13), (II.14) with $(u_c(t), u_d(t_k)) \equiv (0,0)$. Now, the proof follows as in the proof of Theorem III.1 with $t_k$ replaced by $n_k(x_0)$. ■

**Remark III.5.** *To examine the stability of linear state-dependent singularly impulsive dynamical systems with time delay set* $f_c(x) = A_c x$*, and* $f_d(x) = (A_d - E_d)x$ *in Theorem III.2. Considering the quadratic Lyapunov function candidate* $V(E_{c/d}x) = x^T E_{c/d}^T P E_{c/d} x$*, for the argument* $E_c x$ *and* $E_d x$*, respectively where* $P > 0$*, it follows from Theorem III.2 that the conditions*

$$x^T(A_c^T P E_c + E_c^T P A_c)x < 0, \qquad x \notin \mathcal{Z}_x, \tag{III.31}$$

$$x^T(A_d^T P A_d - E_d^T P E_d)x \leq 0, \qquad x \in \mathcal{Z}_x, \tag{III.32}$$

*establish asymptotic stability for linear state-dependent singularly impulsive dynamical systems with time delay. These conditions are implied by* $P > 0$*,* $A_c^T P E_c + E_c^T P A_c < 0$*, and* $A_d^T P A_d - E_d^T P E_d \leq 0$ *which can be solved using Linear Matrix Inequality (LMI) feasibility problem Boyd et al. (1994). See also Haddad, Chellaboina, and Kablar (2001a).*

## IV. Conclusion

In this paper we presented new class of singularly impulsive or generalized impulsive dynamical systems with delay. We gave assupmtions needed for the model and basic division of singularly impulsive dynamical systems into twio classes: time dependenet and state dependent. Next, we developed Lyapunov and asymptotic stability results.

## V. Future Work

It is left to develop invariant set theorem for singularly impulsive dynamical systems. Next, further work will concentrate to specializing this results and developing to time-delay systems. The last is motivated by recognized need in biological applications.

On the other hand finite-time and practical stability results will be developed for the class of impulsive and singularly impulsive dynamical systems with delay.

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## VII. References

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